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# Compact spaces, compact cardinals, and elementary submodels

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**Abstract**

If $M$ is an elementary submodel and $X$ a topological space, then $X_M$ denotes the set $X \cap M$ given the topology generated by the open subsets of $X$ which are members of $M$. Call a compact space *squashable* iff for some $M$, $X_M$ is compact and $X_M \neq X$. The first supercompact cardinal is the least $\kappa$ such that all compact $X$ with $|X| \geqslant \kappa$ are squashable. The first $\lambda$ such that ${}^{\lambda}2$ is squashable is larger than the first 1-extendible cardinal.


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## 1. Introduction

Elementary submodels were first used in set theory by Gödel [5,6] to prove the GCH from $V = L$. They have now become a standard tool in combinatorics and topology; see Dow [4]. In using them, one applies the downward Löwenheim–Skolem–Tarski Theorem to the universe, $V$, to get a small $M \prec V$. This technique simplifies many combinatorial closure arguments. When this technique is formalized in ZFC, one cannot actually define "$M \prec V$", so one replaces it by "$M \prec H(\theta)$", where $\theta$ is a regular cardinal which is large enough that $H(\theta)$ contains all objects under study. $H(\theta)$ can replace $V$ in most applications because $H(\theta) \models \mathrm{ZFC} - \mathrm{P}$ (that is, ZFC minus the power set axiom), so that many elementary combinatorial arguments can be carried out within $H(\theta)$.

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Bandlow [2,3] used these methods prove theorems about Corson compacta. More recently, Junqueira and Tall [7–9,13] have proved a number of general results relating a compact space $X$ to $X \cap M$. We shall follow their notation here, and shall explore the relationship between properties of $X \cap M$ and large cardinals (primarily, supercompact and extendible—see Kanamori [10, §§22,23]).

Formally, a topological space is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a topology (family of open sets) on the set $X$. Then $X = \bigcup \mathcal{T}$, so $\mathcal{T}$ determines $X$, while $X$ does not determine $\mathcal{T}$. Nevertheless, we shall follow the usual abuse of notation and refer to $(X, \mathcal{T})$ as $X$. This abuse extends into elementary submodels, but we must exercise a bit of care here, since to apply $M \prec H(\theta)$ to $X$, we need $\mathcal{T} \in M$, not just $X \in M$.

**Definition 1.1.** Suppose that $(X, \mathcal{T}) \in M \prec H(\theta)$. Then:

- $X_M$ denotes the space $(X \cap M, \mathcal{T}_M)$, where $\mathcal{T}_M$ is the topology on $X \cap M$ which has, as a base, $\{U \cap M\colon\ U \in \mathcal{T} \cap M\}$.
- $X \cap M$ denotes the set $X \cap M$ with the subspace topology, denoted by $\mathcal{T} \restriction M$.

Note that $\mathcal{T}_M \subseteq \mathcal{T} \restriction M$ (that is, $X_M$ is coarser than $X \cap M$). This paper will focus on compact Hausdorff spaces. For these, if $(X, \mathcal{T}) \in M$ and $X \subseteq M$, then $\mathcal{T}_M = \mathcal{T} \restriction M = \mathcal{T}$, since $\mathcal{T}_M$ is coarser than $\mathcal{T}$ and is Hausdorff. We shall investigate the question: When can we have $X_M$ compact and $X \not\subseteq M$? Observe (see [7]) that compactness of $X_M$ implies compactness of $X$.

Consider the example $X = \alpha + 1$, with the usual compact order topology. Then $X \cap M$ is compact only in the trivial case that $\alpha + 1 \subset M$ (since the least $\gamma \neq M$ is a limit ordinal), but $X_M$ is always compact, and is homeomorphic to the order topology on the set $X \cap M$ (whose type is some successor ordinal). More generally,

**Theorem 1.2** (Junqueira and Tall [9]). *Assume that $(X, \mathcal{T}) \in H(\theta)$, $X$ is compact Hausdorff, and $\theta$ is regular. Then the following are equivalent.*

(1) *$X$ scattered.*
(2) *$X_M$ is compact for all $M$ such that $(X, \mathcal{T}) \in M \prec H(\theta)$.*
(3) *$X_M$ is compact for some countable $M$ such that $(X, \mathcal{T}) \in M \prec H(\theta)$.*

If one wants the finer $X \cap M$ to be compact for countable $M$, one needs the stronger assertion that $X$ is scattered *and* Corson compact; equivalently, by Alster [1], strongly Eberlein compact. See Section 3 for some further remarks on this. The main focus of this paper will be on more general compact spaces and the relationships between compactness of $X_M$ and large cardinals. We summarize our results now, with proofs given in Section 2.

**Definition 1.3.** Let $(X, \mathcal{T})$ be compact Hausdorff.

- $(X, \mathcal{T})$ is *squashable* iff for some regular $\theta$ and some $M$: $(X, \mathcal{T}) \in M \prec H(\theta)$, $X_M$ is compact, and $X \not\subseteq M$.

- $(X, \mathcal{T})$ is *$\kappa$-squashable* iff for some regular $\theta$ and some $M$: $(X, \mathcal{T}) \in M \prec H(\theta)$, $X_M$ is compact, and $|M| < \kappa$.

