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Hardy inequalities for Robin Laplacians

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Abstract

In this paper we establish a Hardy inequality for Laplace operators with Robin boundary conditions. For convex domains, in particular, we show explicitly how the corresponding Hardy weight depends on the coefficient of the Robin boundary conditions. We also study several extensions to non-convex and unbounded domains.

Keywords: Robin Laplacian; Hardy inequality

1. Introduction

The classical Hardy inequality states that if $n \geqslant 3$ then for any function u such that $\nabla u \in L^2(\mathbb{R}^n)$ it holds

$$\int_{\mathbb{R}^n} \left|\nabla u(x)\right|^2 dx \geqslant \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2}\, dx. \tag{1.1}$$

It is well known that the constant $(n-2)^2/4$ in (1.1) is sharp but not achieved. The literature concerning different versions of Hardy's inequalities and their applications is extensive and we

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are not able to cover it in this paper. We just mention the classical paper of M.Sh. Birman [6] and the books of E.B. Davies [8,9] and V. Maz'ya [17].

A version of Hardy inequalities was considered by E.B. Davies (see for example [10] or [9, Sect. 5.3]) who, in particular, proved that for convex domains $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$

$$\int_\Omega \left|\nabla u(x)\right|^2 dx \geqslant \frac{1}{4} \int_\Omega \frac{|u(x)|^2}{\delta^2(x)}\, dx, \quad u \in H_0^1(\Omega), \tag{1.2}$$

where δ is the distance function to the boundary $\partial\Omega$,

$$\delta(x) = \operatorname{dist}(x, \partial\Omega) = \min_{y \in \partial\Omega} |x - y|. \tag{1.3}$$

The L^p version of inequality (1.2), for $p > 1$, with the sharp constant was proven in [16] for $n = 2$ and later in [15] for any n.

In the paper [7] H. Brezis and M. Marcus showed that if $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$ is convex, then the inequality (1.2) could be improved to include the L^2-norm:

$$\int_\Omega |\nabla u|^2 \, dx \geqslant \frac{1}{4} \int_\Omega \frac{|u|^2}{\delta^2(x)}\, dx + C(\Omega) \int_\Omega |u|^2 \, dx, \tag{1.4}$$

where $C(\Omega) = c(\operatorname{diam}\Omega)^{-2}$ with some $c > 0$. They also conjectured that $C(\Omega)$ should depend on the Lebesgue measure of Ω. This conjecture was justified in [13] where it was proved that in (1.4) $C(\Omega)$ could be chosen as $c|\Omega|^{-2/n}$ with some $c > 0$ independent of Ω. This result was later generalised to L^p-type inequalities in [19].

If Ω is not convex then generally speaking the constant in front of the first integral in the right-hand side of (1.4) could be less then $1/4$. In 1986 A. Ancona [2] showed using the Koebe one-quarter Theorem, that for a simply-connected planar domain the constant in the Hardy inequality with the distance to the boundary is greater than or equal to $1/16$. In [14] the authors have considered classes of domains for which there is a stronger version of the Koebe Theorem and which in turns implies better estimates for the constant appearing in the Hardy inequality (1.2).

In [11] S. Filippas, V.G. Maz'ya and A. Tertikas (see also F.G. Avkhadiev [4]) have obtained that for convex domains Ω the constant $C(\Omega)$ in (1.4) could be expressed in terms of the inradius of Ω. Namely if

$$R_{in} := \sup\bigl\{\delta(x)\colon\ x \in \Omega\bigr\},$$

then $C(\Omega) = c_0^2 R_{in}^{-2}$ with some $c_0 > 0$. Recently F.G. Avkhadiev and K.-J. Wirths [5] have shown that the best possible constant c_0 equals the first positive zero of the function

$$J_0(t) - 2t J_1(t) = J_0(t) + 2t J_0'(t)$$

where J_0 and J_1 are the Bessel functions of order 0 and 1 respectively. Note, however, that the results obtained in the papers [11,2,5] do not cover the case of non-convex domains whereas it has been showed in [13] that the remainder term $c|\Omega|^{-2/d}$ survives even for non-convex domains.

In this paper we are not going to consider the classical Dirichlet–Laplacian, but the so-called Robin–Laplacian generated by the quadratic form

$$\mathcal{Q}_\sigma[u] = \int_\Omega \left|\nabla u(x)\right|^2 dx + \int_{\partial\Omega} \sigma(y)\left|u(y)\right|^2 d\nu(y), \tag{1.5}$$

where $d\nu$ denotes the surface measure on $\partial\Omega$ and σ is a measurable function which defines the boundary conditions. If $\sigma \in L^\infty(\partial\Omega)$ is non-negative and such that $\sigma > 0$ on a part of the boundary of non-zero surface measure and if Ω is bounded and regular enough, see Section 2, then $\mathcal{Q}_\sigma[\cdot]$ is positive definite on $H^1(\Omega)$ and therefore must satisfy some Hardy type inequality with a positive integral weight.

