

CERTAIN INTEGRALS FOR CLASSES OF p -VALENT MEROMORPHIC FUNCTIONS

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In this paper we introduce two classes, namely $\Gamma_p(m, M)$ and $\Sigma_p(m, M)$, of functions

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \dots + a_{n+p-1}z^n + \dots,$$

regular and p -valent in $D - \{0\}$ where $D = \{z : |z| < 1\}$. We show that, for suitable choices of real constants α, β and γ , the integral operators of the form

$$F(z) = \left[\frac{\gamma - p(\alpha + \beta) + 2}{z^{\gamma - p\beta + 2}} \int_0^z t^{\gamma+1} f(t)^\alpha g(t)^\beta dt \right]^{1/\alpha}$$

map $\Gamma_p^*(\rho) \times \Gamma_p(m, M)$ into $\Gamma_p^*(\rho)$, where $\Gamma_p^*(\rho)$ is the class of p -valent meromorphically starlike functions of order ρ , $0 \leq \rho < 1$. For the classes $\Gamma_p(m, M)$ and $\Sigma_p(m, M)$, we obtain class preserving integral operators of the form

$$F(z) = \left[\frac{\gamma - p\alpha + 1}{z^{\gamma+1}} \int_0^z t^\gamma f(t)^\alpha dt \right]^{1/\alpha},$$

with suitable restrictions on real constants α and γ .

Our results generalize almost all known results obtained so far in this direction.

1. Introduction

Let I^+ denote the set of positive integers. We denote by Γ_p , $p \in I^+$, the set of the functions

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \dots + a_{n+p-1}z^n + \dots,$$

regular and p -valent in $D - \{0\}$, where $D = \{z : |z| < 1\}$ and $f'(z) \neq 0$ there. A function f of Γ_p is said to belong to Γ_p^* , the class of p -valent meromorphically starlike functions, if and only if $\operatorname{Re}\{z(f'(z)/f(z))\} < 0$, $z \in D$. A function f of Γ_p is said to belong to $\Gamma_p^*(\rho)$, the class of p -valent meromorphically starlike functions of order ρ , $0 \leq \rho < 1$, if and only if $\operatorname{Re}\{z(f'(z)/f(z))\} < -p\rho$, $z \in D$. A function f of Γ_p is said to belong to $\Sigma_p(\rho)$, the class of p -valent meromorphically convex functions of order ρ , $0 \leq \rho < 1$, if and only if $\operatorname{Re}\{1+z(f''(z)/f'(z))\} < -p\rho$, $z \in D$. The class Σ_p of p -valent meromorphically convex functions is identified by $\Sigma_p \equiv \Sigma_p(0)$.

Now we define two subclasses, namely $\Gamma_p(m, M)$ and $\Sigma_p(m, M)$, of Γ_p^* and Σ_p respectively.

A function f of Γ_p belongs to the class $\Gamma_p(m, M)$ if and only if $|z(f'(z)/f(z)) + m| < M$, $z \in D$ where

$$(m, M) \in E_p = \{(m, M) : |m-p| < M \leq m\}.$$

A function f of Γ_p belongs to the class $\Sigma_p(m, M)$ if and only if $|1+z(f''(z)/f'(z)) + m| < M$, $z \in D$, where

$$(m, M) \in E_p = \{(m, M) : |m-p| < M \leq m\}.$$

It is clear that $\Gamma_p(m, M) \subset \Gamma_p^*((m-M)/p) \subset \Gamma_p^* \subset \Gamma_p$ and $\Sigma_p(m, M) \subset \Sigma_p((m-M)/p) \subset \Sigma_p \subset \Gamma_p$. Also, a function f belongs to $\Sigma_p(m, M)$ if and only if $zf'(z)/-p$ belongs to $\Gamma_p(m, M)$.

