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Generalized Convexity in Conformal Mappings

P. K. KULSHRESTHA

Department of Mathematics, Louisiana State University, New Orleans, Louisiana 70122

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1. INTRODUCTION

Let S denote the class of all functions f which are regular and univalent in the open disk $E = \{z: |z| < 1\}$, and are normalized by the conditions

$$f(0) = 0, \qquad f'(0) = 1. \tag{1}$$

In [2] we have introduced a subclass $S(\alpha, \gamma) \subset S$ of γ-spiral functions of order α as follows:

DEFINITION 1. Let f be regular in E and satisfy (1); $f \in S(\alpha, \gamma)$ if and only if there exist real numbers α and γ, $0 \leqslant \alpha < 1$, $|\gamma| < \pi/2$, such that

$$\operatorname{Re} e^{i\gamma} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \cos \gamma, \qquad z \in E. \tag{2}$$

This definition is a generalization of Špaček's condition on spirallike functions in E [9] and gives rise to many interesting results in geometric function theory.

Let P denote the class of normalized analytic functions p with positive real parts, i.e., $p \in P$ if and only if p is analytic in E, $p(0) = 1$, and $\operatorname{Re} p(z) > 0$ for $z \in E$.

If $f \in S(\alpha, \gamma)$, we can write with appropriate normalizing factors

$$\sec \gamma \left[e^{i\gamma} \frac{zf'(z)}{f(z)} - \alpha \cos \gamma - \sin \gamma \right]_{z=0} = 1 - \alpha.$$

This enables us to express members of $S(\alpha, \gamma)$ in terms of functions in P. Thus, $f \in S(\alpha, \gamma)$ if and only if there exists a function $p \in P$ such that

$$\frac{zf'(z)}{f(z)} = \frac{(1 - \alpha) \cos \gamma \, p(z) + \alpha \cos \gamma + i \sin \gamma}{\cos \gamma + i \sin \gamma}, \qquad z \in E. \tag{3}$$

Introducing a complex number $h = (\alpha + i \tan \gamma)/(1 - \alpha)$, the condition (3) can be written as

$$\frac{zf'(z)}{f(z)} = \frac{p(z) + h}{1 + h}, \qquad z \in E. \tag{4}$$

By differentiating the representation in (4), we find that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z) + h} + \frac{p(z) + h}{1 + h}, \qquad z \in E. \tag{5}$$

The concept of generalized convexity has been introduced by Mocanu [5] for a particular class of functions in S. These functions are called k-convex if they are regular in E for $0 \leqslant k \leqslant 1$ and satisfy there the condition that $f(z) f'(z)/z \neq 0$ and

$$\operatorname{Re} \left\{ (1 - k) \frac{zf'(z)}{f(z)} + k \left[1 + \frac{zf''(z)}{f'(z)} \right] \right\} \geqslant 0 \tag{6}$$

in E. This definition also holds for $k \geqslant 0$ and, as shown in [6], these functions are starlike in E for $k \geqslant 0$ and convex for $k \geqslant 1$. The class of generalized k-convex functions in E is defined as follows:

DEFINITION 2. Let f be regular in E and satisfy (1); f is a generalized k-convex function in E, $f \in S(k, \alpha, \gamma)$, if and only if f satisfies (2) and there exists a real number $k \geqslant 0$ such that for some $p \in P$

$$(1 - k) \frac{zf'(z)}{f(z)} + k \left(1 + \frac{zf''(z)}{f'(z)} \right) = \frac{p(z) + h}{1 + h} + k \frac{zp'(z)}{p(z) + h}. \tag{7}$$

The defining property (7) follows from (4) and (5).

In this paper we shall establish some mapping properties of functions of the class $S(k, \alpha, \gamma)$, especially those related to their extremum, distortion, and rotation theorems which reduce to some known results for particular values of the parameters k, α, γ.

Another generalization of the Mocanu condition (6) is to replace it by

$$\operatorname{Re} \left\{ e^{i\gamma} \left[(1 - k) \frac{zf'(z)}{f(z)} + k \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right] \right\} > \alpha \cos \gamma, \qquad z \in E, \quad k \geqslant 0. \tag{8}$$

The property (8) then defines a class of generalized convex functions which leads to the Mocanu–Reade spiral functions in E, which are also Bazilevič functions. It may be noted that this latter class differs in general from the class $S(k, \alpha, \gamma)$ considered in this paper.

