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DISTORTION OF SPIRAL-LIKE MAPPINGS

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Abstract

The spiral-like mapping of order α, $0 \leqq \alpha < 1$, of the open unit disk and related distortion properties are investigated by using the Herglotz representation.

1. Introduction

Let P denote the class of functions $p(z)$ which are regular and have a positive real part in the open unit disk $E(|z| < 1)$ and are normalised so that $p(0)=1$. We shall investigate a class of univalent functions which can be expressed in terms of $p(z)$. This class, denoted by $S_\alpha(\gamma)$, consists of spiral-like analytic functions $f(z)$ of order α, $0 \leqq \alpha < 1$, regular in E and normalised by $f(0)=0$, $f'(0)=1$, with the property that

$$\operatorname{Re}\left\{e^{i\gamma}\frac{zf'(z)}{f(z)}\right\} > \alpha\cos\gamma \tag{1}$$

for some real number γ,

$$|\gamma| < \frac{\pi}{2}.$$

The class $S_\alpha(\gamma)$ has been considered by Libera [2] and is denoted by him as $\mathfrak{T}_{\gamma,\alpha}$. We note from (1) that when either or both α and γ are zero, we obtain the following subclasses of $S_\alpha(\gamma)$: $S_\alpha(0)$ which is starlike of order α in E; $S_0(\gamma)$ which is spiral-like in E; and $S_0(0)$ which is starlike in E.

There is not much work done on distortion properties of function of the class $S_\alpha(\gamma)$. Libera [2] has investigated the coefficient bounds and the spiral radius of the class $S_\alpha(\gamma)$. Robertson [5] has considered the class of functions $S_0(\gamma)$ and evaluated bounds for their radius. In the present paper we shall mainly deal with the distortion properties of the class $S_\alpha(\gamma)$.

In view of the definitions given in [2], if $f \in S_\alpha(\gamma)$, we can write

$$\frac{zf'(z)}{f(z)} = \frac{p(z)+h}{1+h}, \tag{2}$$

where

$$h = \frac{\alpha + i \tan \gamma}{1-\alpha}.$$

Using the Herglotz representation

$$p(z) = \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\, \varepsilon P,$$

where $\mu(t)$ is a real-valued non-negative non-decreasing function defined for $t\, \varepsilon[-\pi, \pi]$ with total variation

$$\int_{-\pi}^{\pi} d\mu(t) = 1,$$

the relation (2) gives

$$\log \frac{f(z)}{z} = -2c \int_{-\pi}^{\pi} \log(1 - ze^{-it}) d\mu(t), \tag{3}$$

where $c = (1-\alpha)\cos\gamma\, e^{-i\gamma}$. It may be noted that the relation (3) can be obtained either by a trivial modification of the proof for the case $c = 1$ (i.e., in the class $S_0(0)$) given in Pommerenke [4] and again in Twomey [7], or by a deduction from the case $c = 1$, using the transformation

$$F(z) = z \exp \left\{ \frac{e^{i\gamma}}{(1-\alpha)\cos\gamma} \log \frac{f(z)}{z} \right\}$$

which for f in $S_\alpha(\gamma)$ belongs to the class $S_0(0)$, and then clearly

$$\log \frac{f(z)}{z} = (1-\alpha)\cos\gamma\, e^{-i\gamma} \log \frac{F(z)}{z}$$

is given by (3).

The relation (3) means that if $f \in S_\alpha(\gamma)$, and if

$$\log \frac{f(z)}{z} \text{ and } \log(1-z)$$

are regular in E, the branch of each being that which has the value 0 at $z = 0$, then

$$\log \frac{f(z)}{z}$$

lies in the image, which is convex, of the disk $|z| \leqq r$, $0 < r < 1$, under the mapping

$$w = \log(1-z)^{-2c}. \tag{4}$$

For $c = 1$ (i.e., for $f \in S_\alpha(0)$) the mapping (4) reduces to the well-known result of Marx [3] and Ströhhacker [6]. The fact that the function $f_0(z)$ defined by

$$f_0(z) = \frac{z}{(1-z)^{2c}} \tag{5}$$

belongs to $S_\alpha(\gamma)$ shows that the result (4) is sharp in the sense that the boundary points are attained by this function.