In [1], [2], [3], the integral operators of the forms

$$F(z) = \left[\frac{\gamma - 2\beta + 2}{z^{\gamma - \beta + 2}} \int_0^z t^{\gamma + 1} f(t)^\beta g(t)^\beta dt \right]^{1/\beta}$$

and

$$F(z) = \left[\frac{\gamma - \alpha + 1}{z^{\gamma + 1}} \int_0^z t^\gamma f(t)^\alpha dt \right]^{1/\alpha},$$

with suitable restrictions on the real constants α , β and γ , and for f and g belonging to some favoured classes of meromorphic functions have been studied. The purpose of this paper is to obtain the integral operators that are more general and transform $\Gamma_p^*(\rho) \times \Gamma_p(m, M)$ into $\Gamma_p^*(\rho)$, $\Gamma_p(m, M)$ into $\Gamma_p(m, M)$, and $\Sigma_p(m, M)$ into $\Sigma_p(m, M)$. Our results generalize almost all known results obtained so far in this direction [1], [2], [3].

2. Preliminary lemmas

Let $S(m, M)$ be the class of functions $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ regular

and satisfying

$$(2.1) \quad \left| z \frac{h'(z)}{h(z)} - m \right| < M \text{ in } D,$$

where

$$(2.2) \quad (m, M) \in E = \{(m, M) : m > \frac{1}{2}, |m-1| < M \leq m\}.$$

This class $S(m, M)$ was introduced by Jakubowski [5]. It is worth mentioning here that the requirement $m > \frac{1}{2}$ in (2.2) is superfluous and may be dropped since it follows directly from the inequality $|m-1| < m$ in (2.2).

The proof of the following lemma is based on the lines of a result of Silverman [7, Theorem 1], the only difference is that in the definition of $S(m, M)$ Silverman [7] has taken equality also in (2.1) and restricted m, M by the inequalities

$$(2.3) \quad m + M \geq 1, \quad M \leq m \leq M + 1$$

which are equivalent to the inequalities

$$|m-1| \leq M \leq m .$$

However we follow the definition of $S(m, M)$ given by Jakubowski [5].

LEMMA 2.1. *The function $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $S(m, M)$*

if and only if there exists a function $w(z)$ regular in D and satisfying $w(0) = 0$, $|w(z)| < 1$ for z in D , such that

$$(2.4) \quad z \frac{h'(z)}{h(z)} = \frac{1+a'w(z)}{1-b'w(z)}, \quad z \in D ,$$

where

$$(2.5) \quad a' = \frac{M^2 - m^2 + m}{M} \quad \text{and} \quad b' = \frac{m-1}{M} .$$

Proof. First suppose that $h \in S(m, M)$. Then

$$\left| z \frac{h'(z)}{Mh(z)} - \frac{m}{M} \right| < 1 .$$

Let $g(z) = z(h'(z)/Mh(z)) - m/M$ and

$$(2.6) \quad w(z) = \frac{g(z) - g(0)}{1 - g(0)g(z)} = \frac{z(h'(z)/h(z)) - 1}{M + ((m-1)/M) \{z(h'(z)/h(z)) - m\}} .$$

Then $w(0) = 0$, $|w(z)| < 1$. Rearranging (2.6) and using (2.5) we get (2.4).

Conversely, suppose that $h(z)$ satisfies (2.4). Then

$$(2.7) \quad z \frac{h'(z)}{h(z)} - m = M \frac{((1-m)/M) + w(z)}{1 + ((1-m)/M)w(z)} \\ = MG(z) , \text{ say.}$$

In view of (2.2), $|(1-m)/M| < 1$. Thus the function

$$G(z) = \frac{((1-m)/M) + w(z)}{1 + ((1-m)/M)w(z)}$$

satisfies $|G(z)| < 1$. From (2.7), it follows now that $h \in S(m, M)$.

This completes the proof of the lemma.

LEMMA 2.2. *The function f is in $\Gamma_p(m, M)$ if and only if there exists a function $w(z)$ regular and satisfying $w(0) = 0$, $|w(z)| < 1$, for z in D such that*

$$z \frac{f'(z)}{f(z)} = -p \frac{1+a\omega(z)}{1-b\omega(z)}, \quad z \in D,$$

where $a = (M^2 - m^2 + mp)/Mp$ and $b = (m-p)/M$.

