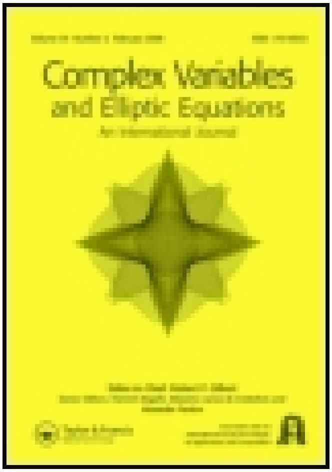

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John's criterion of univalence and a problem of robertson

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John's Criterion of Univalence and a Problem of Robertson

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Dedicated to Professor M. S. Robertson

This paper deals with some univalence criteria related to the geometric structure of the set $\Omega_f = \{\log f'(z) : z \in \mathbb{D}\}$, where f is regular and locally univalent in the unit disk $\mathbb{D}$. One of these criteria is connected with a problem proposed by M. S. Robertson. An estimate of the constant ρ associated with this problem has been given.

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0. INTRODUCTION. NOTATIONS

Let $\mathscr{H}$ be the class of functions f regular and locally univalent in the unit disk $\mathbb{D}$ and let $\mathscr{G}$ be the subclass of $\mathscr{H}$ consisting of functions non-univalent in $\mathbb{D}$. For any $f \in \mathscr{H}$ the derivative f' is regular and does not vanish in $\mathbb{D}$. Consequently, a single-valued function $\log f'$ is uniquely determined by an appropriate choice of $\log f'(0)$. While looking for univalence criteria, no loss in generality is caused by the assumption of local univalence since the equality $f'(z_0) = 0$ eliminates f from the competition, so far as univalence is concerned.

There is a general method of obtaining univalence criteria in $\mathscr{H}$ which is associated with a suitable non-negative functional $\Phi(f)$, $f \in \mathscr{H}$. If $\alpha = \inf\{\Phi(f) : f \in G\} > \inf\{\Phi(f) : f \in \mathscr{H}\}$, then obviously the condition $\Phi(f) < \alpha$ (and even $\Phi(f) \leqslant \alpha$ in many cases) is sufficient for $f \in \mathscr{H}$ to be univalent. One of the most important univalence criteria is associated with the functional

$$\Phi_0(f) = \sup\{(1 - |z|^2)|zf''(z)/f'(z)| : z \in \mathbb{D}\} \tag{0.1}$$

The exact value of the constant

$$\beta = \inf\{\Phi_0(f) : f \in G\} \tag{0.2}$$

is still unknown, nevertheless the estimate $1 \leqslant \beta < 1.21$ (cf. e.g. [5], p. 172) yields Becker's univalence criterion:

$$\Phi_0(f) \leqslant 1 \Rightarrow f \in H \setminus G \tag{0.3}$$

Obviously (0.1) may be expressed in terms of $g = \log f'$. Another univalence criterion where $\log f'$ intervenes, was suggested by F. John [3] and will be discussed in the next section.

In this paper we deal with some related univalence criteria, one of these being connected with an extremal problem proposed by M. S. Robertson. We give an estimate for Robertson's constant ρ associated with this problem. All these criteria are pertinent to a specific geometric structure of the set

$$\Omega_f = \{\log f'(z) : z \in \mathbb{D}\} \tag{0.4}$$

In what follows we aim at exhibiting geometric properties of the set Ω_f involving univalence of $f \in H$.

1. MODULAR AND ANGULAR JOHN CONSTANT

Consider the functional

$$\Phi_1(f) = \sup\{|\log|f'(z_1)| - \log|f'(z_2)|| : z_1, z_2 \in \mathbb{D}\} \tag{1.1}$$

where $f \in \mathscr{H}$. If $\Phi_1(f) \leqslant \pi/2$, then the set Ω_f is contained in a vertical strip of width $\pi/2$. On applying Becker's criterion (0.3) we readily deduce (cf. sect. 3) univalence of f. Hence we infer the existence of a constant $\gamma \geqslant \exp(\pi/2)$ such that $\inf\{\Phi_1(f) : f \in G\} = \log \gamma \geqslant \pi/2$. The exact value of the John constant is still unknown. In what follows we shall call γ for obvious reasons the modular John constant.