2. Main Results

THEOREM 1. *If $f(z) \in S(k, \alpha, \gamma)$, then the set of all possible values of $\log(f(z)/z)$ for a fixed z, $|z| \leqslant r < 1$, lies in the convex image of $|z| = r$ under the mapping*

$$w = \log[(1+h)^k (1-z)^{-2/(1+h)}]. \tag{9}$$

COROLLARY 1. *The extremal function $f_0(z) \in S(k, \alpha, \gamma)$ has the representation of the form*

$$\begin{aligned} f_0(z) &= (1+h)^k z(1-z)^{-2/(1+h)} \\ &= (1+h)^k \left[z + \sum_{n=2}^{\infty} \prod_{m=0}^{n-2} \frac{\frac{2}{1+h} + m}{m+1} z^n \right]. \end{aligned} \tag{10}$$

THEOREM 2. *If $f(z) \in S(k, \alpha, \gamma)$, then for $|z| = r < 1$*

$$T(r, \theta_1, \alpha, \gamma, k) \leqslant \log \left| \frac{f(z)}{z} \right| \leqslant T(r, \theta_2, \alpha, \gamma, k) \qquad \text{if} \qquad \gamma \neq 0, \tag{11}$$

where

$$\begin{aligned} T(r, \theta, \alpha, \gamma, k) = (1-\alpha) \cos\gamma \Big[2 \sin\gamma \tan^{-1} \frac{r \sin\theta}{1 - r\cos\theta} \\ - \cos\gamma \log(1 - 2r\cos\theta + r^2) \Big] - \frac{k}{2} \log[(1-\alpha)\cos\gamma], \end{aligned} \tag{12}$$

$$\theta_{1,2} = 2 \tan^{-1} \left\{ \frac{-\cot\gamma \mp \sqrt{\operatorname{cosec}^2 \gamma - r^2}}{1+r} \right\}. \tag{13}$$

For $\gamma = 0$,

$$\log \left| \frac{f(z)}{z} \right| \leqslant \frac{k}{2} \log \frac{1}{1-\alpha} - 2(1-\alpha) \log(1-r). \tag{14}$$

THEOREM 3. *If $f(z) \in S(k, \alpha, \gamma)$, then for $|z| = r < 1$*

$$S(r, \theta_3, \alpha, \gamma, k) \leqslant \arg \frac{f(z)}{z} \leqslant S(r, \theta_4, \alpha, \gamma, k), \tag{15}$$

where

$$\begin{aligned} S(r, \theta, \alpha, \gamma, k) = (1-\alpha) \cos\gamma \Big[2 \cos\gamma \tan^{-1} \frac{r \sin\theta}{1 - r\cos\theta} \\ + \sin\gamma \log(1 - 2r\cos\theta + r^2) \Big] + k\gamma \end{aligned}$$

$$\theta_{3,4} = 2 \tan^{-1} \left\{ \frac{\tan\gamma \mp \sqrt{\sec^2 \gamma - r^2}}{1+r} \right\}. \tag{16}$$

THEOREM 4. *Let*

$$\zeta(z) = (1-k)\frac{zf'(z)}{f(z)} + k\left[1 + \frac{zf''(z)}{f'(z)}\right].$$

Then the images of all circles $|z| \leqslant r < 1$ *under the mapping* $\zeta(z)$ *lie in the circle* $|\zeta - \zeta_0| \leqslant \rho$, *where*

$$\zeta_0 = \frac{1 - r^2 + 2(1-\alpha) r^2 \cos^2\gamma + i(1-\alpha) r^2 \sin 2\gamma}{1-r^2}$$

$$- \frac{\begin{pmatrix} 8k\alpha(1-\alpha^2)\cos^4\gamma\, r^2[1 - 2(1 - 2\alpha + 2\alpha^2)\cos^2\gamma \\ \qquad - i(1-2\alpha)\sin\gamma\cos\gamma] \end{pmatrix}}{(1-r^2)(1 - r^2 + 2\alpha\cos^2\gamma r^2)(1 - 4\alpha(1-\alpha)\cos^2\gamma)^2},$$

$$\rho = \frac{2(1-\alpha) r\cos\gamma}{1-r^2} - \frac{4kr(1-\alpha)^2\cos^2\gamma}{\sqrt{1 - 4\alpha(1-\alpha)\cos^2\gamma}}$$

$$\times \left\{\frac{1}{1-r^2} + \frac{1}{\sqrt{1 - 4\alpha(1-\alpha)\cos^2\gamma}(1 - r^2 + 2\alpha\cos^2\gamma + r^2)}\right\}.$$