Proof. Since $f \in \Gamma_p(m, M)$, $z(f'(z)/f(z)) = -p + \dots$. Let $(z(f'(z)/f(z)))/-p = z(h'(z)/h(z)) = 1 + \dots$, then from Lemma 2.1, for $|(m/p)-1| < M/p \leq m/p$,

$$\left| \frac{z(f'(z)/f(z))}{-p} - \frac{m}{p} \right| < \frac{M}{p}$$

if and only if

$$\frac{z(f'(z)/f(z))}{-p} = \frac{1+a\omega(z)}{1-b\omega(z)}$$

where

$$a = \frac{(M/p)^2 - (m/p)^2 + (m/p)}{M/p} \quad \text{and} \quad b = \frac{(m/p)-1}{M/p};$$

or, for $|m-p| < M \leq m$,

$$\left| z \frac{f'(z)}{f(z)} + m \right| < M$$

if and only if

$$z \frac{f'(z)}{f(z)} = -p \frac{1+a\omega(z)}{1-b\omega(z)},$$

where

$$a = \frac{M^2 - m^2 + mp}{Mp} \quad \text{and} \quad b = \frac{m-p}{M}.$$

LEMMA 2.3. If the function $w(z)$ is regular for $|z| \leq r < 1$, $w(0) = 0$ and $|w(z_1)| = \max_{|z|=r} |w(z)|$, then

$$z_1 w'(z_1) = k w(z_1), \quad k \geq 1.$$

A proof of Lemma 2.3, which is due to Jack, may be found in [4].

LEMMA 2.4. A function f belongs to $\Gamma_p^*(\rho)$, $0 \leq \rho < 1$, if and only if there exists a function $w(z)$ regular and satisfying $w(0) = 0$, $|w(z)| < 1$ in D such that

$$z \frac{f'(z)}{f(z)} = -p \frac{1+(2\rho-1)w(z)}{1+w(z)} .$$

Proof. Let $P(z) = ((z(f'(z)/f(z))/-p)-\rho)/(1-\rho) = 1 + \dots$; then $\operatorname{Re}\{P(z)\} > 0$ and hence by a well known result [6], $P(z)$ can be written as

$$P(z) = \frac{1-w(z)}{1+w(z)} ,$$

where $w(z)$ is regular and $w(0) = 0$, $|w(z)| < 1$, for z in D . Thus

$$\frac{(z(f'(z)/f(z))/-p)-\rho}{1-\rho} = \frac{1-w(z)}{1+w(z)}$$

and hence the result follows.

3. Integral operators that map $\Gamma_p^*(\rho) \times \Gamma_p(m, M)$ into $\Gamma_p^*(\rho)$

THEOREM 3.1. Let α, β and γ be real constants such that

$$\alpha > 0 , \quad \beta \geq 0 \quad \text{and} \quad \gamma - p(\alpha+\beta) + 1 > -1 .$$

If $f \in \Gamma_p^*(\rho)$ and $g \in \Gamma_p(m, M)$, $(m, M) \in E_p^* = \{(m, M) : |m-p| < M \leq m^*\}$ where $m^* = \min\{m, (m-p) + (\alpha p(1-\rho)/2\beta(\gamma-p\beta-\alpha p p+2))\}$, then

$$(3.1) \quad F(z) = \left[\frac{\gamma-p(\alpha+\beta)+2}{z^{\gamma-p\beta+2}} \int_0^z t^{\gamma+1} f(t)^\alpha g(t)^\beta dt \right]^{1/\alpha} = \frac{1}{z^p} + \dots$$

also belongs to $\Gamma_p^*(\rho)$. In (3.1) all powers are principal ones.

Proof. Let us choose a function $w(z)$ such that

$$(3.2) \quad z \frac{F'(z)}{F(z)} = -p \frac{1+(2\rho-1)w(z)}{1+w(z)} , \quad w(0) = 0 ,$$

and $w(z)$ is either regular or meromorphic in D .

From (3.1) and (3.2), we have

$$(3.3) \quad z^{p\beta} \frac{f(z)^\alpha}{F(z)^\alpha} g(z)^\beta = \frac{1+(\xi/\delta)w(z)}{1+w(z)} ,$$

where $\xi = \gamma + p(\alpha-\beta) + 2(1-\alpha p)$ and $\delta = \gamma - p(\alpha+\beta) + 2$.