See Definition 3.5 for the notion of "squashing". Note that $\kappa$-squashability is trivial when $\kappa > |X|$ and implies squashability when $\kappa \leqslant |X|$. By the following lemma, squashability is a *topological* property (i.e., invariant under homeomorphism), and is not sensitive to the specific $\theta$ used:

**Lemma 1.4.** *Let $(X, \mathcal{T})$ be compact Hausdorff.*

- *$(X, \mathcal{T})$ is squashable iff for all $(Y, \mathcal{U})$ homeomorphic to $(X, \mathcal{T})$, all regular $\theta$, and all sets $a$: If $Y, \mathcal{U}, a \in H(\theta)$, then there is an $M \prec H(\theta)$ such that $Y, \mathcal{U}, a \in M$, $Y_M$ is compact, and $Y \nsubseteq M$.*
- *$(X, \mathcal{T})$ is $\kappa$-squashable iff for all $(Y, \mathcal{U})$ homeomorphic to $(X, \mathcal{T})$, all regular $\theta$, and all sets $a$: If $Y, \mathcal{U}, a \in H(\theta)$, then there is an $M \prec H(\theta)$ such that $Y, \mathcal{U}, a \in M$, $Y_M$ is compact, and $|M| < \kappa$.*

We never use the set $a$ in this paper, but in applications of elementary submodels, it can encode whatever else besides the topology is needed for the argument.

**Theorem 1.5.** *If $(X, \mathcal{T})$ is compact Hausdorff and $\kappa$ is $|X|$-supercompact then $(X, \mathcal{T})$ is $\kappa$-squashable.*

This sharpens a theorem from [13], which used a 2-huge cardinal (a stronger assumption) to get similar results. By Theorem 1.6, supercompactness is the correct order of largeness in Theorem 1.5, although in the special case $|X| = \kappa$, all that is needed is weak compactness, not $\kappa$-supercompactness (i.e., measurability). This is discussed further in Section 2.

**Theorem 1.6.** *Suppose $\sigma$ is such that ${}^{\lambda}2$ is squashable for every $\lambda \geqslant \sigma$. Then there is a supercompact cardinal $\leqslant \sigma$.*

We write ${}^{\lambda}2$ for the product space here to distinguish it from the cardinal exponent $2^{\lambda}$. The first $\lambda$ such that ${}^{\lambda}2$ is squashable is related to the low end of supercompactness and extendibility:

**Definition 1.7.** For $n < \omega$, let $\text{ext}_n(\kappa, \lambda)$ assert that there is an elementary embedding $j : R(\kappa + n) \to R(\lambda + n)$ such that $j(\kappa) = \lambda > \kappa$ and $j \restriction \kappa$ is the identity. $\kappa$ is *$n$-extendible* iff $\text{ext}_n(\kappa, \lambda)$ for some $\lambda$.

Clearly, this property gets stronger as $n$ gets bigger. Extendibility is interleaved with supercompactness. If $\kappa$ is 1-extendible, then $\kappa$ is the $\kappa$th measurable cardinal (see [10, Proposition 23.1]). If $\kappa$ is $2^{\kappa}$-supercompact, then there is an $A \in [\kappa]^{\kappa}$ such that $\text{ext}_1(\alpha, \beta)$ for all $\alpha, \beta \in A$ with $\alpha < \beta$. If $\kappa$ is 2-extendible, then $\kappa$ is the $\kappa$th element of the class $\{\sigma\colon \sigma \text{ is } 2^{\sigma}\text{-supercompact}\}$.

**Theorem 1.8.** *Let $\lambda$ be an infinite cardinal and assume that $X$ is compact Hausdorff and squashable and that $\chi(p, X) = \lambda$ for all $p \in X$. Then there are strongly inaccessible cardinals $\alpha < \beta < \kappa < \lambda$ such that* $\mathrm{ext}_1(\alpha, \beta)$.

**Theorem 1.9.** *If $\kappa$ is the first* 1*-extendible cardinal, $\lambda$ is least such that ${}^{\lambda}2$ is squashable, and $\sigma$ is the least cardinal such that $\sigma$ is $2^{\sigma}$-supercompact, then $\kappa < \lambda < \sigma$.*

These theorems improve a result from [9], which obtains a strongly inaccessible cardinal $\leqslant \lambda$ and the existence of $0^{\#}$ from the squashability of ${}^{\lambda}2$.

Observe that the character assumptions on $X$ in Theorem 1.8 imply that $|X| = 2^{\lambda}$ (by the Čech–Pospíšil Theorem and Arhangel'skiĭ's Theorem), so that $X$ will be squashable whenever some cardinal $\leqslant\lambda$ is $2^{\lambda}$-supercompact. Note that there are many squashable spaces of all sizes whose existence does not entail large cardinals. For example, by Theorem 1.2, a compact scattered space $X$ is squashable iff $|X| > \aleph_0$. Likewise, if $X$ is Corson compact and not scattered, then $X$ is squashable iff $|X| > 2^{\aleph_0}$, since whenever $M$ is countably closed, $X_M = X \cap M$ will be compact.

By Theorem 1.8, if $\kappa$ is the least strong inaccessible such that $R(\kappa)$ is a model for "1-extendible cardinals exist", then also $R(\kappa)$ is a model for "no compact Hausdorff space all of whose points have the same character is squashable".