Our aim is to find out how such a Hardy inequality depends on the function σ and on the geometry of Ω. We will deal with several types of domains in $\mathbb{R}^n$. For convex domains we establish a Hardy inequality with an explicit expression for the associated integral weight, Theorem 3.1. A generalisation to non-convex domains is discussed in Section 4, see Theorem 4.2 and Corollary 4.3. In the closing Section 5 we treat an example of an unbounded non-convex domain.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be an open domain. We will need the following hypothesis.

Assumption 2.1. Ω satisfies the strong local Lipschitz condition. This means that each point $y \in \partial\Omega$ has a neighbourhood U_y such that $U_y \cap \partial\Omega$ is the graph of a Lipschitz continuous function with the Lipschitz constant independent of y, see [1, Chap. 4].

If in addition to Assumption 2.1 we suppose that Ω is bounded and that $\sigma \in L^\infty(\partial\Omega)$, then in view of the trace inequality, see e.g. [1, Sect. 7.5], it follows that

$$\mathcal{Q}_\sigma[u] \leqslant c\|u\|_{H^1(\Omega)} \tag{2.1}$$

for some c and all $u \in H^1(\Omega)$. Hence the quadratic form $\mathcal{Q}_\sigma[u]$ defined on $H^1(\Omega)$ is closed. The unique self-adjoint operator generated by $\mathcal{Q}_\sigma$ is then the Robin–Laplacian which formally satisfies the boundary conditions

$$\frac{\partial u}{\partial \eta_y}(y) + \sigma(y)u(y) = 0, \quad y \in \partial\Omega,$$

where η_y denotes the unit outer normal vector at $y \in \partial\Omega$.

Finally, let us denote by $S \subset \Omega$ the subset of points in Ω for which there exist at least two points $y_1, y_2 \in \partial\Omega$ where the minimum in (1.3) is achieved. Usually this set is called the *singular* set of Ω and it is known that its Lebesgue measure is zero (see for example [18]). We introduce the projection $p : \Omega \setminus S \to \partial\Omega$ by

$$p(x) := y \in \partial\Omega : \quad \delta(x) = |x - y|, \ x \in \Omega \setminus S. \tag{2.2}$$

3. Convex domains

3.1. Bounded convex domains

Our main result is the following

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be open bounded and convex. Then for any $0 \leqslant \sigma \in L^\infty(\partial\Omega)$ and all $u \in H^1(\Omega)$ it holds*

$$\int_\Omega |\nabla u(x)|^2\,dx + \int_{\partial\Omega} \sigma(y)|u(y)|^2\,d\nu(y) \geqslant \frac{1}{4}\int_\Omega \left(\delta(x) + \frac{1}{2\sigma(p(x))}\right)^{-2} |u(x)|^2\,dx + \frac{1}{4}\int_\Omega \left(R_{in} + \frac{1}{2\sigma(p(x))}\right)^{-2} |u(x)|^2\,dx + \frac{1}{2}\int_{\partial\Omega} \left(R_{in} + \frac{1}{2\sigma(y)}\right)^{-1} |u(y)|^2\,d\nu(y). \tag{3.1}$$

Proof of Theorem 3.1 is given after Lemma 3.8.

Remark 3.2. Note that $p(x)$ is defined on $\Omega \setminus S$ and therefore almost everywhere in Ω.

Remark 3.3. Assumption 2.1 is satisfied for all bounded convex domains.

Remark 3.4. Let us discuss the optimality of the first term on the right-hand side of inequality (3.1). Assume that

$$\int_\Omega |\nabla u|^2\,dx + \sigma \int_{\partial\Omega} u^2\,d\nu \geqslant c \int_\Omega \frac{|u|^2}{(\delta + \frac{1}{2\sigma})^2}\,dx \tag{3.2}$$

holds true for a constant $\sigma > 0$ and some c. Let $\Omega \subset \mathbb{R}^n$ be a ball of radius R centred in the origin and consider the function

$$u(x) = \left(R + \frac{1}{2\sigma} - |x|\right)^{1/2}, \quad x \in \Omega.$$

By inserting u into (3.2) and letting $R \to \infty$ we get

$$(1-4c)R^{n-1}\log(1+2\sigma R) + O\left(R^{n-1}\right) \geqslant 0, \quad R \to \infty.$$

This shows that $c \leqslant 1/4$ and hence the constant $1/4$ on the firs line of (3.1) is sharp. Another way to see this is to look at the limit $\sigma \to \infty$, cf. Corollary 3.6 below, in which we recover the classical Hardy inequality where the constant $1/4$ is optimal.