COROLLARY 2. *If* $k = 0$, *it follows from Theorem* 4 *that for* $|z| \leqslant r < 1$

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \leqslant \tan^{-1}\frac{(1-\alpha) r^2 \sin 2\gamma}{1 - r^2 + 2(1-\alpha) r^2\cos^2\gamma} + \tan^{-1}\frac{2(1-\alpha) r\cos\gamma}{\sqrt{(1-r^2)[1 - r^2 + 4\alpha(1-\alpha) r^2\cos^2\gamma]}}, \tag{17}$$

where equality holds for

$$p(z) = \frac{1+z}{1-z} \in P.$$

THEOREM 5. *Let* $\zeta(z)$ *be defined as in Theorem* 4. *Then for* $|z| \leqslant r < 1$

$$|\zeta(z)| \leqslant 1 + \frac{2(1-\alpha) r\cos\gamma}{1-r}\left\{1 + \frac{k}{1 - r + 2(1-\alpha) r\cos^2\gamma}\right\}. \tag{18}$$

Equality in Theorems 4 and 5 holds for the extremal function

$$\zeta^*(z) = \frac{p^*(z) + h}{1+h} + \frac{kzp^{*\prime}(z)}{p^*(z) + h}, \qquad p^*(z) = \frac{1+z}{1-z} \in P.$$

THEOREM 6. *Let*

$$F(z) = \log\left\{\left(\frac{zf'(z)}{f(z)}\right)^k \frac{f(z)}{z}\right\}.$$

Then for $|z| \leqslant r < 1$

$$\begin{aligned}\operatorname{Re}\{1 - k + zF'(z)\} &= \operatorname{Re}\left\{(1-k)\frac{zf'(z)}{f(z)} + k\frac{zf''(z)}{f'(z)}\right\} \\ &\leqslant 1 - k + \frac{2(1-\alpha)\, r\cos\gamma}{1-r}\left\{1 + \frac{k}{1 - r + 2(1-\alpha)\, r\cos^2\gamma}\right\}.\end{aligned} \tag{19}$$

COROLLARY 3. *Denoting the right-hand side of* (19) *by* $\psi(r, \gamma, k)$, *we have for* $|z| \leqslant r < 1$

$$1 - \psi(r, \gamma, k) \leqslant 1 + \operatorname{Re}\{1 - k + zF'(z)\} \leqslant 1 + \psi(r, \gamma, k). \tag{20}$$

This result is sharp for

$$\begin{aligned}r = r_0 \geqslant [&k + (1+k)(1-\alpha)\cos\gamma - k(1-\alpha)\cos^2\gamma + \{2k^2(1-\alpha)\cos\gamma \\ &+ (1+k)^2(1-\alpha)^2\cos^2\gamma + 2k(1-k)(1-\alpha)^2\cos^3\gamma \\ &+ k^2(1-\alpha)^2\cos^4\gamma\}^{1/2}]^{-1}.\end{aligned} \tag{21}$$

3. PROOFS

The condition (7) when integrated from 0 to z gives

$$\log\left[\left(\frac{f(z)}{z}\right)^{1-k}(f'(z))^k\right] = \frac{1}{1+h}\int_0^z \frac{p(x)-1}{x}\,dx + k\log[p(z)+h],$$

which in view of the relation

$$p(z) + h = (1+h)\frac{zf'(z)}{f(z)}$$

becomes

$$\log\left[\frac{f(z)}{(1+h)^k z}\right] = \frac{1}{1+h}\int_0^z \frac{p(x)-1}{x}\,dx. \tag{22}$$

Using the Herglotz representation of function $p(z) \in P$ which satisfies (7) as

$$p(z) = \int_{-\pi}^{\pi} \frac{1+ze^{it}}{1-ze^{it}}\,d\mu(t), \qquad t \in [-\pi, \pi], \tag{23}$$

where $\mu(t)$ is a function of bounded variation with $\int_{-\pi}^{\pi} d\mu(t) = 1$, we find from (22) that

$$\log\frac{f(z)}{z} = \int_{-\pi}^{\pi} \log[(1+h)^k(1-ze^{it})^{-2/(1+h)}]\,d\mu(t). \tag{24}$$

Let

$$q(z, t) = \log[(1 + h)^k (1 - ze^{it})^{-2/(1+h)}], \qquad t \in [-\pi, \pi].$$