## 2. Compactness and large cardinals

We begin by proving a strengthening of Lemma 1.4 which also makes it clear that for a compact Hausdorff $(X, \mathcal{T})$, the squashability of $X$ is equivalent to a statement about objects of size $|X|$, although the original definition involved $(X, \mathcal{T}) \in H(\theta)$, where $\theta$, and possibly also $|\mathcal{T}|$, exceeds $|X|$. To see this, we restate the definition of $X_M$, replacing $\mathcal{T}$ by a base or a subbase (noting that $\mathrm{w}(X) \leqslant |X|$), and replacing $H(\theta)$ by an arbitrary transitive model of ZFC − P:

**Definition 2.1.** If $X$ is any set and $\mathcal{A} \subseteq \mathcal{P}(X)$, then $\mathcal{T}_{\mathcal{A}}$ denotes the topology on $X$ which has $\mathcal{A}$ for a subbase. $(X, \mathcal{A}) \cong (Y, \mathcal{B})$ means that $(X, \mathcal{T}_{\mathcal{A}})$ and $(Y, \mathcal{T}_{\mathcal{B}})$ are homeomorphic. If $(X, \mathcal{A}) \in M \prec N$, where $N$ is a transitive model of ZFC − P, then $X_M$ denotes the set $X \cap M$ with the topology which has, as a subbase, $\{U \cap M\colon U \in \mathcal{A} \cap M\}$.

This is the same topology we get by applying Definition 1.1 to $(X, \mathcal{T}_{\mathcal{A}})$ in the case that $\mathcal{T}_{\mathcal{A}} \in N$. Lemma 1.4 is immediate from the following:

**Lemma 2.2.** *For compact Hausdorff $(X, \mathcal{T})$, the following are equivalent*:

(1) *$(X, \mathcal{T})$ is squashable.*
(2) *For all transitive $N \models$ ZFC − P, all $(Y, \mathcal{B}) \in N$ such $(Y, \mathcal{B}) \cong (X, \mathcal{T})$, and all sets $a \in N$, there is an $M \prec N$ such that $Y, \mathcal{B}, a \in M$, $Y_M$ is compact, and $Y \not\subseteq M$.*
(3) (2) *restricted to $N$ with $|N| = |X|$.*
(4) (2) *restricted to $N$ of the form $H(\theta)$ for regular $\theta$.*

*Also, for each fixed $\kappa$, we get the same four equivalents when "squashable" is replaced by "$\kappa$-squashable" in* (1) *and "$Y \not\subseteq M$" is replaced by "$|M| < \kappa$" in* (2).

**Proof.** We shall only prove the "squashable" version of the lemma, since the "$\kappa$-squashable" proof is almost identical. Since (2) $\Rightarrow$ (4) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are obvious, it is sufficient to prove (3) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).

For (3) $\Rightarrow$ (2), fix $N, Y, \mathcal{B}, a$ as in (2). Let $P \prec N$ with $|P| = |Y| = |X|$, $Y, \mathcal{B}, a \in P$, and $Y \subset P$. Let $Q$ be the transitive collapse of $P$ with $j : Q \to P$ the Mostowski isomorphism, so that $j : Q \to N$ is an elementary embedding. Let $j(\widetilde{Y}) = Y$, $j(\widetilde{\mathcal{B}}) = \mathcal{B}$, and $j(\tilde{a}) = a$. Since $|Q| = |X|$ and $(\widetilde{Y}, \widetilde{\mathcal{B}}) \cong (X, \mathcal{T})$ (by compactness of $X$), we can apply (3) to $Q$ to get $\widetilde{M} \prec Q$ such that $\widetilde{Y}, \widetilde{\mathcal{B}}, \tilde{a} \in \widetilde{M}$, $\widetilde{Y}_{\widetilde{M}}$ is compact, and $\widetilde{Y} \not\subseteq \widetilde{M}$. Now, let $M = j``\widetilde{M}$.

For (1) $\Rightarrow$ (3), first apply "squashable" to fix a regular $\theta$ and an $M$ such that $(X, \mathcal{T}) \in M \prec H(\theta)$, $X_M$ is compact, and $X \not\subseteq M$. Assume that (3) fails, and fix $(N, Y, \mathcal{B}, a, f)$ which is a counter-example in the sense that $(N, Y, \mathcal{B}, a)$ is a counter-example to (2), $|N| = |X|$, and $f : X \to Y$ is a homeomorphism. Then $(N, Y, \mathcal{B}, a, f) \in H(\theta)$, and the statement that it is a counter-example can be expressed within $H(\theta)$, so that by $(X, \mathcal{T}) \in M \prec H(\theta)$, we may assume that $(N, Y, \mathcal{B}, a, f) \in M$. But then $M \cap N \prec N$, $Y, \mathcal{B}, a \in M \cap N$, $Y_{M \cap N} = Y_M$ is compact, and $Y \not\subseteq M \cap N$ (since $f \in M$), so that $(N, Y, \mathcal{B}, a, f)$ is not a counter-example. □