However, it turns out that for a fixed convex domain Ω and certain values of σ, with σ being equal to a constant, it is possible to obtain a better weight than the one given by the first term in (3.1), see Remark 3.9 for more details.

Theorem 3.1 can be applied also in situations with Dirichlet boundary condition on some part of the boundary. In order to formulate the corresponding statement, we introduce the space

$$C^1_{0,\Gamma}(\overline{\Omega}) := \{u \in C^1(\overline{\Omega}):\ u|_\Gamma = 0,\ \Gamma \subset \partial\Omega\}.$$

We then have

Corollary 3.5. *Let Ω be as in Theorem* 3.1 *and let $\Gamma \subset \partial\Omega$ be closed. Then for all $u \in C^1_{0,\Gamma}(\overline{\Omega})$ and any $\tilde{\sigma} \in L^\infty(\partial\Omega \setminus \Gamma)$ inequality* (3.1) *holds true with*

$$\sigma(y) = \begin{cases} \tilde{\sigma}(y) & \text{if } y \in \partial\Omega \setminus \Gamma, \\ +\infty & \text{if } y \in \Gamma. \end{cases}$$

Proof. Let $u \in C^1_{0,\Gamma}(\overline{\Omega})$ and define the sequence of functions $\sigma_n : \partial\Omega \to \mathbb{R}_+$ by

$$\sigma_n(y) = \begin{cases} \tilde{\sigma}(y) & \text{if } y \in \partial\Omega \setminus \Gamma, \\ n & \text{if } y \in \Gamma, \end{cases} \qquad n \in \mathbb{N}.$$

Then $\sigma_n \in L^\infty(\partial\Omega)$ for each $n \in \mathbb{N}$ and therefore we can apply Theorem 3.1. The statement then follows from the monotone convergence theorem by letting $n \to \infty$. □

When $\Gamma = \partial\Omega$, then the above corollary yields an improvement of the classical Hardy inequality (1.2) for Dirichlet Laplacians, see [3–5,7,11,13] for more results in this direction.

Corollary 3.6. *Let Ω be as in Theorem* 3.1*. Then for all $u \in H^1_0(\Omega)$ it holds*

$$\int_\Omega |\nabla u(x)|^2\, dx \geqslant \frac{1}{4} \int_\Omega \frac{|u(x)|^2}{\delta^2(x)}\, dx + \frac{1}{4R_{in}^2} \int_\Omega |u(x)|^2\, dx. \tag{3.3}$$

Remark 3.7. In [7] and [13] the constant in front of the second term on the right-hand side is expressed in terms of the diameter and volume of Ω respectively. A sharp constant involving the inradius R_{in} is due to [5].

The key idea of the proof of Theorem 3.1 is to establish inequality (3.1) first for convex polytops, and then to approximate the domain Ω by a sequence of such polytops. We start with a one-dimensional result.

Lemma 3.8. *Let $b > 0$ and assume that u belongs to $AC[0, b]$, the space of absolutely continuous functions on $[0, b]$. Then for any $\sigma \geqslant 0$ we have*

$$\int_0^b |u'(t)|^2\, dt + \sigma |u(0)|^2 \geqslant \frac{1}{4} \int_0^b \left(\left(t + \frac{1}{2\sigma}\right)^{-2} + \left(b + \frac{1}{2\sigma}\right)^{-2} \right) |u(t)|^2\, dt + \frac{1}{2}\left(b + \frac{1}{2\sigma}\right)^{-1} |u(0)|^2. \tag{3.4}$$

Proof. The inequality is trivial for $\sigma = 0$. Hence we may assume that $\sigma > 0$. Let $f \in C^1[0, b]$. Integration by parts and Cauchy–Schwartz inequality show that

$$\left(\int_0^b \big(f(t) - f(b)\big)'|u|^2\,dt - \big(f(b) - f(0)\big)\big|u(0)\big|^2\right)^2$$

$$= \left(\int_0^b \big(f(b) - f(t)\big)\big(u'\bar{u} + u\bar{u}'\big)\,dt\right)^2$$

$$\leqslant 4\left(\int_0^b \big|u'\big|^2\,dt\right)\left(\int_0^b \big(f(b) - f(t)\big)^2|u|^2\,dt\right).$$

