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Harmonic automorphisms of the unit disk ☆

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Dedicated to Professor Haakon Waadeland on the occasion of his 70th birthday

Abstract

Let $\mathscr{H}$ be the class of harmonic automorphisms of the unit disk $\mathbb{D}$. The function $F=h-g$ associated with $f=h+\bar{g}\in\mathscr{H}$ maps $\mathbb{D}$ conformally onto a horizontally convex domain Ω. Conversely, given Ω both $f\in\mathscr{H}$ and F with $F(\mathbb{D})=\Omega$ can be retrieved (Theorem 1). Compact subclasses $\mathscr{H}(M)\subset\mathscr{H}$ consisting of Poisson extensions of M-quasisymmetric automorphisms of $\partial\mathbb{D}$ span $\mathscr{H}$ (Lemma 1). For $f(re^{it})=\sum_{n=-\infty}^{+\infty}c_n r^{|n|}e^{int}\in\mathscr{H}(M)$ the bounds of $|c_n|$ (upper one for $n=0,2$, lower one for $n=1$) and $\sum_{n=-\infty}^{+\infty}|c_n|$ are given (Theorems 2–4). © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction. Statement of results

Let G be an orientable manifold. Then $\operatorname{Aut} G$ will stand for the class of homeomorphic sense-preserving self-mappings of G. The main object of this paper is the class

$$\mathscr{H}=\{f\in\operatorname{Aut}\mathbb{D}:\ \ f_{z\bar{z}}=0\},\tag{1.1}$$

i.e. the class of harmonic, univalent and sense-preserving self-mappings f of the unit disk $\mathbb{D}$.

Given a bounded convex domain Ω in the finite plane $\mathbb{C}$, let γ denote a homeomorphic sense-preserving map of $\mathbb{T}=\partial\mathbb{D}$ onto $\Gamma=\partial\Omega$. Then, according to the well-known Radó–Kneser–Choquet

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(RKC) Theorem, the Poisson extension $P[\gamma]$ of γ to the unit disk, i.e.

$$P[\gamma](z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}\gamma(e^{it})\,dt,\quad z\in\mathbb{D}, \tag{1.2}$$

is a univalent, sense-preserving harmonic mapping of $\mathbb{D}$ onto Ω, cf. e.g. [2]. In particular, for $\Omega=\mathbb{D}$ and $\gamma\in\operatorname{Aut}\mathbb{T}$ we have $P[\gamma]\in\mathscr{H}$.

A continuous mapping γ of $\mathbb{T}$ onto Γ is said to be a *weak homeomorphism* if for any $\zeta\in\Gamma$ its inverse image $\gamma^{-1}(\zeta)$ is either a point or a closed subarc of $\mathbb{T}$. Then we write $\gamma\in\operatorname{Hom}^*(\mathbb{T},\Gamma)$. The RKC Theorem remains true for $\gamma\in\operatorname{Hom}^*(\mathbb{T},\Gamma)$, cf. [4].

If Ω is a bounded strictly convex domain and f is a univalent harmonic mapping of $\mathbb{D}$ onto Ω then f has a continuous extension to $\overline{\mathbb{D}}$ whose restriction to $\mathbb{T}$ is a weak homeomorphism of $\mathbb{T}$ onto $\Gamma=\partial\Omega$, cf. [4, p. 156]. In case $\Omega=\mathbb{D}$ we obtain the following statement:

$$\mathscr{H}=\{P[\gamma]\colon \gamma\in\operatorname{Hom}^*\mathbb{T}\}, \tag{1.3}$$

where $\operatorname{Hom}^*\mathbb{T}=\operatorname{Hom}^*(\mathbb{T},\mathbb{T})$.

If $\gamma\in\operatorname{Hom}^*\mathbb{T}$ then $\gamma(e^{it})=\exp i\varphi(t)$, where φ is a so-called *circle mapping*, cf. [4, 5]. After setting

$$\varphi(t+2\pi)=\varphi(t)+2\pi \tag{1.4}$$

φ becomes a continuous, real-valued and nondecreasing function on $\mathbb{R}$.

Since γ is a continuous function of bounded variation, its Fourier series is convergent and

$$\gamma(e^{it})=\sum_{n=-\infty}^{+\infty}c_n e^{int}. \tag{1.5}$$

Moreover, $f=P[\gamma]$ has the representation

$$f(re^{it})=\sum_{n=-\infty}^{+\infty}c_n e^{int}r^{|n|}. \tag{1.6}$$

The classes $\mathscr{H}$ and $\operatorname{Hom}^*\mathbb{T}$ are not compact. Consequently, some extremal problems for $\mathscr{H}$ may have no solution in $\mathscr{H}$ which is due to the effect to "collapsing". If the extremal mapping $\gamma=\exp i\varphi(t)$ has a discontinuity point then $P[\gamma](\mathbb{D})$ omits the convex hull of some subarc of $\mathbb{T}$ and consequently $P[\gamma]\notin\mathscr{H}$.