2. Distortion Properties

The main result on the distortion properties of the functions $f \in S_\alpha(\gamma)$ is as follows:

Theorem. If $f \in S_\alpha(\gamma)$, then for $|z| = r < 1$

$$\phi_1(r) = \min_{|z|=r} \left[2(1-\alpha)\cos\gamma \operatorname{Re}\left\{ e^{-i\gamma} \log \frac{1}{1-z} \right\} \right] \leqq \log \left| \frac{f(z)}{z} \right| \leqq$$

$$\leqq \max_{|z|=r} \left[2(1-\alpha)\cos\gamma \operatorname{Re}\left\{ e^{-i\gamma} \log \frac{1}{1-z} \right\} \right] = \phi_2(r), \tag{6}$$

$$\psi_1(r) = \min_{|z|=r} \left[2(1-\alpha)\cos\gamma \, Im \left\{ e^{-i\gamma} \log \frac{1}{1-z} \right\} \right] \leqq \arg \frac{f(z)}{z} \leqq$$

$$\leqq \max_{|z|=r} \left[2(1-\alpha)\cos\gamma \operatorname{Im}\left\{ e^{-i\gamma} \log \frac{1}{1-z} \right\} \right] = \psi_2(r). \tag{7}$$

The bounds in (6) and (7) are attained by the extremal function (5) and so they are the best possible.

The explicit expressions for $\phi_1(r)$, $\phi_2(r)$, $\psi_1(r)$, $\psi_2(r)$ are given below by (12)–(15).

PROOF. We can write (3) as

$$\log \left| \frac{f(z)}{z} \right| = 2(1-\alpha)\cos\gamma \int_{-\pi}^{\pi} \operatorname{Re}\left\{ e^{-i\gamma} \log \frac{1}{1-ze^{-it}} \right\} d\mu(t),$$

$$\arg \frac{f(z)}{z} = 2(1-\alpha)\cos\gamma \int_{-\pi}^{\pi} \operatorname{Im}\left\{ e^{-i\gamma} \log \frac{1}{1-ze^{-it}} \right\} d\mu(t).$$

It is then immediate from the properties of μ that if $\phi_1(r)$, $\phi_2(r)$ are the minimum and maximum respectively of

$$2(1-\alpha)\cos\gamma \operatorname{Re}\left\{ e^{-i\gamma} \log \frac{1}{1-z} \right\} \tag{8}$$

on $|z| = r$, and $\psi_1(r)$, $\psi_2(r)$ are the minimum and maximum respectively of

$$2(1-\alpha)\cos\gamma \operatorname{Im}\left\{ e^{i\gamma} \log \frac{1}{1-z} \right\} \tag{9}$$

on $|z| = r$, then (6) and (7) hold. For the extremal function $f_0(z)$, defined by (5), it is clear that

$$\log \left| \frac{f(z)}{z} \right| \text{ and } \arg \frac{f(z)}{z}$$

are given by (8) and (9) respectively, so that the bounds in (6) and (7) are attained in the case of $f_0(z)$ and so they are best possible.

To derive more explicit expressions for the bounds in (6) and (7), which will yield the order of growth of

$$\log \left| \frac{f(z)}{z} \right|$$

and show whether

$$\arg \frac{f(z)}{z}$$

is bounded or not, we put, from (5),

$$\log \frac{f_0(z)}{z} = u(\theta) + iv(\theta)$$

for a fixed r, $z = re^{\theta}$, $0 < r < 1$, $-\pi < \theta < \pi$, and get

$$u(\theta) = (1-\alpha) \sin 2\gamma \arc \tan \frac{r \sin \theta}{1 - r \cos \theta} - \\ -(1-\alpha) \cos^2 \gamma \log (1 - 2r \cos \theta + r^2),$$

$$v(\theta) = 2(1-\alpha) \cos^2 \gamma \arc \tan \frac{r \sin \theta}{1 - r \cos \theta} + \\ +(1-\alpha) \sin \gamma \cos \gamma \log (1 - 2r \cos \theta + r^2).$$

We thus see that for any $f \in S_\alpha(\gamma)$, the bounds for

$$\log \left| \frac{f(z)}{z} \right| \text{ and } \arg \frac{f(z)}{z}$$

will be attained by the extrema of the functions $u(\theta)$ and $v(\theta)$ respectively.