This together with the inequality

$$\frac{A^2}{B} \geqslant 2A - B, \qquad B > 0,$$

gives

$$\int_0^b \big|u'(t)\big|^2\,dt + \frac{f(b) - f(0)}{2}\big|u(0)\big|^2 \geqslant \frac{1}{2}\int_0^b f'(t)|u|^2\,dt - \frac{1}{4}\int_0^b \big(f(b) - f(t)\big)^2|u|^2\,dt. \tag{3.5}$$

By inserting

$$f(t) = -\left(t + \frac{1}{2\sigma}\right)^{-1}$$

into (3.5) and using the fact that

$$\left(t + \frac{1}{2\sigma}\right)^{-1}\left(b + \frac{1}{2\sigma}\right)^{-1} \geqslant \left(b + \frac{1}{2\sigma}\right)^{-2}, \quad 0 \leqslant t \leqslant b,$$

we arrive at (3.4). □

Remark 3.9. Although the constant $1/4$ on the right-hand side of (3.4) is optimal in the limit $\sigma \to \infty$, the weight in the first term might be improved for σ large enough, depending on b, but finite. This follows from the proof of [7, Lemma A.1] applied to a test function $v(t) = (t + 2/\sigma)^{-1/2}u(t)$ with $u \in C^1[0, b]$. It would be interesting to find a unified formula describing this effect. We thank the referee for pointing out this observation to us.

Proof of Theorem 3.1. We start by proving the statement for $u \in C^1(\overline{\Omega})$. Assume first that σ is continuous. Since $\partial\Omega$ is closed in $\mathbb{R}^n$, by Tietze's extension Theorem there exists a continuous function $\Sigma : \Omega \to \mathbb{R}$ whose restriction to $\partial\Omega$ coincides with σ.

Let $Q \subset \Omega$ be an open convex polytop in $\mathbb{R}^n$ with N sides Γ_j, $j = 1, \dots, N$. Clearly we have $u \in C^1(\bar{Q})$. Denote by n_j the unit inner normal vector to Γ_j. For each side Γ_j we consider the domain P_j attached to Γ_j by including all the points from Q for which the distance to the boundary ∂Q is achieved at a point belonging to Γ_j. More precisely,

$$P_j = \{x \in Q\colon \exists y \in \Gamma_j\colon \operatorname{dist}(x, \partial Q) = |x - y|\}.$$

Then $Q = \bigcup_j \overline{P}_j$ and the singular set of Q is given by $S = \bigcup_{j=1}^N (\partial P_j \setminus \Gamma_j)$. The inradius of Ω obviously satisfies

$$R_{in} \geqslant \max_j \max_{x \in P_j} \operatorname{dist}(x, \Gamma_j). \tag{3.6}$$

Moreover, for each $y \in \Gamma_j$ there is a unique point $x_y \in S$ and t_y such that

$$x_y = y + t_y n_j. \tag{3.7}$$

Denote by p_Q the projection on ∂Q defined in the same way as p in (2.2) with Ω replaced by Q. We apply Lemma 3.8 along the normal vector n_j and, taking into account (3.6), we obtain

$$\begin{aligned}
&\int_0^{t_y} \left|\partial_t u(y + t n_j)\right|^2 dt + \Sigma_Q(y)\left|u(y)\right|^2 \\
&\quad \geqslant \frac{1}{4} \int_0^{t_y} \left(\left(t + \frac{1}{2\Sigma_Q(y)}\right)^{-2} + \left(R_{in} + \frac{1}{2\Sigma_Q(y)}\right)^{-2}\right) \left|u(y + t n_j)\right|^2 dt \\
&\qquad + \frac{1}{2}\left(R_{in} + \frac{1}{2\Sigma_Q(y)}\right)^{-1} \left|u(y)\right|^2,
\end{aligned}$$

where Σ_Q is the restriction of Σ on ∂Q. Next we note that

$$t = \delta(y + t n_j) \quad \forall y \in \Gamma_j,\ \forall t \in (0, t_y).$$

Hence, integrating over variables orthogonal to n_j and using the invariance of the Laplacian with respect to rotations we arrive at

$$\begin{aligned}
&\int_{P_j} \left|\nabla u(x)\right|^2 dx + \int_{\Gamma_j} \Sigma_Q(y)\left|u(y)\right|^2 dy \\
&\quad \geqslant \frac{1}{4} \int_{P_j} \left(\left(\delta(x) + \frac{1}{2\Sigma_Q(p_Q(x))}\right)^{-2} + \left(R_{in} + \frac{1}{2\Sigma_Q(p_Q(x))}\right)^{-2}\right) \left|u(x)\right|^2 dx \\
&\qquad + \frac{1}{2} \int_{\Gamma_j} \left(R_{in} + \frac{1}{2\Sigma_Q(y)}\right)^{-1} \left|u(y)\right|^2 dy,
\end{aligned} \tag{3.8}$$

where dy denotes the $(n-1)$-dimensional Lebesgue measure on Γ_j. Summation of (3.8) over j gives us inequality (3.1) for any convex polytop $Q \subset \Omega$ with σ replaced by Σ_Q and p replaced by p_Q.