While considering $\operatorname{Im} \log f'(z)$ instead of $\operatorname{Re} \log f'(z)$ we are led to the idea of the angular John constant associated with the functional

$$\Phi_2(f) = \sup\{|\arg f'(z_1) - \arg f'(z_2)| : z_1, z_2 \in \mathbb{D}\} \tag{1.2}$$

However, in this case the determination of the exact value of $\inf\{\Phi_2(f): f \in G\}$ which may be called angular John constant, is quite easy and is given in

THEOREM 1 *We have*

$$\inf\{\Phi_2(f) : f \in G\} = \pi \tag{1.3}$$

Proof Suppose that for some $f \in \mathscr{H}$ and all $z_1, z_2 \in \mathbb{D}$ we have

$$|\arg f'(z_1) - \arg f'(z_2)| < \pi \tag{1.4}$$

Obviously (1.4) implies boundedness of $\arg f'(z)$. Suppose that for some fixed branch of $\arg f'(z) = \operatorname{Im} \log f'(z)$ we have $\theta_1 = \inf \arg f'(z)$ and $\theta_2 = \sup \arg f'(z)$, $z \in \mathbb{D}$; $\arg f'(z)$ being continuous on a connected set $\mathbb{D}$, it assumes at some point $z_0 \in D$ an intermediate value $\theta_0 = \frac{1}{2}(\theta_1 + \theta_2)$. It follows from (1.4) that the values of $f'(z)$ are situated in the half-plane $\{w : |\arg w - \theta_0| < \pi/2\}$. Therefore we have $\operatorname{Re}[\exp(-i\theta_0)f'(z)] > 0$ and this implies univalence of f.

In order to show that the value π is best possible we have to construct for each $\epsilon > 0$ a function $f \in G$ such that $|\arg f'(z_1) - \arg f'(z_2)| < \pi(1 + \epsilon)$ for any $z_1, z_2 \in \mathbb{D}$. To this end consider $f(z) = (1 + z)^{2+\epsilon}$. If $z_0 = -1 + r\exp[i\pi/(2 + \epsilon)]$, then $z_0, \bar{z}_0 \in \mathbb{D}$ for $r > 0$ sufficiently small. We have $f(z_0) = f(\bar{z}_0) = -r^{2+\epsilon}$ so that f is not univalent in $\mathbb{D}$, while $|\arg f'(z)| < (\pi/2)(1 + \epsilon)$, i.e. $|\arg f'(z_1) - \arg f'(z_2)| < \pi(1 + \epsilon)$ for any $z_1, z_2 \in \mathbb{D}$. This ends the proof.

The domain Ω_f pertaining to the function f considered above corresponds after rotation by the angle $\pi/2$ to the function $h(z) = \exp\{[(1 + \epsilon)i + 1]\log(1 + z)\}$ which belongs to G. In fact, it is always possible to find $z_1, z_2 \in \mathbb{D}$ $(z_1 \neq z_2)$ satisfying the system of equations:

$$\log\left|\frac{1 + z_1}{1 + z_2}\right| - (1 + \epsilon)\arg\frac{1 + z_1}{1 + z_2} = 0,$$

$$(1 + \epsilon)\log\left|\frac{1 + z_1}{1 + z_2}\right| + \arg\frac{1 + z_1}{1 + z_2} = 2\pi,$$

which implies $h(z_1) = h(z_2)$. On the other hand,

$$|\mathrm{Re}\{\log h'(z_1) - \log h'(z_2)\}| = (1+\epsilon)\left|\arg\frac{1+z_1}{1+z_2}\right| < \pi(1+\epsilon).$$

This implies the estimate $\log\gamma \leqslant \pi$ obtained earlier and in a different way by Yamashita [7].

2. A PROBLEM OF ROBERTSON

Consider the functional

$$\Phi_3(f) = \sup\{|\log f'(z_1) - \log f'(z_2)| : z_1, z_2 \in \mathbb{D}\} \tag{2.1}$$

Obviously $\Phi_3(f)$ is the diameter $\operatorname{diam}\Omega_f$ of the set Ω_f. As an immediate consequence of Theorem 1 we obtain

COROLLARY 1 *If $f \in G$ and the inequality*

$$\operatorname{diam}\Omega_f \leqslant \pi \tag{2.2}$$

holds, then f is univalent in $\mathbb{D}$.