In finding the extrema of the function $u(\theta)$ we are then required to solve the equation

$$\sin (\theta - \gamma) = -r \sin \gamma,$$

so that $\theta - \gamma = \arc \sin (-r \sin \gamma)$, and we can take

$$\theta_1 = \gamma + \arc \sin (r \sin \gamma) - \pi, \\ \theta_2 = \gamma - \arc \sin (r \sin \gamma), \tag{10}$$

where θ_1 minimises $u(\theta)$ while θ_2 maximises it. Similarly for $v(\theta)$ we have to solve the equation

$$\cos (\theta - \gamma) - r \cos \gamma,$$

and we take

$$\theta_3 = \gamma - \arc \cos (r \cos \gamma), \\ \theta_4 = \gamma + \arc \cos (r \cos \gamma), \tag{11}$$

where θ_3 minimises $v(\theta)$ while θ_4 maximises it. Substituting from (10) and (11) into $u(\theta)$ and $v(\theta)$ we obtain

$$\phi_1(r) = 2(1-\alpha) \cos \gamma \{ -\tfrac{1}{2} \cos \gamma \log [1 + 2r \cos \gamma \sqrt{1 - r^2 \sin^2 \gamma} + r^2 \cos 2\gamma] - \\ - \sin \gamma \arc \tan \left[\frac{r \sin \gamma (\sqrt{1 - r^2 \sin^2 \gamma} + r \cos \gamma)}{1 + r \cos \gamma \sqrt{1 - r^2 \sin^2 \gamma} - r^2 \sin^2 \gamma} \right] \}, \tag{12}$$

$$\phi_2(r) = 2(1-\alpha) \cos \gamma \{ -\tfrac{1}{2} \cos \gamma \log [1 - 2r \cos \gamma \sqrt{1 - r^2 \sin^2 \gamma} + r^2 \cos 2\gamma] + \\ + \sin \gamma \arc \tan \left[\frac{r \sin \gamma (\sqrt{1 - r^2 \sin^2 \gamma} + r \cos \gamma)}{1 - r \cos \gamma \sqrt{1 - r^2 \sin^2 \gamma} - r^2 \sin^2 \gamma} \right] \}, \tag{13}$$

$$\psi_1(r)=2(1-\alpha)\cos\gamma\,\{-\cos\gamma\ \mathrm{arc\,tan}\left[\frac{r\cos\gamma(\sqrt{1-r^2\cos^2\gamma}-r\sin\gamma)}{1-r\sin\gamma\sqrt{1-r^2\cos^2\gamma}-r^2\cos^2\gamma}\right]+$$

$$+\tfrac{1}{2}\sin\gamma\log[1-2r\sin\gamma\sqrt{1-r^2\cos^2\gamma}-r^2\cos 2\gamma]\},\qquad(14)$$

$$\psi_2(r)=2(1-\alpha)\cos\gamma\,\{\cos\gamma\ \mathrm{arc\,tan}\left[\frac{r\cos\gamma(\sqrt{1-r^2\cos\gamma}+r\sin\gamma)}{1+r\sin\gamma\sqrt{1-r^2\cos^2\gamma}-r^2\cos^2\gamma}\right]+$$

$$+\tfrac{1}{2}\sin\gamma\log[1+2r\sin\gamma\sqrt{1-r^2\cos^2\gamma}-r^2\cos 2\gamma]\}.\qquad(15)$$

The expressions (12), (13) exhibit the growth of

$$\log\left|\frac{f(z)}{z}\right|.$$

The expressions (14) and (15) imply that

$$\arg\frac{f(z)}{z}\text{ is bounded above if }\gamma\,\varepsilon\left(0,\frac{\pi}{2}\right)\text{ and is bounded below if }\gamma\,\varepsilon\left(-\frac{\pi}{2},0\right),$$

as we see by making $r\to 1$ and examining the logarithmic terms in them.

If $f\in S_0(0)$, we put $\alpha=\gamma=0$ in (12)–(15) and obtain the corresponding results for starlike functions (see, e.g. [4] and [1]).

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