JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **126**, 70–77 (1987)

Quasi-Hadamard Product of Certain Univalent Functions

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Submitted by R. P. Boas

Received January 20, 1986

The author improves some recent results due to Shigeyoshi Owa (*Tamkang J. Math.* **14** (1983), 15–21) concerning the quasi-Hadamard product of certain starlike and convex univalent functions. © 1987 Academic Press, Inc.

1. INTRODUCTION

Throughout the paper, let the functions of the form

$$f(z)=a_1 z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_1>0,\ a_n\geqslant 0),$$

$$g(z)=b_1 z-\sum_{n=2}^{\infty} b_n z^n \qquad (b_1>0,\ b_n\geqslant 0),$$

$$f_i(z)=a_{1,i} z-\sum_{n=2}^{\infty} a_{n,i} z^n \qquad (a_{1,i}>0,\ a_{n,i}\geqslant 0),$$

and

$$g_j(z)=b_{1,j} z-\sum_{n-2}^{\infty} b_{n,j} z^n \qquad (b_{1,j}>0,\ b_{n,j}\geqslant 0),$$

be analytic in the unit disc $U=\{z\colon |z|<1\}$.

Let $ST_0^*(\alpha)$ and $C_0(\alpha)$ denote the classes of functions $f(z)$ which satisfy $\operatorname{Re}\{zf'(z)/f(z)\}>\alpha$ and $\operatorname{Re}\{1+zf''(z)/f'(z)\}>\alpha$ respectively, where $z\in U$ and $0\leqslant\alpha<1$. Clearly, the functions in $ST_0^*(\alpha)$ and $C_0(\alpha)$ are starlike and convex of order α, respectively. It is well known that such functions are univalent. Evidently,

$$ST_0^*(\alpha)\subset ST_0^*(\beta) \qquad \text{and} \qquad C_0(\alpha)\subset C_0(\beta) \qquad \text{when } 0\leqslant\beta<\alpha<1.$$

0022-247X/87 $3.00

Silverman [6] proved that $f(z) \in ST_0^*(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} [(n-\alpha)\, a_n] \leqslant (1-\alpha)\, a_1;$$

and $f(z) \in C_0(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} [n(n-\alpha)\, a_n] \leqslant (1-\alpha)\, a_1.$$

We now introduce the following class of analytic functions which plays an important role in the discussion that follows.

A function $f(z)$ belongs to the class $S_k^*(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} [n^k(n-\alpha)\, a_n] \leqslant (1-\alpha)\, a_1, \tag{1}$$

where $0 \leqslant \alpha < 1$ and k is any fixed nonnegative real number.

Evidently, $S_0^*(\alpha) \equiv ST_0^*(\alpha)$ and $S_1^*(\alpha) \equiv C_0(\alpha)$. Further, $S_k^*(\alpha) \subset S_h^*(\alpha)$ if $k > h \geqslant 0$, the containment being proper. Whence it follows that the functions in $S_k^*(\alpha)$ are starlike of order α, for all $k \geqslant 0$. Moreover, for any positive integer k, we have the inclusion relation

$$S_k^*(\alpha) \subset S_{k-1}^*(\alpha) \subset \cdots \subset S_2^*(\alpha) \subset C_0(\alpha) \subset ST_0^*(\alpha).$$

We note that for every nonnegative real number k, the class $S_k^*(\alpha)$ is non-empty as the functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} n^{-k}\{(1-\alpha)/(n-\alpha)\}\, a_1 \lambda_n z^n,$$

where $0 \leqslant \alpha < 1$, $a_1 > 0$, $\lambda_n \geqslant 0$ and $\sum_{n=2}^{\infty} \lambda_n \leqslant 1$, satisfy the inequality (1).

Let us define the quasi-Hadamard product of the functions $f(z)$ and $g(z)$ by

$$f * g(z) = a_1 b_1 z - \sum_{n=2}^{\infty} a_n b_n z^n.$$

Similarly, we can define the quasi-Hadamard product of more than two functions. It should be noted that Owa [4] used the phrase "Hadamard product" instead of "quasi-Hadamard product" in this definition. But the usual Hadamard product will give

$$f * g(z) = a_1 b_1 + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Very recently, Owa [4] has established the following theorems for the quasi-Hadamard product. The numbering of the theorems here is the same as in [4].