It is an open question whether the constant π is best possible in the univalence criterion (2.2). The answer to this question might be obtained by the determination of the constant

$$\kappa = \inf\{\Phi_3(f) : f \in G\} \tag{2.3}$$

A related problem was proposed a few years ago by M. S. Robertson (private communication to the present author): If $f \in \mathscr{H}$ and $|\log f'(z)| < \pi/2$, then obviously f is univalent because $\mathrm{Re}\, f'(z) > 0$. Is the constant $\pi/2$ best possible? Thus putting

$$\Phi_4(f) = \sup\{|\log f'(z)| : z \in \mathbb{D}\} \tag{2.4}$$

we are led to the determination of

$$\rho = \inf\{\Phi_4(f) : f \in G\} \tag{2.5}$$

which may be called Robertson's constant. There is a simple relation between the constants κ and ρ. We have obviously $\Phi_3(f) \leqslant 2\Phi_4(f)$

and this implies

$$\kappa \leqslant 2\rho \tag{2.6}$$

If we were able to show that $\kappa > \pi$, this would provide us with two essentially new criteria of univalence associated with functionals Φ_3 and Φ_4, resp.

We shall now derive an upper estimate for ρ which also, according to (2.6), gives an upper estimate for κ.

THEOREM 2 *Let $A \geqslant \pi/2 = 1.57079\ldots$ be a constant such that each $F \in \mathscr{H}$ satisfying the inequality $|\log F'(z)| < A$ in $\mathbb{D}$, is univalent in $\mathbb{D}$. Then*

$$A < 1.940 \tag{2.7}$$

Proof Consider the p-symmetric function

$$F(z) = \int_0^z \exp(At^p)\,dt = z + (p+1)^{-1}Az^{p+1} + \ldots. \tag{2.8}$$

We have

$$\sup_{z \in \mathbb{D}} |F(z)| = \int_0^1 \exp(Ax^p)\,dx = M(A, p). \tag{2.9}$$

From the well-known estimate of the second coefficient of a bounded univalent function (cf. e.g. [5], p. 23, Ex. 8) we readily obtain a corresponding estimate for a bounded p-symmetric function $F(z) = (f(z^p))^{1/p} = z + A_{p+1}z^{p+1} + \ldots$:

$$|A_{p+1}| \leqslant (2/p)(1 - M^{-p}), \qquad \text{where} \quad M = \sup_{z \in \mathbb{D}} |F(z)|.$$

For F as given by (2.8) a corresponding inequality takes the form $A \leqslant 2(1 + p^{-1})[1 - M(A, p)^{-p}]$. In case the opposite inequality

$$A > 2(1 + p^{-1})\left[1 - M(A, p)^{-p}\right] \tag{2.10}$$

holds for some $A > \pi/2$ and $p \in \mathbb{N}$, it means that F, as given by (2.8), is not univalent in $\mathbb{D}$.

We shall now prove that

$$\lim_{p\to\infty}\left[M(A,p)\right]^p = \exp\left(\int_0^A \frac{e^x-1}{x}\,dx\right). \tag{2.11}$$

We have from (2.9)

$$M(A,p) = \int_0^1\left(1+\frac{Ax^p}{1!}+\frac{A^2x^{2p}}{2!}+\dots\right)dx$$

$$= 1+\frac{A}{1!(p+1)}+\frac{A^2}{2!(2p+1)}+\dots$$

and hence

$$\lim_{p\to\infty} p\left[M(A,p)-1\right]$$

$$= \lim_{p\to\infty}\left[\frac{A}{1!}\,\frac{p}{p+1}+\frac{A^2}{2!}\,\frac{p}{2p+1}+\frac{A^3}{3!}\,\frac{p}{3p+1}+\dots\right]$$

$$= \frac{A}{1!}+\frac{A^2}{2\cdot 2!}+\frac{A^3}{3\cdot 3!}+\dots = \int_0^A\frac{e^x-1}{x}\,dx.$$

In particular $\lim_{p\to\infty}M(A,p)=1$ and this implies

$$\lim_{p\to\infty}\frac{\log M(A,p)-\log 1}{M(A,p)-1} = 1 = \lim_{p\to\infty}\frac{p\log M(A,p)}{p\left[M(A,p)-1\right]}.$$

Consequently, $\lim_{p\to\infty}p\log M(A,p)=\int_0^A x^{-1}(e^x-1)\,dx$ and by exponentiation we obtain (2.11).