Then since

$$1 + \operatorname{Re} \frac{zq''(z, t)}{q'(z, t)} = \operatorname{Re} \frac{1}{1 - ze^{it}} > \frac{1}{2},$$

the function $q(z, t)$ maps the disk $|z| \leqslant r < 1$ univalently onto a convex domain E^* which is independent of t. Thus the relation (24) means that the points $\log(f(z)/z)$, for a fixed z, $|z| \leqslant r < 1$, lie in the convex hull of E^*, and hence they lie in the image of $|z| \leqslant r$ under the mapping

$$\log[(1 + h)^k (1 - \epsilon z)^{-2/(1+h)}], \qquad |\epsilon| = 1,$$

which yields (9). The extremal function is given by

$$\log \frac{f_0(z)}{z} = \log[(1 + h)^k (1 - z)^{-2/(1+h)}], \tag{25}$$

which in view of (24) gives (10).

The results in Theorems 2 and 3 are obtained by evaluating the extrema of the extremal function (25) for $z = re^{i\theta}$, $|z| \leqslant r < 1$.

The function $\zeta(z)$ defined in Theorem 4 can be written as

$$\zeta(z) = \frac{p(z) + h}{1 + h} + \frac{kzp'(z)}{p(z) + h}.$$

Since $\zeta(z)$ is subordinate to the function $\zeta^*(z)$, we find that

$$\zeta^*(z) = \frac{1}{\beta} \frac{z + \beta}{1 - z} - \frac{4k}{(1 - h)^2} \left[\frac{1}{z - 1} + \frac{\beta}{z + \beta}\right], \qquad \beta = \frac{1 + h}{1 - h},$$

and Theorem 4 follows by superposition of linear mappings.

Using (23), the function $\zeta(z)$ can be written as

$$\zeta(z) = \int_{-\pi}^{\pi} \left\{1 + \frac{2ze^{it}}{(1 + h)(1 - ze^{it})}\right\} d\mu(t) + \frac{k}{1 + h} \cdot \frac{\displaystyle\int_{-\pi}^{\pi} \frac{2ze^{it}}{(1 - ze^{it})^2} d\mu(t)}{\displaystyle\int_{-\pi}^{\pi} \left\{1 + \frac{2ze^{it}}{(1 + h)(1 - ze^{it})}\right\} d\mu(t)}. \tag{26}$$

Then (18) follows from (26) by using the following result [2]:

LEMMA. *Let $f_1(t)$ and $f_2(t)$ be single-valued continuous complex functions of a real variable $t \in [-\pi, \pi]$ and let $\mu(t)$ be the function as defined in* (23). *If* $\operatorname{Re} f_2(t) > 0$, *then*

$$\left| \frac{\int_{-\pi}^{\pi} f_1(t)\, d\mu(t)}{\int_{-\pi}^{\pi} f_2(t)\, d\mu(t)} \right| \leqslant \max_{t \in [-\pi, \pi]} \frac{|f_1(t)|}{\operatorname{Re} f_2(t)} .$$

Theorem 6 follows from Theorem 5, and then its corollary (20) is obvious. By considering the roots of $1 - \psi(r, \gamma, k) = 0$, the bound (21) follows.

4. PARTICULAR CASES

We note that the class $S(0, \alpha, \gamma)$ is the class of spirallike functions of order α, while $S(0, \alpha, 0)$ is the class of starlike functions of order α and $S(0, 0, 0)$ is the class of starlike functions in E. For these classes the following results hold [2]:

COROLLARY 4. *If $f(z) \in S(0, \alpha, \gamma)$, the mapping function in Theorem* 1 *reduces to*

$$w = \log(1 - z)^{-2c}, \tag{27}$$

where $c = (1 - \alpha)/(1 + i \tan \gamma)$. If $f(z) \in S(0, \alpha, 0)$, this mapping function is $w = \log(1 - z)^{-2(1-\alpha)}$. For $f(z) \in S(0, 0, 0)$, (27) *reduces to the well-known mapping $w = \log(1 - z)^{-2}$ for starlike functions* ([4], [10]).