THEOREM 1. *Let the functions* $f_i(z)$ *be in* $ST_0^*(\alpha_i)$ *for each* $i=1, 2, \ldots, m$, *respectively. And let* $\sum_{i=1}^m \alpha_i \leqslant 1$. *Then, the quasi-Hadamard product* $f_1 * f_2 * \cdots * f_m(z)$ *belongs to* $ST_0^*(\prod_{i=1}^m \alpha_i)$.

THEOREM 2. *Let the functions* $f_i(z)$ *be in* $C_0(\alpha_i)$ *for each* $i=1, 2, \ldots, m$, *respectively. And let* $\sum_{i=1}^m \alpha_i \leqslant 1$. *Then, the quasi-Hadamard product* $f_1 * f_2 * \cdots * f_m(z)$ *belongs to* $C_0(\prod_{i=1}^m \alpha_i)$.

THEOREM 3. *Let the functions* $f_i(z)$ *be in* $ST_0^*(\alpha_i)$ *for each* $i=1, 2, \ldots, m$, *respectively. And let the functions* $g_j(z)$ *be in* $C_0(\beta_j)$ *for each* $j=1, 2, \ldots, q$, *respectively. Furthermore, let* $\sum_{i=1}^m \alpha_i + \sum_{j=1}^q \beta_j \leqslant 1$. *Then, the quasi-Hadamard product* $f_1 * f_2 * \cdots * f_m * g_1 * g_2 * \cdots * g_q(z)$ *belongs to* $C_0((\prod_{i=1}^m \alpha_i)(\prod_{j=1}^q \beta_j))$.

THEOREM 4. *Let the functions* $f_i(z)$ *be in the same class* $C_0(\alpha)$ *for every* $i=1, 2, \ldots, m$, *and let* $0 \leqslant \alpha \leqslant r_0$, *where* r_0 *is a root of* $2^m(1-mr)-(1-r)^m=0$ *in the interval* $(0, 1/m)$. *Then the quasi-Hadamard product* $f_1 * f_2 * \cdots * f_m(z)$ *belongs to* $C_0(m\alpha)$.

Problems concerning the quasi-Hadamard product of two functions have been considered by many researchers (e.g., see [1, 2, 3, 5]). In Theorems 1 4, Owa has introduced an interesting modification of the Hadamard product, which we have named the quasi-Hadamard product. However, the stringent restrictions $\sum_{i=1}^m \alpha_i \leqslant 1$ and $\sum_{i=1}^m \alpha_i + \sum_{j=1}^q \beta_j \leqslant 1$ in Theorems 1, 2, and 3 diminish the utility of his results. The author finds that the proofs given by Owa [4] fail when $\sum_{i=1}^m \alpha_i > 1$ and $\sum_{i=1}^m \alpha_i + \sum_{j=1}^q \beta_j > 1$. It is therefore natural to ask whether his results can be extended to these complementary cases.

The object of this paper is to establish Theorems 1, 2, and 3 in these complementary cases. In fact, by employing a different technique, we prove these theorems without restricting $\sum_{i=1}^m \alpha_i$ and $\sum_{i=1}^m \alpha_i + \sum_{j=1}^q \beta_j$. Moreover, the classes, to which the quasi-Hadamard product belongs, determined by us are smaller than those given by Owa [4]. Evidently our results are more inclusive as well as applicable, and thus improve Theorems 1, 2 and 3 of Owa [4]. We improve Theorem 4 also in some sense.

2. The Main Theorems

First we prove:

Theorem A. *For each* $i=1,2,\ldots,m$, *let the functions* $f_i(z)$ *belong to the classes* $ST_0^*(\alpha_i)$, *respectively. Then, the quasi-Hadamard product* $f_1 * f_2 * \cdots * f_m(z)$ *belongs to the class* $S_{m-1}^*(\alpha^*)$, *where* $\alpha^* = \max\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$.