As shown in [4], for any $f\in\mathscr{H}$ satisfying (1.6) we have

$$|c_{-m}|\leqslant\frac{m+1}{m\pi}\sin\frac{\pi}{m+1},\quad m\in\mathbb{N}. \tag{1.7}$$

The bound is sharp and for $m\geqslant 2$ the extremal function maps $\mathbb{D}$ univalently onto the inside of a regular $(m+1)$-gon inscribed in $\mathbb{T}$. In this case the circle mapping has $m+1$ equidistributed discontinuity points with jumps $2\pi/(m+1)$.

We may eliminate some drawbacks of $\mathscr{H}$ (noncompactness, effect of collapsing) by introducing one-parameter families $\mathscr{H}(M)\subset\mathscr{H}$ which in some sense span the whole class $\mathscr{H}$. Moreover, for M near 1 the functions $f\in\mathscr{H}(M)$ are close to $z\to e^{i\alpha}z$, whereas for large M, the elements of $\mathscr{H}(M)$ approximate arbitrarily chosen elements of $\mathscr{H}$.

We start with the following.

Definition. A mapping $\gamma\in\operatorname{Aut}\mathbb{T}$ is said to be M-quasisymmetric (M-qs) iff $\gamma(e^{it})=\exp i\varphi(t)$ where the circle mapping φ extended to $\mathbb{R}$ by (1.4) is M-qs on $\mathbb{R}$ in the sense of Beurling–Ahlfors, cf. [1, 6]. Moreover, the class $\mathscr{H}(M)$ is defined as $\{P[\gamma]\colon \gamma\in \mathrm{QS}(M)\}$, where $\mathrm{QS}(M)$ denotes the collection of all M-qs $\gamma\in\operatorname{Aut}\mathbb{T}$.

Obviously, $\mathrm{QS}(M)$ is a compact subclass of $\mathrm{Hom}^*\,\mathbb{T}$ for any $M\geqslant 1$.

While trying to evaluate the maximal dilatation of the Douady–Earle extension to $\mathbb{D}$ of $\gamma\in\mathrm{QS}(M)$, Partyka [11] obtained as by-products the following estimates for $f\in\mathrm{QS}(M)$:

$$|c_0|\leqslant\cos\frac{\pi}{M+1},\qquad |c_{-1}|\leqslant\cos\left(\frac{\pi}{4}+\frac{\pi}{(M+1)^2}\right). \tag{1.8}$$

Note that $|c_{-1}|<\sqrt{2}/2=0.7071\ldots$ for any $f\in\mathscr{H}(M)$, $M\geqslant 1$, while the sharp bound is $2/\pi=0.6366\ldots$, cf. (1.7).

We now prove that the classes $\mathrm{QS}(M)$ span in some sense the class $\mathrm{Hom}^*\,\mathbb{T}$. We have the following.

Lemma 1. *A necessary and sufficient condition for $\gamma\in\operatorname{Aut}\mathbb{T}$ to be a weak homeomorphism of $\mathbb{T}$ is the existence of a sequence $\{\gamma_n\}$, $\gamma_n\in\mathrm{QS}(M_n)$, which converges to γ uniformly on $\mathbb{T}$.*

Proof. Sufficiency is almost obvious. Let φ and φ_n stand for the circle mappings extended to $\mathbb{R}$ and corresponding to γ and γ_n, resp. Any φ_n is continuous and strictly increases by 2π on $[0,2\pi]$. Since φ_n tends to φ uniformly, φ must be continuous, nondecreasing and increases by 2π on $[0,2\pi]$ which means that $\gamma=\exp i\varphi\in\mathrm{Hom}^*\,\mathbb{T}$. Suppose now $\gamma\in\mathrm{Hom}^*\,\mathbb{T}$ and φ is the corresponding circle mapping extended to $\mathbb{R}$. We may assume $\varphi(0)=0$. Since φ is continuous, nondecreasing and $\varphi(2\pi)=2\pi$, the lines $\operatorname{Im} w=2\pi k/n$, $n\geqslant 3$, $1\leqslant k\leqslant n-1$, intersect the graph of φ either at a single point $w_k=t_k+i\varphi(t_k)$, or along a segment of constancy of φ situated over the interval $[t_k',t_k'']$. In the latter case put $t_k=t_k''$ and assume again w_k as $t_k+i\varphi(t_k)$. We now define φ_n as a strictly increasing function whose graph over $[0,2\pi]$ is the polygonal line with vertices $0,w_1,w_2,\ldots,w_n,2\pi(1+i)$ and extend φ_n on $\mathbb{R}$ by setting $\varphi_n(t\pm 2\pi)=\varphi_n(t)\pm 2\pi$. Obviously $|\varphi(t)-\varphi_n(t)|\leqslant 2\pi/n$ and the slope of the kth segment is equal to $2\pi/[n(t_k-t_{k-1})]$ which implies φ_n to be M_n-qs for some $M_n\geqslant 1$, cf. [6]. Therefore $\gamma_n(e^{it})=\exp i\varphi_n(t)\in\mathrm{QS}(M_n)$. Moreover,