**Proof of Theorem 1.5.** Let $j : V \to W$ be a $|X|$-supercompact embedding. Then $W$ is a transitive class, $j$ is an elementary embedding, $j(\kappa) > |X|$, $j \upharpoonright \kappa$ is the identity, and $W^{|X|} \subset W$. Fix $N, Y, \mathcal{B}, a$ as in (2) of the "$\kappa$-squashable" version of Lemma 2.2, with $|N| = |X|$, as in (3). Abbreviate the conclusion of (2) as $\exists M \Phi(\kappa, M, N, Y, \mathcal{B}, a)$. Now $j``N \in W$ and $\Phi^W(j(\kappa), j``N, j(N), j(Y), j(\mathcal{B}), j(a))$ holds, where $\Phi^W$ denotes relativization to the model $W$. Hence, we have also $\exists M \Phi^W(j(\kappa), M, j(N), j(Y), j(\mathcal{B}), j(a))$, and thus $\exists M \Phi(\kappa, M, N, Y, \mathcal{B}, a)$. □

A similar argument shows that in the case $|X| = \kappa$, we do not need $\kappa$-supercompactness (i.e., measurability), but only weak compactness; see Theorems 2.11, 2.12, and 2.13.

To reverse Theorem 1.5 and obtain extendibility and supercompactness from squashability assumptions, we need to obtain elementary embeddings between transitive models with suitable closure properties. Now, starting from $(X, \mathcal{T}) \in M \prec H(\theta)$, we can let $N$ be the transitive collapse of $M$ and get an elementary embedding $j : N \to H(\theta)$. If $X_M$ is compact and all points of $X$ have large character, then we can deduce the correct closure properties of $N$ by applying the proof of the Čech–Pospíšil Theorem:

**Definition 2.3.** A $\lambda$-*Čech–Pospíšil tree* in a space $X$ is a tree $\mathcal{K} = \langle K_s : s \in {}^{\leqslant\lambda}2 \rangle$ satisfying:

(1) Each $K_s$ is non-empty and is closed in $X$.
(2) $s \subseteq t \to K_s \supseteq K_t$.
(3) $K_{s^\frown 0} \cap K_{s^\frown 1} = \emptyset$.
(4) If $\mathrm{lh}(s) = \gamma$, a limit ordinal, then $K_s = \bigcap_{\alpha < \gamma} K_{s \upharpoonright \alpha}$.

Note that when discussing sequences $s, t \in {}^{\leqslant\lambda}2$, $\mathrm{lh}(s) = \mathrm{dom}(s)$, and $s \subseteq t$ means that $s$ is an initial segment of $t$.

**Theorem 2.4** (Čech and Pospíšil). *If $X$ is compact Hausdorff and $\chi(p, X) \geqslant \lambda$ for all $p \in X$, then there is a $\lambda$-Čech–Pospíšil tree in $X$, and hence $|X| \geqslant 2^\lambda$.*

Applied within a transitive model $N$, these trees can be used to prove that $\mathcal{P}(\lambda) \subset N$:

**Definition 2.5.** Let $N$ be a transitive model of ZFC − P, $(X, \mathcal{T}) \in N$, and $N \models$ "$\mathcal{T}$ is a topology on $X$". Then $(X, \mathcal{T})$ is *truly compact* iff, in $V$, the topology on $X$ which has $\mathcal{T}$ for a base is compact.

**Lemma 2.6.** *Let $N$ be a transitive model of* ZFC − P *with $X, \mathcal{T}, \mathcal{K} \in N$. Assume that $N \models$ "$\mathcal{T}$ is a topology on $X$ and $\mathcal{K}$ is a $\lambda$-Čech–Pospíšil tree in $X$". Assume that $(X, \mathcal{T})$ is truly compact. Then $\mathcal{P}(\lambda) \subset N$.*

**Proof.** Prove by induction on $\gamma \leqslant \lambda$ that ${}^\gamma 2 \subset N$. Assume that $\gamma$ is a limit (otherwise the induction step is trivial), and fix $s \in {}^\gamma 2$. Then for $\alpha < \gamma$, induction gives us $s{\restriction}\alpha \in N$, so that $K_{s\restriction\alpha}$ is defined and is closed in $X$. Since $(X, \mathcal{T})$ is truly compact, $\bigcap_{\alpha<\gamma} K_{s\restriction\alpha}$ is non-empty, so fix $x \in \bigcap_{\alpha<\gamma} K_{s\restriction\alpha}$. Then, $s = \bigcup\{t\colon\ x \in K_t\ \&\ \mathrm{lh}(t) < \gamma\} \in N$. □

Next, note that these truly compact $(X, \mathcal{T})$ occur naturally when we collapse an elementary submodel:

**Lemma 2.7.** *Suppose that $(X, \mathcal{T}) \in M \prec H(\theta)$ and $X_M$ is compact. Let $j : N \to M$ be the Mostowski isomorphism from a transitive $N$ onto $M$, and let $j(\widetilde{X}) = X$ and $j(\widetilde{\mathcal{T}}) = \mathcal{T}$. Then $N$ is a transitive model of* ZFC − P *and $(\widetilde{X}, \widetilde{\mathcal{T}})$ is truly compact. Furthermore, if $X$ has no isolated points, $\lambda = \min\{\chi(p, X)\colon\ p \in X\}$, and $j(\tilde{\lambda}) = \lambda$, then $\mathcal{P}(\tilde{\lambda}) \subset N$.*

**Proof of Theorem 1.8.** We have $(X, \mathcal{T}) \in M \prec H(\theta)$ with $X_M$ compact, $X \nsubseteq M$, and $\chi(p, X) = \lambda$ for all $p \in X$. Now, follow the notation in Lemma 2.7. If $\lambda \subset M$, we would have $\tilde{\lambda} = \lambda$ and $\mathcal{P}(\lambda) \subset M$, which would imply $X \subset M$ because $|X| = 2^\lambda$ by Arhangel'skiĭ's Theorem. Thus, if $\kappa$ is the first ordinal not in $M$, then $\kappa < j(\kappa) \leqslant \lambda$ and $\kappa \leqslant \tilde{\lambda}$.

