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ON P-VALENT STARLIKE FUNCTIONS WITH REFERENCE TO THE BERNARDI INTEGRAL OPERATOR

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Let $S_p^*(A,B)$ denote the class of certain p-valent starlike functions. Recently G. Lakshma Reddy and K.S. Padmanabhan [Bull. Austral. Math. Soc. 25(1982), 387-396] have shown that the function g defined by

$$g(z) = (c+p)z^{-c}\int_0^z t^{c-1}f(t)dt\ ,\quad c = 1,2,3,\ldots,$$

belongs to the class $S_p^*(A,B)$ if $f \in S_p^*(A,B)$. The technique used by them fails when c is any positive real number. In this paper, by employing a more powerful technique, we improve their result to the case when c is any real number such that $c \geq -p(1+A)/(1+B)$.

1. Introduction

Let $S_p^*(A,B)$ denote the class of functions of the form

$$f(z) = a_p z^p + \sum_{n=p+1}^{\infty} a_n z^n,\quad p \geq 1\ , \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$, and satisfy

$$z\frac{f'(z)}{f(z)} = p\,\frac{1+Aw(z)}{1+Bw(z)}\ ,\quad z \in E\ , \tag{1.2}$$

where $-1 \leq A < B \leq 1$, w in analytic in E, and satisfies $w(0) = 0$

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and $|w(z)| < 1$ for $z \in E$. Evidently, the functions in $S_p^*(A,B)$ are p-valent starlike in E.

Bernardi [1] has shown that, if the function f is univalent starlike in E, then so is the function g given by

$$g(z) = (c+1)z^{-c} \int_0^z t^{c-1} f(t)dt$$

where c is a positive integer. This result has been improved by Miller et al. [5,Theorem 2] to the case when c is a positive real number. Recently, Reddy and Padmanabhan [6,Theorem 1] have extended the result of Bernardi [1] by proving that, if $f \in S_p^*(A,B)$, then so does the function g given by

$$g(z) = (c+p)z^{-c} \int_0^z t^{c-1} f(t)dt$$

where c is a positive integer. The classical technique used by Reddy and Padmanabhan [6] fails when c is any positive real number. It is therefore natural to ask whether their result can be improved for real c.

The object of the present paper is to establish a theorem which improves, in particular, the result of Reddy and Padmanabhan [6,Theorem 1] to the case when c is a real number such that $c \geq -p(1+A)/(1+B)$. It is worth noting that the technique employed to prove our theorem is different from those used by Miller et al. [5] and Reddy and Padmanabhan [6]. In fact our important tool is Lemma 2.1, to be proved in section 2, which provides a geometrical definition of the class $S_p^*(A,B)$.

2. Preliminary lemmas

To establish our main result we require the following lemmas:

LEMMA 2.1. *A function f of the form* (1.1) *belongs to* $S_p^*(A,B)$, $-1 \leq A < B < 1$, *if and only if*

(2.1) $$\left| z\frac{f'(z)}{f(z)} - m \right| < M, \quad z \in E,$$

where

(2.2) $$m = p(1-AB)/(1-B^2) \quad \text{and} \quad M = p(B-A)/(1-B^2).$$

Proof. Let $f \in S_p^*(A,B)$. Then from (1.2) we have

$$z\,\frac{f'(z)}{f(z)} - m = \frac{(p-m)+(Ap-Bm)w(z)}{1+Bw(z)} \tag{2.3}$$

$$= M\,h(z)\,,$$

where $h(z) = -(B+w(z))/(1+Bw(z))$. Since $|h(z)| < 1$, the inequality (2.1) follows from (2.3).

Conversely, let f satisfy (2.1). Then

$$\left| z\,\frac{f'(z)}{Mf(z)} - \frac{m}{M} \right| < 1\,, \quad z \in E.$$

Let

$$q(z) = z\,\frac{f'(z)}{Mf(z)} - \frac{m}{M} \tag{2.4}$$

and we define

$$w(z) = \frac{q(0)-q(z)}{1-q(0)q(z)}\,. \tag{2.5}$$

Clearly the function w is analytic in E, and satisfies $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$. Since $q(0) = -B$, from (2.5) we get

$$q(z) = -(B+w(z))/(1+Bw(z))\,. \tag{2.6}$$

Eliminating $q(z)$ from (2.4) and (2.6) we get (1.2). Hence $f \in S_p^*(A,B)$.