Proof. We need to show that

$$\sum_{n=2}^{\infty} \left[n^{m-1}(n-\alpha^*) \prod_{i=1}^{m} a_{n,i} \right] \leqslant (1-\alpha^*) \prod_{i=1}^{m} a_{1,i}.$$

Without loss of generality we may assume $\alpha^* = \alpha_m$.

Since $f_i(z) \in ST_0^*(\alpha_i)$, we have

$$\sum_{n=2}^{\infty} [(n-\alpha_i)\, a_{n,i}] \leqslant (1-\alpha_i)\, a_{1,i}. \tag{2}$$

Therefore,

$$a_{n,i} \leqslant \left(\frac{1-\alpha_i}{n-\alpha_i}\right) a_{1,i},$$

which implies that

$$a_{n,i} \leqslant n^{-1} a_{1,i}. \tag{3}$$

Using (3) for $i=1,2,\ldots,m-1$, and (2) for $i=m$, we obtain

$$\begin{aligned}\sum_{n=2}^{\infty} \left[n^{m-1}(n-\alpha^*) \prod_{i=1}^{m} a_{n,i} \right] &\leqslant \sum_{n=2}^{\infty} \left[n^{m-1}(n-\alpha^*) \left(n^{-(m-1)} \prod_{i=1}^{m-1} a_{1,i} \right) a_{n,m} \right] \\ &= \left(\prod_{i=1}^{m-1} a_{1,i} \right) \sum_{n=2}^{\infty} [(n-\alpha_m)\, a_{n,m}] \\ &\leqslant (1-\alpha^*) \prod_{i=1}^{m} a_{1,i}.\end{aligned}$$

Hence, $f_1 * f_2 * \cdots * f_m(z) \in S_{m-1}^*(\alpha^*)$.

Remark. In view of the inclusion relation

$$S_{m-1}^*(\alpha^*) \subset S_{m-2}^*(\alpha^*) \subset \cdots \subset S_2^*(\alpha^*) \subset C_0(\alpha^*) \subset ST_0^*(\alpha^*) \subset ST_0^*\left(\prod_{i=1}^{m} \alpha_i\right),$$

we observe that the class, to which the quasi-Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ belongs, determined in Theorem A is much smaller than that in Theorem 1. Moreover, Theorem A is free from the restriction $\sum_{i=1}^{m} \alpha_i \leqslant 1$ required in Theorem 1.

THEOREM B. *For each* $i = 1, 2, \ldots, m$, *let the functions* $f_i(z)$ *belong to the classes* $C_0(\alpha_i)$, *respectively. Then, the quasi-Hadamard product* $f_1 * f_2 * \cdots * f_m(z)$ *belongs to the class* $S^*_{2m-1}(\alpha^*)$, *where* $\alpha^* = \max\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$.

Proof. Without loss of generality we may assume $\alpha^* = \alpha_m$. Since $f_i(z) \in C_0(\alpha_i)$, we have

$$\sum_{n=2}^{\infty} [n(n-\alpha_i)\, a_{n,i}] \leqslant (1-\alpha_i)\, a_{1,i}. \tag{4}$$

Therefore,

$$a_{n,i} \leqslant n^{-2} a_{1,i}. \tag{5}$$

Using (5) for $i = 1, 2, \ldots, m-1$, and (4) for $i = m$, we obtain

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left[n^{2m-1}(n-\alpha^*) \prod_{i=1}^{m} a_{n,i} \right] \\
&\qquad \leqslant \sum_{n=2}^{\infty} \left[n^{2m-1}(n-\alpha^*) \left(n^{-2(m-1)} \prod_{i=1}^{m-1} a_{1,i} \right) a_{n,m} \right] \\
&\qquad = \left(\prod_{i=1}^{m-1} a_{1,i} \right) \sum_{n=2}^{\infty} [n(n-\alpha_m)\, a_{n,m}] \\
&\qquad \leqslant (1-\alpha^*) \prod_{i=1}^{m} a_{1,i}.
\end{aligned}$$

Hence, $f_1 * f_2 * \cdots * f_m(z) \in S^*_{2m-1}(\alpha^*)$.