$$|\gamma_n-\gamma|=|\exp i\varphi(t)-\exp i\varphi_n(t)|=2\sin\frac{|\varphi-\varphi_n|}{2}\leqslant 2\sin\frac{\pi}{n}.$$

Hence γ_n converges to γ uniformly on $\mathbb{T}$ and this ends the proof. □

According to the familiar result of Clunie and Sheil-Small [3], for any $f\in\mathscr{H}$ with the decomposition $f=h+\bar{g}$, $(h,g\in\mathscr{A}(\mathbb{D}))$, $F=h-g$ maps $\mathbb{D}$ conformally onto a horizontally convex domain Ω. Theorem 1 in the next section shows how from a given Ω the functions f and F can be recovered.

In Section 3 we slightly improve the first estimate in (1.8) and obtain the inequality

$$\max\{|c_0|,|c_2|\}\leqslant\cos\frac{\pi}{M+1}. \tag{1.9}$$

Moreover, for $f\in\mathscr{H}(M)$ we have

$$|c_1|\geqslant\left[1-\frac{\pi^2}{16}\left(\frac{M-1}{M+1}\right)^2\right]\sin\frac{\pi}{M+1}. \tag{1.10}$$

In proving (1.9) and (1.10) various norm estimates of $\sigma(t)=\varphi(t)-t$, as given in [7,10], were used. The condition $\gamma\in\mathrm{QS}(M)$ imposes strong restrictions on the Fourier series of γ. In particular, (1.5) is absolutely convergent and its sum has an estimate $1+\mathrm{O}(\sqrt{M-1})$ as $M\to 1^+$, or $\mathrm{O}(M)$ as $M\to+\infty$. This will be proved in Section 4.

The coefficient estimates (1.7) for $f\in\mathscr{H}$ and negative indices as given in [4], as well as estimates for positive indices in [5], coincide with estimates obtained some 20 years earlier by Kühnau [9] for coefficients of Laurent series of certain functions F holomorphic and univalent in $\{z\colon 1<|z|<R\}$, where R may depend on F. However, it is easily verified that both coefficient problems are equivalent to the Fourier coefficient problem of $\gamma\in\mathrm{Aut}\,\mathbb{T}$ represented by Fourier series (1.5).

2. Horizontally convex domains and the class $\mathscr{H}$

A domain Ω is said to be convex in the direction of the real axis (or horizontally convex) if every horizontal line intersects Ω in an interval or not at all. If $f\in\mathscr{H}$ has the decomposition $f=h+\bar{g}$, $(h,g\in\mathscr{A}(\mathbb{D}))$, then $F=h-g$ maps conformally $\mathbb{D}$ onto a horizontally convex domain Ω, cf. [3]. A natural question arises whether a kind of converse statement might be formulated. Since horizontally convex domains may be fairly irregular, some further geometrical conditions must be imposed upon Ω. We have in the context the following.

Theorem 1. *Suppose Ω is a bounded horizontally convex domain supported by the lines $\{\mathrm{Im}\,w=-1\}$, $\{\mathrm{Im}\,w=1\}$. If $\partial\Omega$ is locally connected and G maps the unit disk $\mathbb{D}$ conformally onto Ω then there exist a univalent harmonic self-mapping f of $\mathbb{D}$ and a decomposition $f=h+\bar{g}$ with $g,h\in\mathscr{A}(\mathbb{D})$ such that $G=h-g$.*

Proof. By our assumptions G has a continuous extension to the closure $\overline{\mathbb{D}}$ and the boundary $\Gamma=\partial\Omega$ is a curve admitting the parametrization $w=G(\mathrm{e}^{\mathrm{i}t})$, cf. [12, pp. 20, 21]. We may split $\mathbb{T}$ into four arcs I_k so that the image arcs $G(I_1)$, $G(I_3)$ are sets of support points on the lines $\{\mathrm{Im}\,w=-1\}$, $\{\mathrm{Im}\,w=1\}$, whereas the image acrs $G(I_2)=\Gamma_1$, $G(I_4)=\Gamma_2$ join these support lines. Since G is bounded and horizontally convex, the function $t=\mathrm{Im}\,G(\mathrm{e}^{\mathrm{i}\theta})$ is monotonic and continuous on I_2 and I_4. We may now define a mapping $\Phi:\Gamma\to\mathbb{T}$ by projecting horizontally Γ_1 onto the right-hand side semicircle of $\mathbb{T}$ and Γ_2 onto the left-hand side semicircle; i.e. if $u+\mathrm{i}t\in\Gamma_1$ then $\Phi[u+\mathrm{i}t]=\sqrt{1-t^2}+\mathrm{i}t$, if $u+\mathrm{i}t\in\Gamma_2$ then $\Phi[u+\mathrm{i}t]=-\sqrt{1-t^2}+\mathrm{i}t$, $-1<t<1$. Moreover, $\Phi\circ G(I_1)=-\mathrm{i}$, $\Phi\circ G(I_3)=\mathrm{i}$.