$j : N \to H(\theta)$ is an elementary embedding. $\mathcal{P}(\kappa) \subseteq \mathcal{P}(\tilde{\lambda}) \subset N$, so $\kappa$ is measurable and hence strongly inaccessible, so that $j(\kappa)$ is also strongly inaccessible. Thus, from $\mathcal{P}(\kappa) \subset N$ and $\mathcal{P}(j(\kappa)) \subset H(\theta)$, we get $R(\kappa+1) \subset N$ and $R(j(\kappa)+1) \subset H(\theta)$. Then, $j \restriction R(\kappa+1) : R(\kappa+1) \to R(j(\kappa)+1)$ establishes that $\mathrm{ext}_1(\kappa, j(\kappa))$.

Now, $j \restriction R(\kappa+1) \in H(\theta)$ and $R(j(\kappa)+1) \in H(\theta)$ (since $|R(j(\kappa)+1)| = 2^{j(\kappa)} \leqslant 2^\lambda = |X| < \theta$), so $H(\theta) \models \mathrm{ext}_1(\kappa, j(\kappa))$, so $N \models \exists\alpha < \kappa\, [\mathrm{ext}_1(\alpha, \kappa)]$. Fixing one such $\alpha$, and using $R(\kappa+1) \in N$, we have $\mathrm{ext}_1(\alpha, \kappa)$ is really true (in $V$, and in $H(\theta)$), so using $j$ again, there is a $\beta < \kappa$ such that $\mathrm{ext}_1(\alpha, \beta)$. □

**Proof of Theorem 1.9.** We already have $\kappa < \lambda \leqslant \sigma$ by Theorems 1.8 and 1.5. But now if $j : V \to W$ is a $2^\sigma$-supercompact embedding first moving $\sigma$, then the squashability of ${}^\sigma 2$

implies that $({}^{\sigma}2 \text{ is squashable})^W$ by Lemma 2.2, since this notion can be expressed with objects of size only $2^{\sigma}$. Hence, ${}^{\lambda}2$ is squashable for some $\lambda < \sigma$. □

To prove Theorem 1.6, we need the following minor modification of Theorem 2 of Magidor [12]:

**Definition 2.8.** If $\mathfrak{A}, \mathfrak{B}$ are structures for the same language, then $\mathfrak{B} \prec_2 \mathfrak{A}$ means that $\mathfrak{B}$ is an elementary substructure of $\mathfrak{A}$ with respect to all formulas of second order logic.

**Lemma 2.9.** *For any cardinal $\sigma$, the following are equivalent.*

(1) *Some cardinal $\leqslant \sigma$ is supercompact.*
(2) *Whenever $\mathfrak{A}$ is a structure for a finite language and $|\mathfrak{A}| \geqslant \sigma$, there is a $\mathfrak{B} \prec_2 \mathfrak{A}$ with $|\mathfrak{B}| < |\mathfrak{A}|$.*
(3) *Whenever $\mathfrak{A}$ is a structure for a finite language and $|\mathfrak{A}| \geqslant \sigma$, there is a $\mathfrak{B} \prec_2 \mathfrak{A}$ with $\mathfrak{B} \neq \mathfrak{A}$.*

**Proof.** (1) $\Leftrightarrow$ (2) is from [12] and (2) $\Rightarrow$ (3) is obvious. For (3) $\Rightarrow$ (2), assume (3), and fix $\mathfrak{A}$ with $|\mathfrak{A}| = \lambda \geqslant \sigma$. WLOG, $\mathfrak{A} = (\lambda; R_1, \ldots, R_n)$.

Apply (3) to the structure $(R(\lambda + \omega); \in, \mathfrak{A})$ to get $M \prec_2 R(\lambda + \omega)$ with $\mathfrak{A} \in M$ and $M \neq R(\lambda + \omega)$. Then, if $N$ is the transitive collapse of $M$, we get $j : N \to R(\lambda + \omega)$ with $j$ an elementary embedding with respect to second order logic, $\mathfrak{A} \in \operatorname{ran}(j)$, and $j$ not the identity map. Let $j(\widetilde{\mathfrak{A}}) = \mathfrak{A}$. Then $\widetilde{\mathfrak{A}} = (\tilde{\lambda}; \widetilde{R}_1, \ldots, \widetilde{R}_n)$, where $j(\tilde{\lambda}) = \lambda$.