From (2.10) and (2.11) we conclude that the function (2.8) is not univalent if p is large enough and A satisfies the inequality

$$A > 2\left[1-\exp\left(-\int_0^A x^{-1}(e^x-1)\,dx\right)\right] \tag{2.12}$$

The inequality (2.12) can be written in the following equivalent form:

$$\log\left(1-\frac{A}{2}\right)^{-1} > \int_0^A x^{-1}(e^x-1)\,dx = \sum_{n=1}^{\infty}\frac{A^n}{n\cdot n!} \equiv \sigma(A), \tag{2.13}$$

which is very convenient for numerical evaluation. For $A=1.939$ we have

$$\sigma(A) > \sum_{n \leqslant 7} A^n/(n \cdot n!) = 3.4920\ldots > 3.49003\ldots = \log\left(1 - \frac{A}{2}\right)^{-1},$$

whereas for $A = 1.940$ we have

$$\sigma(A) < \sum_{n \leqslant 6} A^n/(n \cdot n!) + (\text{estimate of remainder})$$

$$< 3.497 < 3.506\ldots = \log\left(1 - \frac{A}{2}\right)^{-1}$$

so that the inequality (2.13) holds for $A = 1.940$ and does not hold for $A = 1.939$. This implies the estimate (2.7).

3. UNIVALENCE DOMAINS

A domain Ω in the finite plane $\mathbb{C}$ will be called a univalence domain (for short: a U-domain) if the inclusion $\Omega_f = \{\log f'(z) : z \in \mathbb{D}\} \subset \Omega$ for $f \in \mathscr{H}$ and some branch of $\log f'$ implies the univalence of f in $\mathbb{D}$.

Obviously the strip $\{w : |\mathrm{Im}\, w| < \pi/2\}$ is a U-domain. Moreover, any subdomain and any translation of a U-domain is also a U-domain. Each U-domain corresponds to a particular criterion of univalence. A more detailed discussion of U-domains will be preceded by some preliminaries.

Let Ω be a domain in the finite plane possessing a Green's function (in the generalized sense) $g(w, w_0; \Omega) = g(w, w_0)$, (cf. e.g. [2], p. 97). The finite limit $\lim_{w \to w_0}[g(w, w_0) + \log|w - w_0|]$ which necessarily exists, is called the Robin's constant $\gamma(w_0; \Omega)$, while $r(w_0; \Omega) = \exp \gamma(w_0; \Omega)$ is called the inner radius of Ω at w_0.

We shall need a lemma which is actually a variant of Theorem 4.7 in [1].

LEMMA 1 *Suppose φ is analytic in $\mathbb{D}$ and the values of φ are contained in a domain Ω possessing a generalized Green's function. Then for any $z \in \mathbb{D}$*

$$(1 - |z|^2)|\varphi'(z)| \leqslant r(\varphi(z); \Omega) \tag{3.1}$$

The sign of equality at some point $z_0 \in \mathbb{D}$ holds only for the univalent function φ and for a simply connected domain $\Omega = \varphi(\mathbb{D})$.

Proof We may obviously assume that $\varphi \neq$ const. For a fixed $z_0 \in \mathbb{D}$ consider the function

$$\psi(z) = \log\left|\frac{z - z_0}{1 - z\bar{z}_0}\right| + g(\varphi(z), \varphi(z_0); \Omega) \tag{3.2}$$

which is harmonic in $\mathbb{D}$ apart from its isolated singularities. Evidently ψ has a removable singularity at z_0 and positive logarithmic poles at all $a \neq z_0$ where $\varphi(a) = \varphi(z_0)$. Moreover, $\liminf_{|z|\to 1}\psi(z) \geqslant 0$, and consequently $\psi(z) \geqslant 0$ in $\mathbb{D}$. We have

$$g(\varphi(z), \varphi(z_0); \Omega) = -\log|\varphi(z) - \varphi(z_0)| + \log r(\varphi(z_0); \Omega) + o(1)$$

$$\text{as} \quad z \to z_0$$

and this implies

$$\lim_{z\to z_0}\psi(z) = \lim_{z\to z_0}\left[\log\left|\frac{z - z_0}{1 - z\bar{z}_0}\right| - \log|\varphi(z) - \varphi(z_0)| + \log r(\varphi(z_0); \Omega)\right]$$

$$= -\log(1 - |z|^2)|\varphi'(z_0)| + \log r(\varphi(z_0); \Omega) \geqslant 0.$$

The inequality (3.1) is proved.