Let $x(u,v)$ be a solution of the Dirichlet problem for Ω and boundary values $\sqrt{1-v^2}$ on Γ_1, $-\sqrt{1-v^2}$ on Γ_2 and 0 on $G(I_1)$, $G(I_3)$. The boundary segments of Γ may be either free boundary arcs or slits. In both cases they are images of suitable subarcs of $\mathbb{T}$ and it is easily verified that $\Phi\circ G\in\mathrm{Hom}^*\,\mathbb{T}$. This implies $f=P[\Phi\circ G]\in\mathscr{H}$. The composition $f\circ G^{-1}$ results in a univalent harmonic map of Ω onto $\mathbb{D}$ with boundary values Φ and $f\circ G^{-1}$ may be also denoted by Φ. It is easily verified that

$$\Phi(u,v)=x(u,v)+\mathrm{i}v,\quad u+\mathrm{i}v\in\Omega, \tag{2.1}$$

and consequently

$$f\circ G^{-1}(u,v)=\Phi(u,v),\quad u+\mathrm{i}v\in\Omega. \tag{2.2}$$

Set $\alpha_t=(-\sqrt{1-t^2}+\mathrm{i}t,\sqrt{1-t^2}+\mathrm{i}t)$, $-1<t<1$, and $\gamma_t=f^{-1}(\alpha_t)$, or $f(\gamma_t)=\alpha_t$. It follows from (2.1) that an open segment $\beta_t=\{w\in\Omega\colon\ \operatorname{Im} w=t\}$ is mapped on α_t under Φ. Hence $\beta_t=\Phi^{-1}(\alpha_t)=G\circ f^{-1}(\alpha_t)=G(\gamma_t)$ and consequently $\operatorname{Im}\{f(\zeta)\colon\ \zeta\in\gamma_t\}=t=\operatorname{Im}\{G(\zeta)\colon\ \zeta\in\gamma_t\}$. Since the arcs γ_t sweep out the disk $\mathbb{D}$ as t ranges over $(-1,1)$, we have $\operatorname{Im} f=\operatorname{Im} G$ on $\mathbb{D}$. Suppose now f has a decomposition $f=h+\bar g$ with $h,g\in\mathscr{A}(\mathbb{D})$. Then $F=h-g$ maps $\mathbb{D}$ conformally onto a domain horizontally convex. Since $\operatorname{Im} G=\operatorname{Im} f=\operatorname{Im}(h+\bar g)=\operatorname{Im}(h-g)=\operatorname{Im} F$, $G-F$ is equal to a real constant. We have $f=h+\bar g=h+c+\bar g-c=h_1+\bar g_1$ with $c\in\mathbb{R}$. The function $F_1=h_1-g_1$ corresponding to the new decomposition of f takes the form $h+c-(g-c)=F+2c$. A suitable choice of c yields the identity $G=F_1$ and this ends the proof. □

3. Some coefficient estimates in QS(M)

Let $E_0(M,a)$, $a>0$, denote the class of all real-valued and a-periodic functions $\sigma(t)$ defined on $\mathbb{R}$ such that $\varphi(t)=t+\sigma(t)$ is M-qs on $\mathbb{R}$ and $\int_0^a\sigma(t)\,\mathrm{d}t=0$. It is easy to see that

$$\tilde\sigma(x)\in E_0(M,1)\Leftrightarrow\sigma(t)=a\tilde\sigma(t/a)\in E_0(M,a). \tag{3.1}$$

In what follows we will need some estimates in case $a=\pi$ and 2π which may be readily derived from corresponding estimates for $a=1$ and the following.