Logarithmic differentiation of (3.3) yields

$$(3.4) \quad z \frac{f'(z)}{f(z)} = \frac{\beta}{\alpha} (m-p) - \frac{\beta}{\alpha} \left\{ z \frac{g'(z)}{g(z)} + m \right\} - p \frac{1+(2\rho-1)w(z)}{1+w(z)} + \frac{2p(1-\rho)zw'(z)}{\delta\{1+w(z)\}\{1+(\xi/\delta)w(z)\}} .$$

Let r^* be the distance from the origin of the pole of $w(z)$ nearest the origin. Then $w(z)$ is regular in $|z| < r_0 = \min\{r^*, 1\}$. By Lemma 2.3, for $|z| \leq r$ ($r < r_0$) there is a point z_0 such that

$$(3.5) \quad z_0 w'(z_0) = kw(z_0), \quad k \geq 1 .$$

From (3.4) and (3.5) we have

$$(3.6) \quad \operatorname{Re} \left\{ z_0 \frac{f'(z_0)}{f(z_0)} \right\} \geq \frac{\beta}{\alpha} (m-p) - \frac{\beta}{\alpha} \left| z_0 \frac{g'(z_0)}{g(z_0)} + m \right| - p \frac{\operatorname{Re} [\{1+(2\rho-1)w(z_0)\}\{1+\overline{w(z_0)}\}]}{|1+w(z_0)|^2} + \frac{2p(1-\rho)k \operatorname{Re} [w(z_0)\{1+\overline{w(z_0)}\}\{1+(\xi/\delta)\overline{w(z_0)}\}]}{\delta |1+w(z_0)|^2 |1+(\xi/\delta)w(z_0)|^2} > \frac{\beta}{\alpha} \{(m-p)-M\} - p \frac{1+2\rho \operatorname{Re} w(z_0) + (2\rho-1)|w(z_0)|^2}{1+2 \operatorname{Re} w(z_0) + |w(z_0)|^2} + \frac{2p(1-\rho)k \operatorname{Re} [w(z_0) + (2\{\gamma-p\beta-\alpha p p+2\}/\delta)|w(z_0)|^2 + (\xi/\delta)|w(z_0)|^2 \overline{w(z_0)}]}{\delta [1+2 \operatorname{Re} w(z_0) + |w(z_0)|^2] [1+(2\xi/\delta) \operatorname{Re} w(z_0) + (\xi/\delta)^2 |w(z_0)|^2]} .$$

Now suppose that it were possible to have $M(r, w) = \max_{|z|=r} |w(z)| = 1$ for some r ($r < r_0 \leq 1$). At the point z_0 where this occurred, we would have $|w(z)| = 1$. Then, from (3.6),

$$\operatorname{Re} \left\{ z_0 \frac{f'(z_0)}{f(z_0)} \right\} > -p\rho + \frac{2\{\gamma-p\beta-\alpha p p+2\}[(1-\rho)p + (2\beta/\alpha)\{(m-p)-M\}\{\gamma-p\beta-\alpha p p+2\}]}{[\delta^2 + 2\delta\xi \operatorname{Re} w(z_0) + \xi^2]} \geq -p\rho, \text{ provided } M \leq (m-p) + \frac{\alpha p(1-\rho)}{2\beta(\gamma-p\beta-\alpha p p+2)} .$$

But this is contrary to the fact that $f \in \Gamma_p^*(\rho)$. So we cannot have $M(r, w) = 1$. Thus $|w(z)| \neq 1$ in $|z| < r_0$. Since $w(0) = 0$, $|w(z)|$

is continuous in $|z| < r_0$ and $|w(z)| \neq 1$ there, $w(z)$ can not have a pole at $|z| = r_0$. Therefore $|w(z)| < 1$ and $w(z)$ is regular in D .

Hence from (3.2) and Lemma 2.4, $F \in \Gamma_p^*(\rho)$.

REMARK. Taking $p = 1$, the undermentioned results follow from Theorem 3.1:

- (i) for $\alpha = 1$, $\beta = 0$, $\gamma = 0$ and $\rho = 0$, a result of Bajpai [1];
- (ii) for $\alpha = \beta = 1$, a result of Dwivedi [2];
- (iii) for $\alpha = \beta$, a result of Dwivedi, Bhargava and Shukla [3].