Note: (i) The condition (2.1) can be written in the form

$$\left| \frac{(zf'(z)/f(z))-(p(1+A)/(1+B))}{p-(p(1+A)/(1+B))} - \frac{1}{1-B} \right| < \frac{1}{1-B}\,, \quad z \in E.$$

Now as $B \to 1$ and $A = -(1-2\beta)$, $0 \le \beta < 1$, this inequality reduces to $\mathrm{Re}\{zf'(z)/f(z)\} > p\beta$, $z \in E$, which is precisely a necessary and sufficient condition for $f \in S_p^*(2\beta-1,1)$. Thus including the limiting case $B \to 1$, the results proved with the help of Lemma 2.1 will hold for $-1 \le A < B \le 1$.

(ii) Throughout this paper m and M are given by (2.2).

LEMMA 2.2. *If the function w is analytic for $|z| \le r < 1$, $w(0) = 0$ and $|w(z_0)| = \max_{|z|=r} |w(z)|$, then $z_0 w'(z_0) = kw(z_0)$, where k is a real number such that $k \ge 1$.*

The above lemma is due to Jack [3].

3. Main result

THEOREM. *If* $f \in S_p^*(A,B)$ *and* g *is defined by*

$$g(z) = [(c+p\alpha)z^{-c} \int_0^z t^{c-1} f^{\alpha}(t)dt]^{1/\alpha} \tag{3.1}$$

where α *and* c *are real numbers such that* $\alpha > 0$ *and* $c \geq -p\alpha(1+A)/(1+B)$. *Then the function* g *also belongs to* $S_p^*(A,B)$.

In (3.1) powers denote principal ones.

Proof. Let us define a function w such that

$$w(z) = \frac{zg'(z)/g(z)-p}{Ap-Bzg'(z)/g(z)}$$

so that

$$z\frac{g'(z)}{g(z)} = p\frac{1+Aw(z)}{1+Bw(z)}, \tag{3.2}$$

where w is either analytic or meromorphic in E. Clearly $w(0) = 0$. We claim that w is analytic in E, and $|w(z)| < 1$ for $z \in E$, which we will prove by contradiction.

From (3.1) and (3.2) we have

$$(c+p\alpha)\left\{\frac{f(z)}{g(z)}\right\}^{\alpha} = \frac{(c+p\alpha)+(Ap\alpha+Bc)w(z)}{1+Bw(z)}. \tag{3.3}$$

Logarithmic differentiation of (3.3) yields

$$z\frac{f'(z)}{f(z)} - m = \frac{(p-m)+(Ap-Bm)w(z)}{1+Bw(z)} - \frac{p(B-A)zw'(z)}{\{1+Bw(z)\}\{(c+p\alpha)+(Ap\alpha+Bc)w(z)\}}. \tag{3.4}$$

Let r^* be the distance, from the origin, of the pole of w nearest the origin. Then w is analytic in $|z| < r_0 = \min\{r^*,1\}$. By Lemma 2.2, for $|z| \leq r$ $(r \leq r_0)$, there exists a point z_0 such that

$$z_0 w'(z_0) = kw(z_0), \quad k \geq 1. \tag{3.5}$$

From (3.4) and (3.5) we have

$$z_0\frac{f'(z_0)}{f(z_0)} - m = \frac{N(z_0)}{D(z_0)} \tag{3.6}$$

where

$$N(z_0) = (p-m)(c+p\alpha) + \{(c+p\alpha)(Ap-Bm) + (Ap\alpha+Bc)(p-m) - kp(B-A)\}w(z_0)$$
$$+ \{(Ap\alpha+Bc)(Ap-Bm)\}w^2(z_0)$$

and

$$D(z_0) = (c+p\alpha) + (Ap\alpha+2Bc+Bp\alpha)w(z_0) + B(Ap\alpha+Bc)w^2(z_0).$$