COROLLARY 5. *If $f(z) \in S(0, 0, 0)$, then Theorems* 2 *and* 3 *give the following well-known bound for starlike functions* [7]:

$$\frac{r}{1 + r^2} \leqslant |f(z)| \leqslant \frac{r}{1 - r^2}, \qquad \left| \arg \frac{f(z)}{z} \right| \leqslant 2 \sin^{-1} r.$$

COROLLARY 6. *If $f(z) \in S(0, 0, 0)$, it follows from* (17) *and* (18) *that for* $|z| \leqslant r < 1$

$$\left| \arg \frac{zf'(z)}{f(z)} \right| \leqslant \tan^{-1} \frac{2(1 - \alpha) r}{\sqrt{(1 - r^2)\,[1 - r^2 + 4\alpha(1 - \alpha) r^2]}}, \tag{28}$$

$$\left| \frac{zf'(z)}{f(z)} \right| \leqslant 1 + \frac{2(1 - \alpha) r}{1 - r}. \tag{29}$$

If $f(z) \in S(0, 0, 0)$, we obtain the well-known results from (28) *and* (29) *for starlike functions in E* [7].

Using the method developed in [1] and applied in [3] it is easy to find the coefficient bounds for functions $f \in S(0, \alpha, \gamma)$ as the following results shows:

THEOREM 7. *If $f \in S(0, \alpha, \gamma)$ and*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in E,$$

then

$$|a_n| \leqslant \prod_{m=0}^{n-2} \frac{\left|\frac{2}{1+h} + m\right|}{1+h}, \qquad n = 2, 3, \ldots. \tag{30}$$

The bounds in (30) are sharp and equality holds for the extremal function (10) with $k = 0$. The estimate in (30) is the same as found by Libera [3]. The β-spiral radius of $f \in S(0, \alpha, \gamma)$, $|\beta| < \pi/2$, also turns out to be the same as in [3].

COROLLARY 7. *If $f(z) \in S(1, \alpha, \gamma)$ which is the class of convex functions of order α, we find from* (19) *that* [2]

$$\left| \frac{zf''(z)}{f'(z)} \right| \leqslant \frac{2(1-\alpha) r \cos \gamma}{1-r} \left\{1 + \frac{1}{1 - r + 2(1-\alpha) r \cos^2 \gamma}\right\}. \tag{31}$$

This result is sharp for

$$r = r_0 \geqslant [1 + (1-\alpha)(2\cos\gamma - \cos^2\gamma) + \{2(1-\alpha)\cos\gamma + 4(1-\alpha)^2\cos^2\gamma + (1-\alpha)^2 \cos^4\gamma\}^{1/2}]^{-1}. \tag{32}$$

If $f(z) \in S(1, \alpha, 0)$, the estimate (31) becomes

$$\left| \frac{zf''(z)}{f'(z)} \right| \leqslant \frac{2(1-\alpha) r}{1-r} \left\{1 + \frac{1}{1 + r - 2\alpha r}\right\}$$

which holds for $r_0 \geqslant [2 - \alpha + \sqrt{5\alpha^2 - 12\alpha + 7}]^{-1}$. If $f(z) \in S(1, 0, 0)$, we get $|zf''(z)/f'(z)| \leqslant (4r + 2r^2)/(1 - r^2)$, which holds for

$$r > \frac{1}{2 + \sqrt{7}} = 0.213, \ldots,$$

which is a well-known result [7].

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References

1. J. Clunie, On meromorphic schlicht functions, *J. London Math. Soc.* **34** (1959), 215–216.
2. P. K. Kulshrestha, Distortion of spiral-like mappings, Tech. Rept. No. 53, LSUNO, October 1971.
3. R. J. Libera, Univalent α-spiral functions, *Can. J. Math.* **19** (1967), 449–456.
4. A. Marx, Untersuchungen über schlichte Abbildungen, *Math. Ann.* **107** (1932), 40–67.
5. P. T. Mocanu, Une propriété de convexité généralisée dans la théorie de la représentation conforme, *Mathematica* **11** (1969), 127–133.
6. P. T. Mocanu and Maxwell O. Reade, The order of starlikeness of certain univalent functions, *Mathematica* (to be published).
7. Z. Nehari, *Conformal Mapping* (McGraw-Hill, New York, 1952), 215–225.
8. M. S. Robertson, Extremal problems for analytic functions with positive real part and applications, *Trans. Amer. Math. Soc.* **106** (1963), 236–253.
9. L. Špaček, Contribution à la théorie des fonctions univalentes, *Časopis Pest. Mat. Fys.* **62** (1933), 12–19.
10. E. Strohhäcker, Beitrage zur Theorie der schlichten Funktionen, *Math. Zeit.* **37** (1933), 356–380.