THEOREM 3.2. Let β and γ be real constants such that $\beta \geq 0$ and $\gamma > 0$. If $f \in \Gamma_p^*(\rho)$ and $g \in \Gamma_p(m, M)$,

$$(m, M) \in E_p^{**} = \{(m, M) : |m-p| < M \leq m^*\}$$

where

$$m^* = \min\left\{m, (m-p) + \frac{p(1-\rho)}{2\beta\{\gamma+p(1-\rho)\}}\right\}.$$

Then

$$(3.7) \quad F(z) = \frac{\gamma}{z^{\gamma+p}} \int_0^z t^{\gamma+p(1+\beta)-1} f(t) \cdot g(t)^\beta dt = \frac{1}{z^p} + \dots$$

also belongs to $\Gamma_p^*(\rho)$. In (3.7) all powers are principal ones.

Proof. Let us choose a function $w(z)$ such that

$$(3.8) \quad z \frac{F'(z)}{F(z)} = -p \frac{1+(2\rho-1)w(z)}{1+w(z)}, \quad w(0) = 0,$$

$w(z)$ is either regular or meromorphic in D .

From (3.7) and (3.8), we have

$$(3.9) \quad \frac{z^{p\beta} f(z) g(z)^\beta}{F(z)} = \frac{1+\{(\gamma+2p(1-\rho))/\gamma\}w(z)}{1+w(z)}.$$

Logarithmic differentiation of (3.9) yields

$$(3.10) \quad z \frac{f'(z)}{f(z)} = -p\beta - \beta z \frac{g'(z)}{g(z)} - p \frac{1+(2\rho-1)w(z)}{1+w(z)} + \frac{2p(1-\rho)zw'(z)}{[1+w(z)][\gamma+\{\gamma+2p(1-\rho)\}w(z)]}.$$

Let r^* be the distance from the origin of the pole of $w(z)$ nearest the origin. Then $w(z)$ is regular in $|z| < r_0 = \min\{r^*, 1\}$. By Lemma 2.3, for $|z| \leq r$ ($r < r_0$) there is a point z_0 such that

$$(3.11) \quad z_0 w'(z_0) = kw(z_0), \quad k \geq 1.$$

From (3.10) and (3.11), we have

$$\operatorname{Re} \left\{ z_0 \frac{f'(z_0)}{f(z_0)} \right\} \geq \beta(m-p) - \beta \left| z_0 \frac{g'(z_0)}{g(z_0)} + m \right| - p \frac{\operatorname{Re} [\{1+(2\rho-1)w(z_0)\} \{1+\overline{w(z_0)}\}]}{|1+w(z_0)|^2} + \frac{2p(1-\rho)k \operatorname{Re} [w(z_0) \{\gamma+\{\gamma+2p(1-\rho)\}w(z_0)\} \{1+\overline{w(z_0)}\}]}{|1+w(z_0)|^2 |\gamma+\{\gamma+2p(1-\rho)\}w(z_0)|^2},$$

or

$$(3.12) \quad \operatorname{Re} \left\{ z_0 \frac{f'(z_0)}{f(z_0)} \right\} > -\beta\{M-(m-p)\} - p \frac{1+2\rho \operatorname{Re} w(z_0) + (2\rho-1)|w(z_0)|^2}{1+2 \operatorname{Re} w(z_0) + |w(z_0)|^2} + \frac{2p(1-\rho)k \operatorname{Re} [\gamma w(z_0) + 2\{\gamma+p(1-\rho)\}|w(z_0)|^2 + \{\gamma+2p(1-\rho)\}|w(z_0)|^2 \overline{w(z_0)}]}{[1+2 \operatorname{Re} w(z_0) + |w(z_0)|^2] [\gamma^2 + 2\gamma\{\gamma+2p(1-\rho)\} \operatorname{Re} w(z_0) + \{\gamma+2p(1-\rho)\}^2 |w(z_0)|^2]}.$$