Since Ω is convex, there exists a sequence of n-dimensional convex polytops $Q_m \subset \Omega$, $m \in \mathbb{N}$, which approximates Ω. More precisely, for every ε there exists an m_ε such that the Hausdorff distance between Ω and Q_{m_ε} satisfies $d_H(\Omega, Q_{m_\varepsilon}) < \varepsilon$, see [12, §9]. Since Q_m approximates Ω also in the surface measure, [12, §14], from the continuity of u and Σ it follows that

$$\Sigma_{Q_m}\big(p_{Q_m}(x)\big) \to \sigma\big(p(x)\big) \quad \text{a.e. } x \in \Omega,$$

and

$$\int_{\partial Q_m} \Sigma_{Q_m}(y)\big|u(y)\big|^2\, dy \to \int_{\partial\Omega} \sigma(y)\big|u(y)\big|^2\, d\nu(y)$$

as $m \to \infty$. Hence, by the dominated convergence we can pass to the limit to get

$$\begin{aligned}
&\int_{\Omega} \big|\nabla u(x)\big|^2\, dx + \int_{\partial\Omega} \sigma(y)\big|u(y)\big|^2\, d\nu(y) \\
&\quad \geqslant \frac{1}{4}\int_{\Omega}\left(\left(\delta(x) + \frac{1}{2\sigma(p(x))}\right)^{-2} + \left(R_{in} + \frac{1}{2\sigma(p(x))}\right)^{-2}\right)\big|u(x)\big|^2\, dx \\
&\qquad + \frac{1}{2}\int_{\partial\Omega}\left(R_{in} + \frac{1}{2\sigma(y)}\right)^{-1}\big|u(y)\big|^2\, d\nu(y)
\end{aligned} \tag{3.9}$$

for all $u \in C^1(\overline{\Omega})$ and any σ continuous.

Next we note that if $\sigma \in L^\infty(\partial\Omega)$, then in view of the regularity of $\partial\Omega$ there exists a sequence of continuous functions σ_k on $\partial\Omega$ which converges to σ in $L^1(\partial\Omega)$ as $k \to \infty$. Then σ_k admits a subsequence, which we still denote by σ_k, such that $\sigma_k \to \sigma$ almost everywhere on $\partial\Omega$. Hence $\sigma_k(p(x)) \to \sigma(p(x))$ for almost every $x \in \Omega$. From inequality (3.9) it follows that (3.1) holds for all σ_k. Since $u|_{\partial\Omega} \in L^\infty(\partial\Omega)$, we can pass to the limit as $k \to \infty$ and using the dominated convergence we obtain inequality (3.1) for any $\sigma \in L^\infty(\partial\Omega)$ and all $u \in C^1(\overline{\Omega})$.

Finally, if $u \in H^1(\Omega)$, then by density there exists a sequence $u_j \in C^1(\overline{\Omega})$ such that $\|u - u_j\|_{H^1(\Omega)} \to 0$ as $j \to \infty$. By the regularity of Ω it follows that $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ [1, Sect. 7.5]. Hence, after applying inequality (3.9) to u_j and letting $j \to \infty$ we conclude that (3.1) holds for all $u \in H^1(\Omega)$. □

3.2. Unbounded convex domains

For unbounded domains we need to impose some decay conditions on the test functions. Let $\rho > 0$ and define

$$\dot{C}^1(B_\rho) = \big\{u \in C^1\big(\mathbb{R}^n\big):\ u(x) = 0\ \forall x:\ |x| \geqslant \rho\big\}, \tag{3.10}$$

where $B_\rho = \{x \in \mathbb{R}^n:\ |x| < \rho\}$. We have

Theorem 3.10. *Let Ω be open and convex and let $0 \leqslant \sigma \in L^\infty(\partial\Omega)$. Let $\rho > 0$. Then inequality* (3.1) *holds for all $u \in \dot{C}^1(B_\rho)$.*

Proof. Let $u \in \dot{C}^1(B_\rho)$. Define $\Omega_\rho = \Omega \cap B_\rho$. Then Ω_ρ is convex and bounded. Now define $\hat{\sigma} : \partial\Omega_\rho \to \mathbb{R}_+$ by

$$\hat{\sigma}(y) = \begin{cases} \sigma(y) & \text{if } y \in \partial\Omega \setminus \partial B_\rho, \\ +\infty & \text{elsewhere.} \end{cases}$$