Since $j$ is second-order elementary, $N$ must really equal $R(\tilde{\lambda} + \omega)$. It follows that $\tilde{\lambda} < \lambda$, since there is no non-trivial elementary embedding of $R(\lambda + \omega)$ into itself (see [10], Corollary 23.14). Now, let $\mathfrak{B}$ be the restriction of $\mathfrak{A}$ to $j``\lambda$. Then $\mathfrak{B} \prec_2 \mathfrak{A}$ and $|\mathfrak{B}| = \tilde{\lambda} < |\mathfrak{A}|$. □

**Proof of Theorem 1.6.** Assume that ${}^{\lambda}2$ is squashable for all $\lambda \geqslant \sigma$. By Lemma 2.9, it is sufficient to prove that for all $\lambda \geqslant \sigma$ and all structures $\mathfrak{A} = (\lambda; R_1, \ldots, R_n)$, there is a proper elementary substructure $\mathfrak{B} \prec_2 \mathfrak{A}$. So, assume, for some fixed $\lambda \geqslant \sigma$, some $\mathfrak{A}$ is a counter-example to this.

Applying squashability, fix $\theta, M$ with $(X, \mathcal{T}) \in M \prec H(\theta)$, where $X = {}^{\lambda}2$, $X_M$ is compact, and $X \not\subseteq M$. Obtain $j, \tilde{\lambda}, N$ as in the proof of Theorem 1.8. Note that $\theta \geqslant (2^{\lambda})^+$. By $M \prec H(\theta)$, we may assume that our counter-example $\mathfrak{A}$ is in $M$. Now, fix $\mathfrak{B}$ so that $B = j``\tilde{\lambda}$. Since $\mathcal{P}(\tilde{\lambda}) \subset N$, we have $\mathfrak{B} \prec_2 \mathfrak{A}$, and $\mathfrak{B} \neq \mathfrak{A}$, since $j``\tilde{\lambda} = \lambda$ would imply that $X \subset M$. □

Finally, we consider the case where $|X|$ itself is inaccessible. Here, to prove squashability of $|X|$, weak compactness, or even just $\Pi^1_1$-indescribability, suffices:

**Definition 2.10.** A cardinal $\kappa$ is *$\Pi^1_1$-indescribable* iff whenever $\varphi$ is a $\Pi^1_1$ sentence, $\mathfrak{A} = (\kappa; R_1, \ldots, R_n)$, and $\mathfrak{A} \models \varphi$, there is an $\alpha < \kappa$ such that $\mathfrak{B} := (\alpha; R_1 \restriction \alpha, \ldots, R_n \restriction \alpha) \prec \mathfrak{A}$ and $\mathfrak{B} \models \varphi$.

This implies that $\kappa$ is weakly inaccessible, but not necessarily strongly inaccessible. $\Pi^1_1$-indescribability plus strong inaccessibility is equivalent to weak compactness. Many authors include strong inaccessibility as part of the definition of $\Pi^1_1$-indescribability; see [10, §6] for more details.

**Theorem 2.11.** *Assume that $\kappa = |X|$ is $\Pi^1_1$-indescribable and that $X, \mathcal{T}, a \in H(\theta)$, where $(X, \mathcal{T})$ is compact Hausdorff and $\theta$ is regular. Then there is an $M$ with $X, \mathcal{T}, a \in M \prec H(\theta)$ such that $|M| < \kappa$, $M \cap \kappa \in \kappa$, and $X_M$ is compact.*

**Proof.** As in (3) ⇒ (2) of Lemma 2.2, it is sufficient to show that for any transitive $N \models \mathrm{ZFC} - \mathrm{P}$ with $|N| = \kappa$ and $Y, \mathcal{B}, a \in N$ such $(Y, \mathcal{B}) \cong (X, \mathcal{T})$, there is an $M \prec N$ such that $Y, \mathcal{B}, a \in M$, $Y_M$ is compact, $|M| < \kappa$, and $M \cap \kappa \in \kappa$. But since $(N, \in)$ may be coded by relations on $\kappa$, and compactness of $Y$ expressed by a $\Pi^1_1$ sentence, we may apply $\Pi^1_1$-indescribability to get $M$. □

Note that if $\kappa = 2^\rho$, then $\kappa$ cannot be $\Pi^1_1$-indescribable, and the space $X = {}^\rho 2$ is a counter-example to the theorem. That is ${}^\rho 2 \in M$ implies that $\rho \in M$ (by $M \prec H(\theta)$), and hence $\rho \subset M$ (by $M \cap \kappa \in \kappa$). But then $|M| \geqslant 2^\rho = \kappa$ would follow from compactness of $X_M$ (see Lemma 2.7). Without the "$M \cap \kappa \in \kappa$" in the conclusion, we would get an $M$ as in Theorem 2.11 by Theorem 1.5 whenever $\kappa$ is above the first supercompact cardinal.

It is consistent for a $\kappa < 2^{\aleph_0}$ to be $\Pi^1_1$-indescribable (see [11]), but the theorem for these $\kappa$ is trivial by Theorem 1.2, since then $X$ must be scattered. It is also consistent to have a $\Pi^1_1$-indescribable $\kappa$ with $2^{\aleph_0} < \kappa < 2^{\aleph_1}$, and the theorem does have non-vacuous content in this case. For strongly inaccessible $\kappa$, the theorem fails whenever $\kappa$ is not weakly compact:

**Theorem 2.12.** *Assume that $\kappa$ is strongly inaccessible and not $\Pi^1_1$-indescribable. Then there is a compact Hausdorff $X$ of size $\kappa$ such that no $M$ satisfies $(X, \mathcal{T}) \in M \prec H(\theta)$, $M \cap \kappa \in \kappa$, and $X_M$ is compact.*

**Proof.** Since $\kappa$ is strongly inaccessible and not weakly compact, there is a $\kappa$-Aronszajn tree, $T \subset {}^{<\kappa}2$. Let $X$ be the corresponding Aronszajn line, which is the space of all maximal chains in $T$; this is a compact LOTS under its lexical ordering. Then $X$ is compact, $|X| = \kappa$ (since $\kappa$ is strongly inaccessible), and there are no increasing or decreasing $\kappa$-sequences in $X$. Now, assume that $(X, \mathcal{T}) \in M \prec H(\theta)$, $\gamma = M \cap \kappa \in \kappa$, and $X_M$ is compact. We shall derive a contradiction.