Now suppose that it were possible to have $M(r, w) = \max_{|z|=r} |w(z)| = 1$ for some r ($r < r_0$). At the point z_0 where this occurred we would have $|w(z_0)| = 1$. Then, from (3.12),

$$\operatorname{Re} \left\{ z_0 \frac{f'(z_0)}{f(z_0)} \right\} > -p\rho + \frac{2\{\gamma+p(1-\rho)\}[p(1-\rho) - 2\beta\{M-(m-p)\}\{\gamma+p(1-\rho)\}]}{[\gamma^2 + 2\gamma\{\gamma+2p(1-\rho)\} \operatorname{Re} w(z_0) + \{\gamma+2p(1-\rho)\}^2]} \geq -p\rho, \quad \text{provided } M \leq (m-p) + \frac{p(1-\rho)}{2\beta\{\gamma+p(1-\rho)\}}.$$

But this is contrary to the fact that $f \in \Gamma_p^*(\rho)$. So we can not have $M(r, w) = 1$. Thus $|w(z)| \neq 1$ in $|z| < r_0$. Since $|w(0)| = 0$,

$|w(z)|$ is continuous in $|z| < r_0$ and $|w(z)| \neq 1$ there, $w(z)$ can not have a pole at $|z| = r_0$. Therefore $|w(z)| < 1$ and $w(z)$ is regular in D .

Hence from (3.8) and Lemma 2.4, $F \in \Gamma_p^*(\rho)$.

4. Class preserving integral operators for $\Gamma_p(m, M)$ and $\Sigma_p(m, M)$

THEOREM 4.1. *If $f \in \Gamma_p(m, M)$ and $F(z)$ is defined by*

$$(4.1) \quad F(z) = \left[\frac{\gamma - p\alpha + 1}{z^{\gamma+1}} \int_0^z t^\gamma f(t)^\alpha dt \right]^{1/\alpha} = \frac{1}{z^p} + \dots,$$

$\alpha > 0$. Then $F \in \Gamma_p(m, M)$ if $\gamma > \max\{(p\alpha(1+\alpha)+b-1)/(1-b), p\alpha-1\}$.

In (4.1) all powers are principal ones.

Proof. From (4.1) we have

$$(4.2) \quad \alpha z \frac{F'(z)}{F(z)} + (\gamma+1) = (\gamma-p\alpha+1) \left\{ \frac{f(z)}{F(z)} \right\}^\alpha.$$

Let us choose a function $w(z)$ such that

$$(4.3) \quad z \frac{F'(z)}{F(z)} = -p \frac{1+\alpha w(z)}{1-bw(z)}, \quad w(0) = 0,$$

and $w(z)$ is either regular or meromorphic in D . From (4.2) and (4.3), we have

$$(4.4) \quad (\gamma-p\alpha+1) \left\{ \frac{f(z)}{F(z)} \right\}^\alpha = \frac{(\gamma-p\alpha+1) - (\alpha p + b + b\gamma)w(z)}{1-bw(z)}.$$

Logarithmic differentiation of (4.4) yields

$$(4.5) \quad z \frac{f'(z)}{f(z)} + m = \frac{[(m-p)\kappa - \{(m-p)\theta + (\alpha p + b)m\}w(z) + (\alpha p + b)m\theta w^2(z) - (\alpha + b)pzw'(z)]}{[\kappa - \{\alpha p + b(2-p\alpha) + 2b\gamma\}w(z) + b\theta w^2(z)]}$$

where $\kappa = \gamma - p\alpha + 1$ and $\theta = \alpha p + b + b\gamma$.

Let r^* be the distance of the pole of $w(z)$ nearest the origin. Then $w(z)$ is regular in $|z| < r_0 = \min\{r^*, 1\}$. By Lemma 2.3, for

$|z| \leq r$ ($r < r_0$), there is a point z_0 such that

$$(4.6) \quad z_0 w'(z_0) = k w(z_0), \quad k \geq 1.$$

From (4.5) and (4.6), we have

$$(4.7) \quad z_0 \frac{f'(z_0)}{f(z_0)} + m = \frac{N(z_0)}{D(z_0)},$$

where

$$N(z_0) = (m-p)(\gamma-p\alpha+1) - \{(m-p)(a\alpha p+b+b\gamma)+(ap+bm)(\gamma-p\alpha+1)+(a+b)pk\}w(z_0) \\ + \{(ap+bm)(a\alpha p+b+b\gamma)\}w^2(z_0)$$

and

$$D(z_0) = (\gamma-p\alpha+1) - \{a\alpha p+b(2-p\alpha)+2b\gamma\}w(z_0) + b(a\alpha p+b+b\gamma)w^2(z_0).$$