Let p_ρ be the projection on $\partial\Omega_\rho$ defined in the same way as p in (2.2) with Ω replaced by Ω_ρ. Accordingly, let $\delta_\rho(x) = \operatorname{dist}(x, \partial\Omega_\rho)$. Then

$$\hat{\sigma}\big(p_\rho(x)\big) \geqslant \sigma\big(p(x)\big), \qquad \delta_\rho(x) \leqslant \delta(x) \quad \text{for a.e. } x \in \Omega_\rho.$$

Moreover, the inradius of Ω_ρ is less or equal to the inradius R_{in} of Ω. Since $u|_{\Omega_\rho} \in H^1(\Omega_\rho)$ we can apply Theorem 3.1 to Ω_ρ with $\hat{\sigma}$ defined as above. This gives the result. □

4. General bounded domains

In this section we will establish a version of the Hardy inequality on general open domains $\Omega \subset \mathbb{R}^n$ satisfying Assumption 2.1. We will follow the approach of [8, Sect. 1.5], see also [10]. As in the case of convex domains we start with an auxiliary one-dimensional result.

Lemma 4.1. *Let $u \in AC[0, b]$. Then for any $\sigma_1 \geqslant 0$ and $\sigma_2 \geqslant 0$ we have*

$$\int_0^b \big|u'(t)\big|^2\,dt + \sigma_1\big|u(0)\big|^2 + \sigma_2\big|u(b)\big|^2 \geqslant \frac{1}{4}\int_0^{b/2}\left(t + \frac{1}{2\sigma_1}\right)^{-2}\big|u(t)\big|^2\,dt + \frac{1}{4}\int_{b/2}^{b}\left(b - t + \frac{1}{2\sigma_2}\right)^{-2}\big|u(t)\big|^2\,dt. \tag{4.1}$$

Proof. This follows immediately from inequality (3.4). □

Given $x \in \Omega$ and some $e \in \mathbb{R}^n$ with $\|e\|_n = 1$, we introduce

$$d_e(x) = \min\big\{|s|\colon\ s \in \mathbb{R}\colon\ x + se \notin \Omega\big\}. \tag{4.2}$$

Hence, $d_e(x)$ is the distance from x to the boundary of Ω in the direction of the vector e. Denote by $m(e, x)$ the set on which the minimum in (4.2) is achieved. Clearly, $m(e, x)$ contains either one or two elements. Let us define

$$\sigma_e(x) := \max_{s \in m(e,x)} \sigma(x + se), \qquad x \in \Omega. \tag{4.3}$$

Finally, let $d\mathcal{L}$ be the normalised surface measure on the unit sphere in $\mathbb{R}^n$ and define

$$\mu_\sigma(x) := \int\limits_{e:\ \|e\|=1} \left(d_e(x) + \frac{1}{2\sigma_e(x)}\right)^{-2} d\mathcal{L}(e). \tag{4.4}$$

We have

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying Assumption* 2.1. *Then for all $u \in H^1(\Omega)$ and any $\sigma \in L^\infty(\partial\Omega)$ it holds*

$$\int\limits_\Omega \left|\nabla u(x)\right|^2 dx + \int\limits_{\partial\Omega} \sigma(y)\left|u(y)\right|^2 d\nu(y) \geqslant \frac{1}{4}\int\limits_\Omega \left|u(x)\right|^2 \mu_\sigma(x)\, dx. \tag{4.5}$$

Proof. By Assumption 2.1 and [1, Thm. 3.22] it suffices to prove inequality (4.5) for $u \in C^1(\overline{\Omega})$. We denote by ∂_e the partial differentiation in direction e. From Lemma 4.1 we find out that

$$\frac{1}{4}\int\limits_\Omega \left(d_e(x) + \frac{1}{2\sigma_e(x)}\right)^{-2} \left|u(x)\right|^2 dx \leqslant \int\limits_\Omega \left|\partial_e u(x)\right|^2 dx + \int\limits_{\partial\Omega} \sigma(y)\left|u(y)\right|^2 d\nu(y) \tag{4.6}$$

holds for all e with $\|e\| = 1$. The result then follows by integrating the last inequality with respect to e over the unit sphere. □

Corollary 4.3. *Let Ω satisfy conditions of Theorem* 4.2. *Assume that $\sigma(y) = \sigma > 0$ is constant. Then there exists a constant $K = K(\Omega, n)$, independent of σ, such that for all $u \in H^1(\Omega)$ it holds*

$$\int\limits_\Omega \left|\nabla u(x)\right|^2 dx + \sigma\int\limits_{\partial\Omega} \left|u(y)\right|^2 d\nu(y) \geqslant K\int\limits_\Omega \frac{|u(x)|^2}{(\delta(x) + \frac{1}{4\sigma})^2}\, dx. \tag{4.7}$$