Remark. In view of the inclusion relation

$$S^*_{2m-1}(\alpha^*) \subset S^*_{2m-2}(\alpha^*) \subset \cdots \subset S^*_2(\alpha^*) \subset C^*_0(\alpha^*) \subset C^*_0\left(\prod_{i=1}^{m} \alpha_i \right),$$

it follows that Theorem B provides a better estimate when compared with Theorem 2. Moreover, Theorem B is free from the restriction $\sum_{i=1}^{m} \alpha_i \leqslant 1$ required in Theorem 2.

THEOREM C. *For each* $i = 1, 2, \ldots, m$, *let the functions* $f_i(z)$ *belong to the* *classes* $ST_0^*(\alpha_i)$, *respectively; and for each* $j = 1, 2, \ldots, q$, *let the functions* $g_j(z)$ *belong to the classes* $C_0(\beta_j)$, *respectively. Then, the quasi-Hadamard product* $f_1 * f_2 * \cdots * f_m * g_1 * g_2 * \cdots * g_q(z)$ *belongs to the class* $S^*_{m+2q-1}(\gamma)$, *where*

$$\gamma = \max\{\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_q\}. \tag{6}$$

Proof. Since $f_i(z) \in ST_0^*(\alpha_i)$, the inequalities (2) and (3) hold. Further, since $g_j(z) \in C_0(\beta_j)$, we have

$$\sum_{n=2}^{\infty} [n(n-\beta_j)\, b_{n,j}] \leqslant (1-\beta_j)\, b_{1,j}. \tag{7}$$

Therefore,

$$b_{n,j} \leqslant n^{-2} b_{1,j}. \tag{8}$$

From (6), it follows that either $\gamma = \max\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ or $\gamma = \max\{\beta_1, \beta_2, \ldots, \beta_q\}$.

Case I. When $\gamma = \max\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. In this case we may assume $\gamma = \alpha_m$. Then, using (3) for $i = 1, 2, \ldots, m-1$; (8) for $j = 1, 2, \ldots, q$; and (2) for $i = m$, we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left[n^{m+2q-1}(n-\gamma) \prod_{i=1}^{m} a_{n,i} \prod_{j=1}^{q} b_{n,j} \right] \\
&\quad \leqslant \sum_{n=2}^{\infty} \left[n^{m+2q-1}(n-\gamma)\, n^{-(m-1)} n^{-2q} a_{n,m} \prod_{i=1}^{m-1} a_{1,i} \prod_{j=1}^{q} b_{1,j} \right] \\
&\quad = \left(\prod_{i=1}^{m-1} a_{1,i} \prod_{j=1}^{q} b_{1,j} \right) \sum_{n=2}^{\infty} [(n-\alpha_m)\, a_{n,m}] \\
&\quad \leqslant (1-\gamma) \prod_{i=1}^{m} a_{1,i} \prod_{j=1}^{q} b_{1,j}.
\end{aligned}$$

Case II. When $\gamma = \max\{\beta_1, \beta_2, \ldots, \beta_q\}$. In this case we may assume $\gamma = \beta_q$. Then, using (3) for $i = 1, 2, \ldots, m$; (8) for $j = 1, 2, \ldots, q-1$; and (7) for $j = q$, we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left[n^{m+2q-1}(n-\gamma) \prod_{i=1}^{m} a_{n,i} \prod_{j=1}^{q} b_{n,j} \right] \\
&\quad \leqslant \sum_{n=2}^{\infty} \left[n^{m+2q-1}(n-\gamma)\, n^{-m} n^{-2(q-1)} b_{n,q} \prod_{i=1}^{m} a_{1,i} \prod_{j=1}^{q-1} b_{1,j} \right]
\end{aligned}$$

$$= \left(\prod_{i=1}^{m} a_{1,i} \prod_{j=1}^{q-1} b_{1,j} \right) \sum_{n=2}^{\infty} [n(n-\beta_q)\, b_{n,q}]$$

$$\leqslant (1-\gamma) \prod_{i=1}^{m} a_{1,i} \prod_{j=1}^{q} b_{1,j}.$$

In both the cases we conclude that

$$f_1 * f_2 * \cdots f_m * g_1 * g_2 * \cdots * g_q(z) \in S^*_{m+2q-1}(\gamma).$$