Now suppose that it were possible to have $\max_{|z|=r} |w(z)| = |w(z_0)| = 1$ for some r, $r < r_0 \le 1$. Then by using the identities $Ap - Bm = -M$ and $B - A = (M^2-(m-p)^2)/(Mp)$, we have

$$|N(z_0)|^2 - M^2|D(z_0)|^2 = a + 2b\ \mathrm{Re}\{w(z_0)\} \tag{3.7}$$

where

$$a = kp(B-A)\{kp(B-A) + 2M(c+p\alpha) + 2MB(Ap\alpha+Bc)\}$$

and

$$b = kp(B-A)M\{(Ap\alpha+Bc) + B(c+p\alpha)\}.$$

From (3.7) we have

$$|N(z_0)|^2 - M^2|D(z_0)|^2 > 0, \tag{3.8}$$

provided $a \pm 2b > 0$. Now

$$a + 2b = kp(B-A)[kp(B-A) + 2M(1+B)\{c(1+B) + p\alpha(1+A)\}]$$
$$> 0, \quad \text{provided} \quad c \ge -p\alpha(1+A)/(1+B),$$

and

$$a - 2b = kp(B-A)[kp(B-A) + 2M(1-B)\{c(1-B) + p\alpha(1-A)\}]$$
$$> 0, \quad \text{provided} \quad c \ge -p\alpha(1-A)/(1-B).$$

Thus from (3.6) and (3.8) it follows that

$$\left| z_0 \frac{f'(z_0)}{f(z_0)} - m \right| > M$$

provided

$$c \ge \max\{-p\alpha(1+A)/(1+B),\ -p\alpha(1-A)/(1-B)\}$$
$$= -p\alpha(1+A)/(1+B).$$

But this is, in view of Lemma 2.1, contrary to our assumption $f \in S_p^*(A,B)$. Therefore we cannot have $|w(z)| = 1$ in $|z| < r_0$. Since

$|w(0)| = 0$, $|w(z)|$ is continuous and $|w(z)| \neq 1$ in $|z| < r_0$, w cannot have a pole at $|z| = r_0$. Since r_0 is arbitrary, we conclude that w is analytic in E, and satisfies $|w(z)| < 1$ for $z \in E$.

Hence, from (3.2), $g \in S_p^*(A,B)$.

Remark. It is evident that, for $\alpha = 1$, the above theorem improves the result of Reddy and Padmanabhan [6, Theorem 1].

If we set $A = -(1-2\beta)$, where $0 \leq \beta < 1$, and $a_1 = p = B = 1$, the class $S_p^*(A,B)$ reduces to the well known class $S^*(\beta)$ of univalent starlike functions of order β. For the class $S^*(\beta)$, the under-mentioned corollary follows immediately from the above theorem.

COROLLARY. *Let* α *and* c *be real numbers such that* $\alpha > 0$ *and* $c \geq -\alpha\beta$. *If* $f \in S^*(\beta)$, *then the function* g *defined by*

$$g(z) = \left[\frac{c+\alpha}{z^c} \int_0^z t^{c-1} f^\alpha(t)dt \right]^{1/\alpha} \tag{3.9}$$

is also an element of $S^*(\beta)$.

Remarks (i) A result of Miller et al. [5, Theorem 2] turns out to be a particular case of the above corollary when $\beta = 0$.

(ii) Gupta and Jain [2, Theorem 1] have also shown that the function g defined by (3.9) belongs to the class $S^*(\beta)$ if $f \in S^*(\beta)$. However, as an example, the integral operator

$$g(z) = \left[\frac{11}{4z^2} \int_0^z t \; f^{3/4}(t)dt \right]^{4/3}$$

can be studied by the above corollary and not by the result of Gupta and Jain, since the technique followed by them fails when at least one of α and c is not a positive integer.

Problem. Very recently, Kumar and Shukla [4, Theorem 1(i)] have shown that the function g given by (3.9) belongs to $S^*(\beta)$ even when c is a complex number such that $\text{Re}(c) \geq -\alpha\beta$. It would be interesting to show that the function g given by (3.1) belongs to $S_p^*(A,B)$ when c is a complex number such that $\text{Re}(c) \geq -p\alpha(1+A)/(1+B)$.

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