Lemma 2 (Nowak [10]; Lemma 2.1). *If $\tilde\sigma\in E_0(M,1)$ then the following estimates hold*:

$$\sup\{|\tilde\sigma(x)|\colon x\in\mathbb{R}\}\leqslant\frac{1}{2}\frac{M-1}{M+1}, \tag{3.2}$$

$$\sup\left\{\left|\tilde\sigma\left(x+\frac{1}{2}\right)-\tilde\sigma(x)\right|\colon x\in\mathbb{R}\right\}\leqslant\frac{1}{2}\frac{M-1}{M+1}; \tag{3.3}$$

$$\int_0^1|\tilde\sigma(x)|^2\mathrm{d}x\leqslant\frac{1}{8}\left(\frac{M-1}{M+1}\right)^2. \tag{3.4}$$

We now prove

Theorem 2. *If $\gamma\in\mathrm{QS}(M)$ and*

$$\gamma(\mathrm{e}^{\mathrm{i}t})=\exp\mathrm{i}[t+\sigma(t)]=\sum_{n=-\infty}^{\infty}c_n\mathrm{e}^{\mathrm{i}nt}, \tag{3.5}$$

then

$$\max\{|c_0|,|c_2|\}\leqslant\cos\frac{\pi}{M+1}. \tag{3.6}$$

Proof. We have

$$|c_2| = \frac{1}{2\pi}\left|\int_0^{2\pi} \exp\mathrm{i}(\sigma(t)-t)\,\mathrm{d}t\right| \leqslant \frac{1}{2\pi}\int_0^{\pi} |\exp\mathrm{i}[\sigma(t)-t] - \exp\mathrm{i}[\sigma(t+\pi)-t]|\,\mathrm{d}t$$

$$= \frac{1}{2\pi}\int_0^{\pi} |1-\exp\mathrm{i}[\sigma(t+\pi)-\sigma(t)]|\,\mathrm{d}t = \frac{1}{\pi}\int_0^{\pi} \sin\frac{1}{2}|\sigma(t+\pi)-\sigma(t)|\,\mathrm{d}t.$$

Note that

$$|\sigma(t+\pi)-\sigma(t)| \leqslant \pi\frac{M-1}{M+1} \quad \text{for any } t\in\mathbb{R} \tag{3.7}$$

by (3.3) and (3.1) with $a=2\pi$. Since sin is concave in $[0,\pi]$, we obtain by Jensen's inequality (cf. e.g. [13, p. 63]) and (3.7)

$$\int_0^{\pi} \sin\frac{1}{2}\,|\sigma(t+\pi)-\sigma(t)|\,\mathrm{d}t/\pi \leqslant \sin\left(\int_0^{\pi}\frac{1}{2}\,|\sigma(t+\pi)-\sigma(t)|\,\mathrm{d}t/\pi\right)$$

$$\leqslant \sin\frac{\pi}{2}\frac{M-1}{M+1} = \cos\frac{\pi}{M+1}.$$

Thus

$$|c_2| \leqslant \cos\frac{\pi}{M+1}. \tag{3.8}$$

Moreover,

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} \exp\mathrm{i}[t+\sigma(t)]\,\mathrm{d}t$$

and hence

$$|c_0| \leqslant \frac{1}{2\pi}\int_0^{\pi} |\exp\mathrm{i}[\sigma(t)+t] - \exp\mathrm{i}[\sigma(t+\pi)+t]|\,\mathrm{d}t$$

$$= \frac{1}{2\pi}\int_0^{\pi} |1-\exp\mathrm{i}[\sigma(t+\pi)-\sigma(t)]|\,\mathrm{d}t.$$

However, the last expression was already estimated by $\cos\pi/(M+1)$ and this ends the proof. □

The same estimate of $|c_0|$ was also obtained by a different method in [11].

It is well known that for any $f\in\mathscr{H}$ the coefficient c_1 in (1.6) never vanishes but it can be arbitrarily close to 0, consider e.g. $f_r(z)=(z+r)(1+rz)^{-1}=r+(1-r^2)z+\cdots$, $0\leqslant r<1$. We now prove that c_1 is uniformly bounded away from 0 in any $\mathscr{H}(M)$. We have

Theorem 3. *If* $\gamma\in\mathrm{QS}(M)$ *and* (3.5) *holds then*

$$|c_1| \geqslant \left[1-\frac{\pi^2}{16}\left(\frac{M-1}{M+1}\right)^2\right]\sin\frac{\pi}{M+1}. \tag{3.9}$$

Proof. We have $\gamma(e^{it}) = \exp i(t + \sigma(t))$ and hence $c_1 = 1/2\pi \int_0^{2\pi} \exp i\sigma(t)\,dt$. Since the rotations: $t \to t + t_0, \varphi(t) \to \varphi(t) + \alpha$, do not change $|c_1|$, we may assume, after a suitable choice of t_0 and α, that

$$\int_0^{2\pi} \sigma(t)\,dt = 0, \quad \sigma(0) = \sigma(2\pi) = 0. \tag{3.10}$$