Remark. Regarding the inclusion relation

$$S^*_{m+2q-1}(\gamma) \subset S^*_{m+2q-2}(\gamma) \subset \cdots \subset S^*_2(\gamma) \subset C_0(\gamma)$$
$$\subset ST^*_0(\gamma) \subset ST^*_0\left(\left(\prod_{i=1}^{m} \alpha_i\right)\left(\prod_{j=1}^{q} \beta_j\right)\right),$$

we observe that Theorem C provides a better estimate when compared with Theorem 3. Moreover, Theorem C is free from the restriction $\sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{q} \beta_j \leqslant 1$ required in Theorem 3.

THEOREM D. *For each $i = 1, 2, ..., m$, let the functions $f_i(z)$ belong to the class $C_0(\alpha)$, and let $0 \leqslant \alpha \leqslant r_0$, where r_0 is a root of the equation $2^m(1-mr) - (1-r)^m = 0$ in the interval $(0, 1/m)$. Then, the quasi-Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ belongs to the class $S^*_{m-1}(m\alpha)$.*

Proof. Since $f_i(z) \in C_0(\alpha)$, we have

$$\sum_{n=2}^{\infty} [n(n-\alpha)\, a_{n,i}] \leqslant (1-\alpha)\, a_{1,i}.$$

This inequality implies

$$\sum_{n=2}^{\infty} [(n-\alpha)\, a_{n,i}] \leqslant \tfrac{1}{2}(1-\alpha)\, a_{1,i}, \tag{9}$$

and therefore

$$(n-\alpha)\, a_{n,i} \leqslant \tfrac{1}{2}(1-\alpha)\, a_{1,i}. \tag{10}$$

Also, by mathematical induction on m, we obtain the inequality

$$n^{m-1}(n-m\alpha) \leqslant (n-\alpha)^m, \tag{11}$$

where $0 \leqslant \alpha < 1$, $m \geqslant 1$ and $m\alpha < 1$.

Using (11), (10) for $i=1, 2, \ldots, m-1$; and (9) for $i=m$, we get

$$\sum_{n=2}^{\infty}\left[n^{m-1}(n-m\alpha)\prod_{i=1}^{m}a_{n,i}\right]\leqslant\sum_{n=2}^{\infty}\left[(n-\alpha)^{m}\prod_{i=1}^{m}a_{n,i}\right]$$
$$\leqslant\left\{\left(\frac{(1-\alpha)^{m-1}}{2^{m-1}}\right)\prod_{i=1}^{m-1}a_{1,i}\right\}\sum_{n=2}^{\infty}\left[(n-\alpha)\,a_{n,m}\right]$$
$$\leqslant\left(\frac{(1-\alpha)^{m}}{2^{m}}\right)\prod_{i=1}^{m}a_{1,i}$$
$$\leqslant(1-m\alpha)\prod_{i=1}^{m}a_{1,i},\ \text{for}\ 0\leqslant\alpha\leqslant r_0,$$

where r_0 is a root of the equation $2^m(1-mr)-(1-r)^m=0$.
Hence $f_1 * f_2 * \cdots * f_m(z) \in S^*_{m-1}(m\alpha)$.

Remark. Since $S^*_{m-1}(m\alpha) \subset C_0(m\alpha)$ when $m \geqslant 3$, Theorem D provides a better estimate when compared with Theorem 4.

Note. It is worth noting that the definition of the class $S^*_k(\alpha)$ can be extended to the case when k is any real number. However, the functions in this class are not univalent when $k<0$. In this case the author has obtained some interesting results which will appear elsewhere.

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