Note that $\gamma$ is strong limit cardinal and $|X \cap M| = \gamma$. Since $X$ is a LOTS, the topology of $X_M$ is just the order topology induced by the natural ordering on $X \cap M$, so compactness of $X_M$ implies that $X \cap M$ is Dedekind-complete. Let 0 and 1 be the first and last elements of $X$. $X \cap M$ is not dense in $X$ (since $2^\gamma < \kappa$), so fix $p \in X \setminus \overline{(X \cap M)}$. Let $a = \sup([0, p) \cap M)$ and $b = \inf((p, 1] \cap M)$. Then $a < p < b$.

If $a, b \notin M$, then $[0, p) \cap M$ has no least upper bound in the set $X \cap M$, contradicting the Dedekind completeness of $X \cap M$. So, $a \in M$ or $b \in M$.

Say $b \in M$. Note that $[0, b)$ cannot have a largest element (using $M \prec H(\theta)$ and $[p, b) \cap M = \emptyset$). Let $\sigma$ be the cofinality of the order type $[0, b)$. Then $\sigma \in M$ and $\sigma < \kappa$

(since $X$ is Aronszajn), so $\sigma < \gamma$. Since $\sigma \in M$, there is a $\sigma$-sequence, $\vec{c} = \langle c_\alpha \colon \alpha < \sigma \rangle \nearrow b$, with $\vec{c} \in M$. But then, each $c_\alpha \in M$, contradicting $[p, b) \cap M = \emptyset$. □

**Theorem 2.13.** *Assume that $\kappa$ is strongly inaccessible and not $\Pi^1_1$-indescribable, and that there are no* 1*-extendible cardinals less than $\kappa$. Then there is a compact Hausdorff $Z$ of size $\kappa$ which is not squashable.*

**Proof.** Let $Y_0$ be the disjoint sum of all ${}^{\lambda}2$ for $\lambda$ a cardinal less than $\kappa$, and let $Y$ be the 1-point compactification of $Y_0$. Let $X$ be a $\kappa$-Aronszajn line, as in the proof of Theorem 2.12. Let $Z$ be the disjoint sum of $X$ and $Y$.

If $(Z, \mathcal{T}) \in M \prec H(\theta)$ and $Z_M$ is compact, then $\lambda \subset M$ whenever $\lambda < \kappa$ and $\lambda \in M$ (since ${}^{\lambda}2$ is not squashable by Theorem 1.9). It follows that $M \cap \kappa$ is an ordinal $\leqslant \kappa$. But $M \cap \kappa \in \kappa$ would yield a contradiction, as in the proof of Theorem 2.12, and $M \cap \kappa = \kappa$ implies that $Z \subset M$. □

## 3. Remarks on Corson compacta

A *Corson Compactum* is a space $X$ homeomorphic to some closed $Y \subseteq {}^{\lambda}[0, 1]$ such that $\{\alpha \colon y_\alpha \neq 0\}$ is countable for all $y \in Y$. A *Strong Eberlein Compactum* is a space $X$ homeomorphic to some closed $Y \subseteq {}^{\lambda}\{0, 1\}$ such that $\{\alpha \colon y_\alpha = 1\}$ is finite for all $y \in Y$. By Alster [1], this is equivalent to $X$ being a scattered Corson compactum. We now recall Bandlow's characterization of Corson compacta:

**Definition 3.1.** If $(X, \mathcal{T}) \in M \prec H(\theta)$, then $M$ *separates* $X$ iff for all $a, b \in \overline{X \cap M}$ with $a \neq b$, there is an $f \in C(X) \cap M$ such that $f(a) \neq f(b)$.

Here, $C(X) = C(X, \mathbb{R})$. Note that for Tychonov spaces, we trivially get such an $f$ when $a, b \in X \cap M$. For $X$ compact Hausdorff, Bandlow's terminology for "$M$ separates $X$" was "$\varphi^X_M$ is an $M$-retraction".

**Theorem 3.2** (Bandlow [2]). *Let $(X, \mathcal{T})$ be compact Hausdorff and $(X, \mathcal{T}) \in H(\theta)$. Then the following are equivalent*:

(1) *$X$ is Corson compact.*
(2) *$M$ separates $X$ for all $M$ such that $(X, \mathcal{T}) \in M \prec H(\theta)$.*
(3) *$\{M \in [H(\theta)]^\omega \colon M$ separates $X\}$ contains a club.*

As usual, $\mathcal{C} \subseteq [I]^\omega$ is *club* iff $\mathcal{C}$ is **cl**osed ($\bigcup_{n\in\omega} A_n \in \mathcal{C}$ whenever each $A_n \in \mathcal{C}$ and $A_0 \subseteq A_1 \subseteq \cdots$) and **un**bounded ($\forall A \in [I]^\omega\, \exists B \in \mathcal{C}[A \subseteq B]$). Note that $\{M \in [H(\theta)]^\omega \colon (X, \mathcal{T}) \in M \prec H(\theta)\}$ is always a club.