If the sign of equality in (3.1) holds for some z_0, then by the minimum principle ψ vanishes identically in $\mathbb{D}$, i.e.

$$g(\varphi(z), \varphi(z_0); \Omega) \equiv \log|1 - z\bar{z}_0|\,|z - z_0|^{-1}.$$

Hence the univalence of φ and simple connectedness of Ω immediately follow.

This lemma may be linked together with Becker's univalence criterion and the symmetrization principle in order to obtain some new criteria of univalence. As a matter of example we shall prove

THEOREM 3 *Let Ω be a domain in the finite plane intersecting each line $v = const.$ in the $w = u + iv$ plane in a set of intervals of total length at most $\pi/2$. Then Ω is a U-domain.*

Proof Let Ω satisfy the assumptions stated above and let $f \in \mathscr{H}$ be such that $\Omega_f \subset \Omega$. Suppose that $w_0 = \log f'(z_0)$. We may assume without loss in generality that w_0 is purely imaginary since this may be achieved by a simultaneous horizontal translation of both Ω and Ω_f. After Steiner symmetrization of Ω w.r.t. the imaginary axis we obtain a domain Ω^* contained in the strip $\Omega_0 = \{w : |\mathrm{Re}\, w| < \pi/4\}$. On applying the lemma with $\varphi = \log f'$ and the symmetrization principle (cf. [1], p. 84) we obtain

$$(1 - |z_0|^2)|f''(z_0)/f'(z_0)| \leqslant r(w_0\,; \Omega) \leqslant r(w_0\,; \Omega^*) \leqslant r(w_0\,; \Omega_0) = 1$$

and consequently, f is univalent by Becker's criterion. This ends the proof.

Corollary 2 *If the inequality $\beta > 1$ were true, the Theorem 3 would remain true under the assumption that the total length of intervals on intersection of Ω with horizontal lines does not surpass $\beta\pi/2$. Moreover, we would be able to improve in this case the lower estimate of John's constant as follows*: $\exp(\beta\pi/2) \leqslant \gamma$.

It is an interesting open question, whether the assumption on Ω_f that all the lines $u = \text{const.}$ intersect Ω_f in a set of intervals of total length at most π implies univalence of f. If true, this statement would imply the univalence of $\int_0^z g(u)\,du$, where g is a Gelfer function, i.e. a function regular in $\mathbb{D}$ and such that $g(z_1) + g(z_2) \neq 0$ for any $z_1, z_2 \in \mathbb{D}$.

We shall now prove another generalization of John's criterion. To this end we need following

Lemma 2 *Let Q be the rectangle*: $|\mathrm{Re}\, w| < a$, $|\mathrm{Im}\, w| < b$ *such that* $r(0; Q) = 1$. *Then for the unique $k \in (0; 1)$ satisfying $K'(k)/K(k) = a/b$ we have*

$$a = \tfrac{1}{2}K(k), \qquad b = \tfrac{1}{2}K'(k) = \tfrac{1}{2}K(k'), \qquad k' = \sqrt{1 - k^2} \tag{3.3}$$

where

$$K(k) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \ldots\right] \tag{3.4}$$

is the complete Legendre elliptic integral. Moreover, for any $w \in Q$:

$$r(w;\, Q) \leqslant 1 \tag{3.5}$$

Proof The elliptic function $z = sn(w,k)$, $0 < k < 1$, maps the rectangle $Q_1 : |\mathrm{Re}\, w| < K(k)$, $|\mathrm{Im}\, w| < K'(k)$ onto the z-plane slit along the half-lines $(-\infty; -1]$, $[1; +\infty)$, the latter domain having the inner radius 2 at the origin. Since the inner radius at w_0 is invariant under conformal mappings g satisfying $|g'(w_0)| = 1$, we see that $r(0; Q_1) = 2$. The similarity transformation $w \mapsto \frac{1}{2}w$ yields the rectangle

$$Q' : |\mathrm{Re}\, w| < \tfrac{1}{2}K(k), \qquad |\mathrm{Im}\, w| < \tfrac{1}{2}K'(k) \qquad \text{with} \quad r(0; Q') = 1.$$

