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CONSTRUCTIVE APPROXIMATION

Positive Gegenbauer Polynomial Sums and Applications to Starlike Functions

Stamatis Koumandos and Stephan Ruscheweyh

Abstract. Let $s_n(f,z) := \sum_{k=0}^{n} a_k z^k$ be the nth partial sum of $f(z) = \sum_{k=0}^{\infty} a_k z^k$. We show that $\operatorname{Re} s_n(f/z, z) > 0$ holds for all $z \in \mathbf{D}$, $n \in \mathbf{N}$, and all starlike functions f of order λ iff $\lambda_0 \le \lambda < 1$ where $\lambda_0 = 0.654222...$ is the unique solution $\lambda \in (\frac{1}{2}, 1)$ of the equation $\int_0^{3\pi/2} t^{1-2\lambda} \cos t \, dt = 0$. Here $\mathbf{D}$ denotes the unit disk in the complex plane $\mathbf{C}$. This result is the best possible with respect to λ_0. In particular, it shows that for the Gegenbauer polynomials $C_n^{\mu}(x)$ we have $\sum_{k=0}^{n} C_k^{\mu}(x) \cos k\theta > 0$ for all $n \in \mathbf{N}$, $x \in [-1, 1]$, and $0 < \mu \le \mu_0 := 1 - \lambda_0 = 0.345778....$ This result complements an inequality of Brown, Wang, and Wilson (1993) and extends a result of Ruscheweyh and Salinas (2000).

1. Introduction

Let $\mathbf{D}$ denote the unit disk in the complex plane $\mathbf{C}$, and for $\lambda < 1$ let $\mathcal{S}_\lambda$ be the family of functions f *starlike of order* λ, i.e., f analytic in $\mathbf{D}$ with $f(0) = f'(0) - 1 = 0$ and $\operatorname{Re}(zf'(z)/f(z)) > \lambda$ in $\mathbf{D}$. Note that $f_\lambda(z) = z(1-z)^{2\lambda-2}$ belongs to $\mathcal{S}_\lambda$ and is, indeed, the most important member in that family. For an analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $n \in \mathbf{N}$ we set $s_n(f,z) = \sum_{k=0}^{n} a_k z^k$, the nth partial sum of f. It has been shown in [8] that $\operatorname{Re} s_n(f/z, z) > 0$ holds in $\mathbf{D}$ and for all $f \in \mathcal{S}_{3/4}$, $n \in \mathbf{N}$. However, it was pointed out in [8], that the number $\frac{3}{4}$ can probably be replaced by a smaller number, and numerical tests indicated that something like 0.65... might be closer to the best possible result. Our main result here confirms this observation.

Theorem 1. *Let* $\lambda_0 = 0.654222...$ *be the unique solution* $\lambda \in (\frac{1}{2}, 1)$ *of the equation*

$$\int_0^{3\pi/2} t^{1-2\lambda} \cos t \, dt = 0.$$

Then, for every $f \in \mathcal{S}_{\lambda_0}$, *we have*

$$\operatorname{Re} s_n(f/z, z) > 0, \qquad z \in \mathbf{D}, \quad n \in \mathbf{N}.$$

λ_0 *is the smallest number with this property.*

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Remark. In [8] it was actually shown that, for $\lambda \geq \frac{1}{2}$,

$$|\arg s_n(f/z, z)| \leq 2\pi(1-\lambda), \qquad f \in \mathcal{S}_\lambda, \quad n \in \mathbf{N}. \tag{1}$$

Although this is the best possible as far as $\lambda = \frac{1}{2}$ is concerned, Theorem 1 indicates that this is generally not so for larger λ. Best possible results are not yet available in the cases not covered by Theorem 1.

An important special case of Theorem 1 is as follows: let $C_n^\mu(x)$ be the Gegenbauer polynomial of degree n and order μ, $\mu > 0$, defined by the generating function

$$G_\mu(z, x) := (1 - 2xz + z^2)^{-\mu} = \sum_{n=0}^{\infty} C_n^\mu(x) z^n, \qquad x \in [-1, 1].$$

It is easily seen that $zG_\mu(\cdot, x) \in \mathcal{S}_{1-\mu}$, so that we have the following corollary. We set $\mu_0 := 1 - \lambda_0 = 0.345778...$ (see Theorem 1).

Corollary 1. *For* $0 < \mu \leq \mu_0$ *we have*

$$\operatorname{Re} \sum_{k=0}^{n} C_k^\mu(x) z^k > 0, \qquad z \in \mathbf{D}, \quad n \in \mathbf{N}, \quad x \in [-1, 1],$$

and, *in particular*,

$$\sum_{k=0}^{n} C_k^\mu(x) \cos(k\theta) \geq 0, \qquad \theta \in \mathbf{R}, \quad n \in \mathbf{N}, \quad x \in [-1, 1]. \tag{2}$$