Then

$$|c_1| \geqslant \operatorname{Re} c_1 = \frac{1}{2\pi}\int_0^{2\pi} \cos\sigma(t)\,dt = \frac{1}{2\pi}\int_0^{\pi} [\cos\sigma(t) + \cos\sigma(t+\pi)]\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos\frac{1}{2}[\sigma(t) + \sigma(t+\pi)]\cos\frac{1}{2}[\sigma(t+\pi) - \sigma(t)]\,dt. \tag{3.11}$$

Using the inequality (3.7) we see that $\frac{1}{2}|\sigma(t+\pi) - \sigma(t)| \leqslant (\pi/2)(M-1)/(M+1)$ for any $t \in \mathbb{R}$ and hence

$$\cos\frac{1}{2}|\sigma(t+\pi) - \sigma(t)| \geqslant \cos\frac{\pi}{2}\frac{M-1}{M+1} > 0, \quad t \in \mathbb{R}. \tag{3.12}$$

Consider now $\sigma_1(t) = \frac{1}{2}[\sigma(t) + \sigma(t+\pi)]$. Obviously $t + \sigma_1(t)$ is M-qs. We have by (3.10) $0 = \int_0^{\pi} \sigma(t)\,dt = \int_0^{\pi} 2\sigma_1(t)\,dt$ which implies $\int_0^{\pi} \sigma_1(t)\,dt = 0$. Moreover, σ_1 is π-periodic and hence $\sigma_1 \in E_0(M, \pi)$, or $\sigma_1(\pi\tau)/\pi \in E_0(M, 1)$. By (3.2) $\sup\{|\sigma_1(t)|: t \in \mathbb{R}\} \leqslant (\pi/2)(M-1)/(M+1)$ and consequently

$$\cos\sigma_1(t) \geqslant \cos\frac{\pi}{2}\frac{M-1}{M+1} > 0. \tag{3.13}$$

Moreover, by (3.4) $\int_0^1 |\sigma_1(\pi\tau)/\pi|^2\,d\tau \leqslant \frac{1}{8}((M-1)/(M+1))^2$ and the change of variable $\pi\tau = t$ yields

$$\int_0^{\pi} |\sigma_1(t)|^2\,dt \leqslant \frac{\pi^3}{8}\left(\frac{M-1}{M+1}\right)^2.$$

Hence, because of $\cos x \geqslant 1 - \frac{1}{2}x^2$, we obtain

$$\int_0^{\pi} \cos\sigma_1(t)\,dt \geqslant \pi - \frac{\pi^3}{16}\left(\frac{M-1}{M+1}\right)^2. \tag{3.14}$$

Now, taking (3.12)–(3.14) into account, (3.11) may be written

$$|c_1| \geqslant \frac{1}{\pi}\cos\frac{\pi}{2}\frac{M-1}{M+1}\int_0^{\pi}\cos\sigma_1(t)\,dt \geqslant \left[1 - \frac{\pi^2}{16}\left(\frac{M-1}{M+1}\right)^2\right]\cos\frac{\pi}{2}\frac{M-1}{M+1}.$$

Since $\cos(\pi/2)(M-1)/(M+1) = \sin\pi/(M+1)$, (3.9) follows and this ends the proof. □

4. Absolute convergence of $\sum c_n$

If $\gamma(e^{it}) \in \mathrm{QS}(M)$ then the corresponding circle mapping $\varphi(t)$ is M-qs on the real line. Since $\sigma(t) = \varphi(t) - t$ is continuous, 2π-periodic and of bounded variation on $[0, 2\pi]$, it is represented by its Fourier series whose absolute convergence was established in [7,10]. A natural problem arises: does the exponentiation of σ preserve the absolute convergence of its Fourier series? The affirmative answer was given in [8], however an improved estimate is contained in the following.

Theorem 4. *If* $\gamma \in \mathrm{QS}(M)$ *and*

$$\gamma(\mathrm{e}^{\mathrm{i}t}) = \sum_{n=-\infty}^{+\infty} c_n \mathrm{e}^{\mathrm{i}nt} \tag{4.1}$$

then

$$\sum_{n=-\infty}^{+\infty} |c_n| \leqslant 1 + 2\cos\frac{\pi}{M+1} + 2\pi\sqrt{2}\sum_{n=2}^{+\infty}\left[\left(\frac{M}{M+1}\right)^n - \frac{1}{2^n}\right]^{1/2} = \rho(M). \tag{4.2}$$

We have: $\rho(M) = 1 + \mathrm{O}(\sqrt{M-1})$ *as* $M \to 1$ *and* $\rho(M) = \mathrm{O}(M)$ *as* $M \to +\infty$.