If k is chosen so that $K'(k)/K(k) = b/a$, the similar rectangles Q, Q' are necessarily congruent due to the equality $r(0; Q) = r(0; Q') = 1$. This proves (3.3). The inequality (3.5) is easily obtained by Steiner symmetrization.

We may now state

THEOREM 4 *Suppose Ω is a domain in the finite w-plane, $w = u + iv$, which is contained in a horizontal strip of width $K'(k)$ and intersects any line $v = \text{const.}$ in a set of intervals of total length at most $K(k)$. Then Ω is a U-domain.*

Proof Suppose that $f \in \mathscr{H}$ is such that $\Omega_f \subset \Omega$. In view of Lemma 1 and Becker's criterion it is sufficient to show that $r(w; \Omega) \leqslant 1$ for all $w \in \Omega$. As before we may assume that $\mathrm{Re}\, w = 0$. After Steiner symmetrization w.r.t. the imaginary axis we obtain a domain Ω^* contained in a rectangle Q of Lemma 2. Thus $r(w; \Omega) \leqslant r(w; \Omega^*) \leqslant r(w; Q) \leqslant 1$ in view of (3.5). This ends the proof.

Obviously the rôle of u and v in Theorem 4 may be interchanged. In particular we have for $k_0 = 1/\sqrt{2}$: $a_0 = K(k_0) = K'(k_0) = \frac{1}{4}\pi^{-1/2}\Gamma^2(1/4) = 1.85406\ldots$ (cf. e.g. [4], Problem 4.7.12). This implies Corollary 3. If the set Ω_f is contained in the vertical strip of width a_0 and intersects each line $u = \text{const.}$ in a set of intervals of total length at most a_0, then f is univalent.

The following problem is suggested by Theorem 4: Let Ω be a domain in the finite plane which intersects every horizontal (and every vertical) line in a set of segments of total length at most $K(k)$ (or $K'(k)$, resp.). Is it true that $r(w; \Omega) \leqslant 1$ for any $w \in \Omega$?

The Schwarz symmetrization leads to the following, well-known result: Of all domains Ω having a given area $|\Omega|$ and containing a given point w the disk with center w has the maximum inner radius

$r(w; \Omega)$, cf. [6], p. 192. As a corollary we obtain Theorem 5. If Ω is a domain with area $|\Omega| \leqslant \pi$, then Ω is a U-domain.

4. FINAL REMARKS

Professor Becker called my attention to a paper by C. D. Minda and D. J. Wright recently published in The Rocky Mountain Journal of Mathematics which also dealt with restrictions on Ω_f implying univalence of f. In particular they consider the case $\Omega_f \subset Q$, where Q is a rectangle with sides parallel to the coordinate axes. This turns out to be a special case of Theorem 4.

In connection with Theorem 2, Professor Pommerenke remarked (private communication) that the estimate (2.7) of Robertson's constant can be improved as follows

$$\rho \leqslant 1.765 \tag{4.1}$$

This results from the fact that the function (2.8) with $A = -1.765$ is not typically real (and hence not univalent) in $\mathbb{D}$ for all $n \in \mathbb{N}$.

References

[1] Hayman, W. K., *Multivalent Functions*, Cambridge University Press, Cambridge, 1958.

[2] Heins, M., *Selected Topics in the Classical Theory of Functions of a Complex Variable*, Holt, Rinehart and Winston, New York, 1962.

[3] John, F., A criterion for univalency brought up to date, *Comm. Pure Appl. Math.* **29** (1976), 293–295.

[4] Krzyż, J. G., *Problems in Complex Variable Theory*, American Elsevier Publ. Co., New York, 1971.

[5] Pommerenke, Ch., *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.

[6] Pólya, G. and Szegö, G., *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, Princeton, 1951.

[7] Yamashita, S., On the John constant, *Math. Zeitschr.* **161** (1978), 185–188.