Proof. Denote by ω_n the surface area of the unit ball in $\mathbb{R}^n$. Let $x \in \Omega$ and let $a \in \partial\Omega$ be such that $r := \delta(x) = |x - a|$. Define

$$\Lambda_x = \left\{e \in \mathbb{R}^n\colon\ \|e\| = 1 \wedge \exists s > 0 \text{ such that } x + se \notin \Omega \text{ and } |x + se - a| < r\right\}.$$

By the triangle inequality we have $s < 2r$. It follows that the region $\{y \notin \Omega\colon\ |y - a| < r\}$ is contained in the n-dimensional cone of radius $2r$ with the opening angle $\mathcal{L}(\Lambda_x)\omega_n$ centred in x. Hence

$$c_n \mathcal{L}(\Lambda_x) r^n \geqslant \operatorname{Vol}\left(\left\{y \notin \Omega\colon\ |y - a| < r\right\}\right), \tag{4.8}$$

for some constant c_n which depends only on n. On the other hand, from Assumption 2.1 it follows that there exits $\alpha > 0$ such that

$$\operatorname{Vol}\left(\left\{y \notin \Omega\colon\ |y - a| < r\right\}\right) \geqslant \alpha r^n \quad \forall a \in \partial\Omega,\ \forall r > 0.$$

This together with (4.8) implies that $\mathcal{L}(\Lambda_x) \geqslant c_n^{-1}\alpha$ for every $x \in \Omega$. We thus get

$$\mu_\sigma(x) = \int\limits_{e:\, \|e\|=1} \left(d_e(x) + \frac{1}{2\sigma}\right)^{-2} d\mathcal{L}(e) \geqslant \int\limits_{\Lambda_x} \left(d_e(x) + \frac{1}{2\sigma}\right)^{-2} d\mathcal{L}(e)$$
$$\geqslant \left(2\delta(x) + \frac{1}{2\sigma}\right)^{-2} \mathcal{L}(\Lambda_x) \geqslant \left(2\delta(x) + \frac{1}{2\sigma}\right)^{-2} \frac{\alpha}{c_n},$$

since for every $e \in \Lambda_x$ we have $d_e(x) \leqslant s \leqslant 2\delta(x)$. Inequality (4.7) now follows from Theorem 4.2. □

4.1. The case of sign changing σ

Note that the assumption $\sigma \geqslant 0$ is crucial for the results given in Theorems 3.1 and 4.2. When σ is negative on some part of $\partial\Omega$, then a simple test function argument shows that the resulting Robin–Laplacian may have a negative eigenvalue even if

$$\int\limits_{\partial\Omega} \sigma \, d\nu > 0,$$

provided σ is chosen in a suitable way. This tells us that if σ changes sign, then no Hardy inequality with positive integral weight can hold unless some further restrictions are made.

In order to give an example of a Hardy inequality with a sign changing weight, we consider a class of domains characterised as follows. Suppose that $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is continuous and that there exists an open set $A \subset \mathbb{R}^{n-1}$ such that $f > 0$ in A and $f = 0$ on ∂A. We then define

$$\Omega = \{x := (x', t) \in \mathbb{R}^n:\ x' \in A,\ 0 < t < f(x')\}. \tag{4.9}$$

Let $\sigma \in L^\infty(\partial\Omega)$ be such that $\sigma = 0$ on $\partial\Omega \setminus A$ and denote by $A_\pm \subset A$ the sets on which σ is positive respectively negative. Let $\chi_{A_\pm}$ be the related characteristic functions. We have

Proposition 4.4. *Let Ω and σ be given as above. Then for any $u \in H^1(\Omega)$ it holds*

$$\int\limits_\Omega |\nabla u(x)|^2 \, dx + \int\limits_{\partial\Omega} \sigma(y) |u(y)|^2 \, d\nu(y)$$
$$\geqslant \int\limits_\Omega \rho(x) |u(x)|^2 \, dx + \frac{1}{2} \int\limits_{A_+} \left(f(x') + \frac{1}{2\sigma(x')}\right)^{-1} |u(x', 0)|^2 \, dx', \tag{4.10}$$

where

$$\rho(x) = \rho(x', t) = \frac{1}{2}\left(f(x') + \frac{1}{2\sigma(x')}\right)^{-2} \chi_{A_+}(x') + \mu(x') \chi_{A_-}(x'),$$

and $-\mu(x')$ is the first positive solution to the implicit equation

$$-\sigma(x') = \sqrt{-\mu(x')}\tanh\bigl(f(x')\sqrt{-\mu(x')}\bigr), \quad x' \in A_-. \tag{4.11}$$