Proof. If $\gamma(\mathrm{e}^{\mathrm{i}t}) = \exp \mathrm{i}\varphi(t) = \exp \mathrm{i}\sigma(t)\mathrm{e}^{\mathrm{i}t}$ then

$$\exp \mathrm{i}\sigma(t) = \sum_{n=-\infty}^{+\infty} c_{n+1}\mathrm{e}^{\mathrm{i}nt} = \sum_{n=-\infty}^{+\infty} d_n \mathrm{e}^{\mathrm{i}nt}, \quad d_n = c_{n+1}. \tag{4.3}$$

Put $u(t)=\cos\sigma(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\rho_n\sin(nt+t_n)$, where $\rho_n=\sqrt{a_n^2+b_n^2}$. Similarly, put $v(t)=\sin\sigma(t)=\frac{1}{2}a_0'+\sum_{n=1}^{\infty}(a_n'\cos nt+b_n'\sin nt)=\frac{1}{2}a_0'+\sum_{n=1}^{\infty}\rho_n'\sin(nt+t_n')$, where $\rho_n'=\sqrt{(a_n')^2+(b_n')^2}$. Then we have

$$\exp \mathrm{i}\sigma(t) = \cos\sigma(t) + \mathrm{i}\sin\sigma(t) = \frac{1}{2}(a_0 + a_0')$$
$$+\frac{1}{2}\sum_{n=1}^{\infty}(a_n + b_n' + \mathrm{i}(a_n - b_n'))\mathrm{e}^{\mathrm{i}nt} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - b_n' + \mathrm{i}(a_n' + n_n'))\mathrm{e}^{-\mathrm{i}nt}. \tag{4.4}$$

From (4.3) and (4.4) we obtain

$$|d_n|^2 = \tfrac{1}{4}[(a_n + b_n')^2 + (a_n' - b_n)^2], \qquad |d_{-n}|^2 = \tfrac{1}{4}[(a_n - b_n')^2 + (a_n' + b_n)^2]$$

and hence

$$|d_n|^2 + |d_{-n}|^2 = \tfrac{1}{2}(\rho_n^2 + (\rho_n')^2). \tag{4.5}$$

In [10] the following result was established (cf. Theorem 3.1):
If $\sigma \in E_0(M, 2\pi)$ and $\sigma(t) = c_0 + \sum_{n=1}^{\infty} r_n \sin(t + t_n)$, $r_n \geqslant 0$, then

$$\sum_{n=2}^{\infty} r_n \leqslant \pi\sqrt{2}\sum_{n=2}^{\infty}\left[\left(\frac{M}{M+1}\right)^n - \frac{1}{2^n}\right]^{1/2}. \tag{4.6}$$

The starting point in proving (4.6) was the identity

$$\int_0^{2\pi}[\sigma(t + k\pi/2^n) - \sigma(t + (k-1)\pi/2^n)]^2\,\mathrm{d}t = 4\pi\sum_{k=1}^{\infty} r_k^2\sin^2(k\pi/2^{n+1}), \quad k, n \in \mathbb{N}, \tag{4.7}$$

cf. [14, p. 241] which holds for continuous, 2π-periodic σ represented by its Fourier series. Then using the inequalities

$$|\sigma(x+\pi/2^n)-\sigma(x)| \leqslant 2\pi\left[\left(\frac{M}{M+1}\right)^{n+1}-\frac{1}{2^{n+1}}\right], \quad n\in\mathbb{N},\ x\in\mathbb{R}, \tag{4.8}$$

$$V_0^{2\pi}[\sigma] \leqslant 4\pi, \tag{4.9}$$

and performing suitable calculations suggested by [14, p. 242] the inequality (4.6) could be obtained. Consider now $u(t)=\cos\sigma(t)$ and the identity $u(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\rho_n\sin(nt+t_n')$. Then (4.7) can be also written with u and ρ_k instead of σ and r_k, resp. Now, $|\cos x-\cos y| \leqslant |x-y|$; $x,y\in\mathbb{R}$, and hence $|u(x_1)-u(x_2)| \leqslant |\sigma(x_1)-\sigma(x_2)|$ for any $x_1,x_2\in\mathbb{R}$. Consequently, (4.8) and (4.9) also hold for u instead of σ and we deduce that (4.6) is valid for u, and also for $v(t)=\sin\sigma(t)$, i.e.

$$\max\left\{\sum_{n=2}^{\infty}\rho_n, \sum_{n=2}^{\infty}\rho_n'\right\} \leqslant \pi\sqrt{2}\sum_{n=2}^{\infty}\left[\left(\frac{M}{M+1}\right)^{n}-\frac{1}{2^{n}}\right]^{1/2}.$$

Now taking into account (4.5) we obtain

$$|d_n|+|d_{-n}| \leqslant \sqrt{2}\sqrt{|d_n|^2+|d_{-n}|^2}=\sqrt{\rho_n^2+\rho_n'^2} \leqslant \rho_n+\rho_n'$$

and hence

$$\sum_{n=2}^{\infty}(|d_n|+|d_{-n}|) \leqslant 2\pi\sqrt{2}\sum_{n=2}^{\infty}\left[\left(\frac{M}{M+1}\right)^{n}-\frac{1}{2^{n}}\right]^{1/2}. \tag{4.10}$$

Since $|d_0|=|c_1| \leqslant 1$ and $|d_1|+|d_{-1}| \leqslant 2\cos\pi/(M+1)$ by (3.6), inequality (4.2) follows.