**Corollary 3.3.** *Assume that $(X, \mathcal{T}) \in H(\theta)$, where $X$ is compact Hausdorff and $\theta$ is regular and uncountable. Then the following are equivalent.*

(1) *$X$ is strongly Eberlein compact.*
(2) *$X \cap M$ is compact for all $M$ such that $(X, \mathcal{T}) \in M \prec H(\theta)$.*
(3) *$\{M \in [H(\theta)]^{\omega}\colon X \cap M$ is compact$\}$ contains a club.*

**Proof.** (2) ⇒ (3) is trivial and (1) ⇒ (2) is easy from the definition. For (3) ⇒ (1), observe that whenever $X \cap M$ is compact, we have $\overline{X \cap M} = X \cap M$, so that $M$ separates $X$. Thus, $X$ is Corson compact by Theorem 3.2. Since $X$ is also scattered by Theorem 1.2, we have that $X$ is strongly Eberlein compact by Alster [1]. □

Note that in (3), we cannot replace "contains a club" by the weaker "$\neq \emptyset$", or even "is stationary", as we had in the corresponding (3) of Theorem 1.2:

**Example 3.4.** There is a scattered compactum $X$ which is not strongly Eberlein compact such that $\{M \in [H(\theta)]^{\omega}\colon X \cap M$ is compact$\}$ is stationary.

**Proof.** Let $S \subseteq \omega_1$ be a stationary set of limit ordinals such that $\omega_1 \backslash S$ is also stationary. For $\gamma \in S$, let $E_\gamma \subset \gamma$ be a cofinal set of successor ordinals of order type $\omega$. Define the topology $\mathcal{T}_0$ on $\omega_1$ by: $U \in \mathcal{T}_0$ iff $E_\gamma \setminus U$ is finite for all $\gamma \in S \cap U$. Observe that $\mathcal{T}_0$ is locally compact Hausdorff and finer than the usual order topology on $\omega_1$. Also, $\gamma < \omega_1$ is closed in $\mathcal{T}_0$ iff $\gamma \notin S$. Now, let $\mathcal{T}$ be the one-point compactification topology on $\omega_1 + 1$. If $M$ is countable and $M \prec H(\theta)$, then $M \cap X = \gamma_M \cup \{\omega_1\}$ for some limit $\gamma_M < \omega_1$. If $\gamma_M \in S$, then $X \cap M$ is not compact, so $X$ is not a strong Eberlein compactum by Corollary 3.3. However, $X \cap M$ is compact whenever $\gamma_M \notin S$, and $\{M \in [H(\theta)]^{\omega}\colon \gamma_M \notin S\}$ is stationary. □

Finally, we comment on squashings:

**Definition 3.5.** If $(X, \mathcal{T})$ is compact Hausdorff, $E \subseteq X$, and $\sigma$ is a function from $X$ onto $E$, then $\sigma$ is a *squashing* of $X$ onto $E$ iff $\sigma \circ \sigma = \sigma$ and $\sigma$ is continuous with respect to $\mathcal{T}$ on $X$ and some compact Hausdorff topology $\mathcal{T}' \subseteq \mathcal{T} \upharpoonright E$ on $E$.

Equivalently, $E$ is a section of some (continuous) map from $X$ onto some compact Hausdorff space. A retraction is the special case where $\mathcal{T}' = \mathcal{T} \upharpoonright E$.

Now, suppose that $(X, \mathcal{T}) \in M \prec H(\theta)$. For $x, y \in X$, define $x \sim y$ iff $f(x) = f(y)$ for all $f \in C(X) \cap M$, and let $[x] = \{y \in x\colon x \sim y\}$. Note that $|[x] \cap X \cap M| \leqslant 1 \leqslant |[x] \cap \overline{X \cap M}|$. For the second inequality, use $M \prec H(\theta)$ to show that the family of sets of the form $\{y \in X \cap M\colon |f(y) - f(x)| < \varepsilon\}$, where $\varepsilon > 0$ and $f \in C(X) \cap M$, has the finite intersection property.

Following Bandlow [2], $M$ separates $X$ iff $\forall x \in X[|[x] \cap \overline{X \cap M}| = 1]$, in which case we have a retraction $\rho : X \to \overline{X \cap M}$, where $\rho(x)$ is the (unique) $y \in \overline{X \cap M}$ such that $y \sim x$. Following Junqueira and Tall [8], $X_M$ is compact iff $\forall x \in X[|[x] \cap X \cap M| = 1]$, in which case we have a squashing $\sigma : X \to X_M$, where $\sigma(x)$ is the (unique) $y \in X \cap M$ such that $y \sim x$.

If $M$ separates $X$ *and* $X_M$ is compact, then $\rho$ and $\sigma$ agree, so that $X \cap M$ is closed in $X$ and is homeomorphic to $X_M$. In particular, this applies when $X$ is Corson compact and $X_M$ is compact. However, for Corson compacta, it is easy to see directly that $X \cap M \cong X_M$, whether or not $X_M$ is compact. Other examples where $X \cap M \cong X_M$ are discussed in [8, §2].

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