Proof. Let $x' \in A_-$. A straightforward calculation shows that $\mu(x')$ defined by (4.11) is the lowest eigenvalues of the Laplace operator on interval $(0, f(x'))$ with Neumann boundary condition at $f(x')$ and Robin boundary condition with coefficient $\sigma(x')$ at 0. Therefore we have

$$\int_0^{f(x')} \bigl|\partial_t u(x',t)\bigr|^2\,dt + \sigma(x')\bigl|u(x',0)\bigr|^2 \geqslant \mu(x') \int_0^{f(x')} \bigl|u(x',t)\bigr|^2\,dt \quad \forall x' \in A_-.$$

On the other hand, from Lemma 3.8 we easily deduce that

$$\int_0^{f(x')} \bigl|\partial_t u(x',t)\bigr|^2\,dt + \sigma(x')\bigl|u(x',0)\bigr|^2 \geqslant \frac{1}{2}\left(f(x') + \frac{1}{2\sigma(x')}\right)^{-2} \int_0^{f(x')} \bigl|u(x',t)\bigr|^2\,dt$$
$$+ \frac{1}{2}\left(f(x') + \frac{1}{2\sigma(x')}\right)^{-1} \bigl|u(x',0)\bigr|^2 \tag{4.12}$$

for all $x' \in A_+$. □

5. Unbounded non-convex domains: an example

In this section we give an example of a Hardy inequality on a particular type of an unbounded non-convex domain. Namely, on the complement of a ball.

Theorem 5.1. *Let $B_R = \{x \in \mathbb{R}^n\colon\ |x| < R\}$ and let $B_r^c = \mathbb{R}^n \setminus B_R$ be its complement. Then for any constant $\sigma \geqslant 0$ the inequality*

$$\int_{B_R^c} \bigl|\nabla u(x)\bigr|^2\,dx + \sigma \int_{\partial B_R} \bigl|u(y)\bigr|^2\,d\nu(y) \geqslant \frac{1}{4}\int_{B_R^c} \left(|x| - R + \frac{1}{2\sigma}\right)^{-2} \bigl|u(x)\bigr|^2\,dx$$
$$+ \frac{(n-1)(n-3)}{4} \int_{B_R^c} \frac{|u(x)|^2}{|x|^2}\,dx \tag{5.1}$$

holds true for all $u \in H^1(B_R^c)$.

Proof. Without loss of generality we may assume that u is real-valued. Consider first the case $\sigma > 0$. Let $\delta(x) = |x| - R$. We have $|\nabla\delta(x)| = 1$ and $\Delta\delta(x) = \frac{n-1}{|x|}$. Integration by parts then gives

$$\int_{B_R^c} \frac{u(x)\nabla u(x)\cdot\nabla\delta(x)}{\delta(x)+\frac{1}{2\sigma}}\,dx = \frac{1}{2}\int_{B_R^c}\left(\frac{1}{(\delta(x)+\frac{1}{2\sigma})^2} - \frac{n-1}{|x|(\delta(x)+\frac{1}{2\sigma})}\right)\left|u(x)\right|^2 dx - \sigma\int_{\partial B_R}\left|u(y)\right|^2 d\nu(y), \tag{5.2}$$

and

$$\int_{B_R^c} \frac{u(x)\nabla u(x)\cdot x}{|x|^2}\,dx = -\frac{1}{2R}\int_{\partial B_R}\left|u(y)\right|^2 d\nu(y) - \frac{(n-2)}{2}\int_{B_R^c}\frac{|u(x)|^2}{|x|^2}\,dx. \tag{5.3}$$

Since $\nabla\delta(x)\cdot x = |x|$, using (5.2), (5.3) and a straightforward calculation we obtain

$$\begin{aligned}
&\int_{B_R^c}\left|\nabla u(x) - \frac{\nabla\delta(x)}{2\delta(x)+\frac{1}{\sigma}}u(x) + \frac{(n-1)u(x)x}{2|x|^2}\right|^2 dx \\
&\quad\leqslant \int_{B_R^c}\left|\nabla u(x)\right|^2 dx + \sigma\int_{\partial B_R}\left|u(y)\right|^2 d\nu(y) \\
&\qquad - \frac{1}{4}\int_{B_R^c}\left(\delta(x)+\frac{1}{2\sigma}\right)^{-2}\left|u(x)\right|^2 dx - \frac{(n-1)(n-3)}{4}\int_{B_R^c}\frac{|u(x)|^2}{|x|^2}\,dx.
\end{aligned}$$

This proves the theorem for $\sigma > 0$. The case $\sigma = 0$ then follows from (5.1) by monotone convergence. □

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