We now estimate $\rho(M)$ as $M\to 1$ and $M\to+\infty$. For $0<b<a<1$ we have $a^n-b^n \leqslant n(a-b)a^{n-1}$ and hence

$$\sum_{n=2}^{\infty}(a^n-b^n)^{1/2} \leqslant \sqrt{a-b}\sum_{n=2}^{\infty}\sqrt{n+1}(\sqrt{a})^n<\sqrt{a-b}\sum_{n=2}^{\infty}(n+1)(\sqrt{a})^n<\sqrt{a-b}[(1-\sqrt{a})^{-2}-1].$$

Putting $a=M/(M+1)$, $b=\frac{1}{2}$ we obtain

$$\begin{aligned}\sum_{n=2}^{\infty}\left[\left(\frac{M}{M+1}\right)^{n}-\frac{1}{2^{n}}\right]^{1/2} &< \sqrt{M-1}\frac{1}{\sqrt{2}}\left[\frac{\sqrt{M+1}}{(\sqrt{M+1}-\sqrt{M})^2}-\frac{1}{\sqrt{M+1}}\right]\\ &= O(\sqrt{M-1}) \quad \text{as } M\to 1.\end{aligned} \tag{4.11}$$

Moreover,

$$\begin{aligned}\sum_{n=2}^{\infty}\left[\left(\frac{M}{M+1}\right)^{n}-\frac{1}{2^{n}}\right]^{1/2} &< \sqrt{\frac{M}{M+1}}\sum_{n=1}^{\infty}\left(\sqrt{\frac{M}{M+1}}\right)^n\\ &= \sqrt{\frac{M}{M+1}}(M+\sqrt{M(M+1)})=O(M) \quad \text{as } M\to\infty.\end{aligned} \tag{4.12}$$

The convergence of $\sum(|c_n|+|c_{-n}|)$ was already established in [8] and the following estimate was given there (Theorem 2.5):

$$\sum_{n=2}^{\infty}(|c_n|+|c_{-n}|)\leqslant 2\sqrt{2\pi}[M+\sqrt{M(M+1)}]. \tag{4.13}$$

Obviously the estimate (4.12) is better that (4.13) for all $M\geqslant 1$. Moreover, (4.11) describes the behaviour of the sum on the 1.h.s of (4.13) better than (4.12) for $M\to 1^+$.

References

[1] L.V. Ahlfors, Lectures on quasiconformal mappings, Van Nostrand, Princeton, 1966.
[2] D. Bshouty, W. Hengartner, Univalent harmonic mappings in the plane, Ann. Univ. Mariae Curie–Skłodowska, Sect. A 48 (1994) 1–42.
[3] J. Clunie, T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A.I.Math. 9 (1984) 3–25.
[4] P. Duren, G. Schober, A variational method for harmonic mappings onto convex regions, Complex Variables Theory Appl. 9 (1987) 153–168.
[5] P. Duren, G. Schober, Linear extremal problems for harmonic mappings of the disk, Proc. Amer. Math. Soc. 106 (1989) 967–973.
[6] J.G. Krzyż, Quasicircles and harmonic measure, Ann. Acad. Sci. Fenn. Ser. A.I.Math. 12 (1987) 19–24.
[7] J.G. Krzyż, Harmonic analysis and boundary correspondence under quasiconformal mappings, Ann. Acad. Sci. Fenn. Ser. A.I.Math. 14 (1989) 225–242.
[8] J.G. Krzyż, D. Partyka, Harmonic extensions of quasisymmetric mappings, Complex Variables Theory Appl. 33 (1997) 159–176.
[9] R. Kühnau, Schranken für die Koeffizienten gewisser schlicht abbildender Laurentscher Reihen, Math. Nachr. 41 (1969) 177–183.
[10] M. Nowak, Some new inequalities for periodic quasisymmetric functions, Ann. Univ. Mariae Curie–Skłodowska, Sect. A 43 (1989) 93–100.
[11] D. Partyka, The maximal dilatation of the Douady–Earle extension of a quasisymmetric automorphism of the unit circle, Ann. Univ. Mariae Curie–Skłodowska, Sect. A 44 (1990) 45–57.
[12] Ch. Pommerenke, Boundary Behaviour of Conformal Maps, Springer, Berlin, 1992.
[13] W. Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1974.
[14] A. Zygmund, Trigonometric Series, vol. 1, Cambridge University Press, Cambridge, 1968.