Now suppose that it were possible to have $M(r, w) = \max_{|z|=r} |w(z)|$ for some r ($r < r_0$). At the point z_0 where this occurred we would have $|w(z_0)| = 1$. Then we have

$$(4.8) \quad |N(z_0)|^2 - M^2 |D(z_0)|^2 = A - 2B \operatorname{Re}\{w(z_0)\},$$

where

$$A = kp(a+b)[kp(a+b)+2M(\gamma-p\alpha+1)+2Mb(a\alpha p+b+b\gamma)]$$

and

$$B = kp(a+b)M[a\alpha p+b(2-p\alpha)+2b\gamma].$$

From (4.8), we have

$$(4.9) \quad |N(z_0)|^2 - M^2 |D(z_0)|^2 > 0,$$

provided $A \pm 2B > 0$.

Now

$$A + 2B = kp(a+b)[kp(a+b)+2M(1+b)\{\gamma(1+b)+(1-p\alpha+b+ap\alpha)\}] \\ > 0$$

provided $\gamma \geq (-ap\alpha-b+p\alpha-1)/(1+b)$ and

$$A - 2B = kp(a+b)[kp(a+b)+2M(1-b)\{\gamma(1-b)+(1-p\alpha-b-ap\alpha)\}] > 0,$$

provided $\gamma \geq (ap\alpha+b+p\alpha-1)/(1-b)$.

Thus from (4.7) and (4.9), it follows that

$$\left| z_0 \frac{f'(z_0)}{f(z_0)} + m \right| > M, \text{ if } \gamma \geq \frac{\alpha p(1+\alpha)+b-1}{1-b}.$$

But this is contrary to the fact that $f \in \Gamma_p(m, M)$. So we can not have $M(r, w) = 1$. Thus $|w(z)| \neq 1$ in $|z| < r_0$. Since $|w(0)| = 0$, $|w(z)|$ is continuous and $|w(z)| \neq 1$ in $|z| < r_0$, $w(z)$ can not have a pole at $|z| = r_0$. Therefore $w(z)$ is regular and $|w(z)| < 1$, for z in D .

Hence from (4.3) and Lemma 2.2, $F \in \Gamma_p(m, M)$.

REMARK. Taking $p = 1$, the undermentioned results follow from Theorem 4.1:

- (i) for $\alpha = \gamma = 1$, $m = M$ and $m \rightarrow \infty$, a result of Bajpai [1];
- (ii) for $\alpha = 1$, a result of Dwivedi [2];
- (iii) a result of Dwivedi, Bhargava and Shukla [3].

THEOREM 4.2. *If $f \in \Sigma_p(m, M)$ and $F(z)$ is defined by*

$$(4.10) \quad F(z) = \frac{\gamma-p+1}{z^{\gamma+1}} \int_0^z t^\gamma f(t) dt = \frac{1}{z^p} + \dots$$

Then $F \in \Sigma_p(m, M)$ if $\gamma > \max\{(p(1+\alpha)+b-1)/(1-b), p-1\}$.

In (4.10) all powers are principal ones.

Proof. We can write (4.10) as

$$(4.11) \quad zF'(z) = \frac{\gamma-p+1}{z^{\gamma+1}} \int_0^z t^\gamma t f'(t) dt.$$

Since $f \in \Sigma_p(m, M)$ if and only if $zf'(z)/-p \in \Gamma_p(m, M)$ and hence from

Theorem 4.1, and (4.11), we get $zF'(z)/-p$ belongs to $\Gamma_p(m, M)$. So $F(z)$ belongs to $\Sigma_p(m, M)$.

REMARK. Taking $p = 1$, the undermentioned results follow from Theorem 4.2:

- (i) for $m = M$ and $m \rightarrow \infty$, a result of Bajpai [1].
- (ii) a result of Dwivedi [2].

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