We recall

$$C_k^\mu(1) = \frac{(2\mu)_k}{k!}, \qquad k = 0, 1, 2, \ldots,$$

where, as usual, $(a)_k$ denotes the Pochhammer symbol, defined by $(a)_0 = 1$ and $(a)_k = a(a+1)\ldots(a+k-1)$, $k = 1, 2, \ldots$. Therefore the special case $x = 1$ of (2) is

$$S_n^\mu(\theta) := \sum_{k=0}^{n} \frac{(2\mu)_k}{k!} \cos(k\theta) > 0, \qquad \theta \in \mathbf{R}, \quad n \in \mathbf{N}, \quad 0 < \mu \leq \mu_0, \tag{3}$$

and this appears to be a very remarkable inequality. For instance, since the sequences

$$\frac{k!}{(2\mu)_k k^{1-2\mu}}, \qquad k = 1, 2, 3, \ldots,$$

are strictly decreasing for $0 < \mu < \frac{1}{2}$ (see Lemma 2 below), a summation by parts of (3) can be used to obtain

$$\frac{1}{1-\alpha} + \sum_{k=1}^{n} \frac{\cos(k\theta)}{k^\alpha} > 0, \qquad n \in \mathbf{N}, \quad \theta \in \mathbf{R}, \tag{4}$$

for $1 > \alpha \geq \alpha_0 := 1 - 2\mu_0 = 0.308443\ldots$. The sharper result

$$1 + \sum_{k=1}^{n} \frac{\cos(k\theta)}{k^{\alpha}} > 0, \qquad n \in \mathbf{N}, \quad \theta \in \mathbf{R}, \quad \alpha \geq \alpha_0, \tag{5}$$

has previously been established by G. Brown et al. in [6] as a natural extension of W. H. Young's classical inequality [11], and like (3) it is also sharp with respect to α_0. Numerical evaluations show that (3) alone does not imply (5). This is mainly because the minima of the S_n^{μ} are taken in different places for different n.

In turn, (5) implies (3) in the sharper version

$$S_n^{\mu}(\theta) \geq 1 - 2\mu, \qquad n \in \mathbf{N}, \quad 0 < \mu \leq \mu_1,$$

where $\mu_1 < \mu_0$ is the largest number μ such that the sequences

$$\frac{(2\mu)_k}{k!} k^{\alpha_0}, \qquad k = 1, 2, 3, \ldots,$$

are decreasing. A numerical evaluation gives $\mu_1 = 0.30751\ldots$. Extending this idea, one can show that (5) indeed implies (3) for $\mu \leq 0.345735$ (the upper bound being slightly less than μ_0) but not up to μ_0.

The strength of (3) also becomes clear from the fact that it actually contains the whole of Theorem 1 via a general convolution theorem for starlike functions, and is therefore the key to all results in this paper.

The special case $\mu = \frac{1}{4}$ of (3) had first been proved by Turàn [10] in the context of trigonometric series with nonnegative partial sums and coefficients not in ℓ_2, see also [2, p. 248].

Some very general conditions on the coefficients of trigonometric series in order for their partial sums to be nonnegative have been given by A. S. Belov [3]. We wish to remark that neither (3) nor (5) fall into that category.

In the next section we give the proof of (3). To this end, we employ some of the ideas in [6] and [12, V. 2.29]. An important ingredient of that proof is sharp estimates for the remainder of the infinite series corresponding to (3), for θ close to zero. These inequalities may be of independent interest.

In Section 3 we complete the proof of Theorem 1 and give some related information, including a discussion of possible further work.

2. Proof of (3)

We first show that the upper bound μ_0 in (3) would be sharp. Using the fact that

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^{x-1}}{(x)_n},$$

we obtain the asymptotic formula

$$\lim_{n\to\infty} \left(\frac{\theta}{n}\right)^{2\mu} \sum_{k=0}^{n} \frac{(2\mu)_k}{k!} \cos k\frac{\theta}{n} = \frac{1}{\Gamma(2\mu)} \int_0^{\theta} \frac{\cos t}{t^{1-2\mu}}\, dt. \tag{6}$$

It follows from this that for $\theta = 3\pi/2$ and $\mu > \mu_0$ the right-hand side of (3) will be negative. (See the discussion in [6] and [12, V, 2.29] and compare paper [5] for a more general setting of these results.) Therefore inequality (3) cannot hold for $\mu > \mu_0$, appropriate θ, and n sufficiently large.

Here are three further reductions to the claim in question:

(i) It is sufficient to prove (3) only for $\mu = \mu_0$. Indeed, for $0 < \mu < \mu_0$, the sequence $(2\mu)_k/(2\mu_0)_k$, $k = 1, 2, \ldots$, is strictly decreasing, so that a summation by parts of the $S_n^{\mu_0}(\theta)$ yields the positivity of $S_n^{\mu}(\theta)$.

(ii) It is obvious that $(d/d\theta)\{S_n^{\mu_0}(\theta)\} < 0$ for $0 < \theta < \pi/n$. Hence we need to prove the positivity of $S_n^{\mu_0}(\theta)$ only for $\pi/n \le \theta < \pi$.

(iii) The cases $n = 1, 2, 3$ of (3) are easily checked by straightforward calculations. Therefore, in what follows, we assume that $n \ge 4$.

In order to simplify formulas we shall write μ instead of μ_0 for the rest of Section 2. The Taylor series expansion

$$\frac{1}{(1-z)^{2\mu}} = \sum_{k=0}^{\infty} \frac{(2\mu)_k}{k!} z^k \qquad \text{for} \quad |z| < 1,$$

converges for $z \in \partial\mathbf{D}\backslash\{1\}$ since the coefficients form a monotonically decreasing zero-sequence. Hence Abel's theorem implies

$$\sum_{k=0}^{\infty} \frac{(2\mu)_k}{k!} \cos k\theta = \frac{\cos\mu(\pi-\theta)}{(2\sin\theta/2)^{2\mu}}, \qquad 0 < \theta < \pi. \tag{7}$$

2.1. *The Case* $\pi/2 \le \theta < \pi$

Again, the sequence

$$a_n := \frac{(2\mu)_n}{n!}, \qquad n = 1, 2, \ldots,$$

is decreasing, so that the standard estimate

$$\left| \sum_{k=n+1}^{\infty} a_k \cos k\theta \right| \le \frac{a_{n+1}}{\sin\theta/2}$$

obtains. Using this together with (7) we get

$$\begin{aligned} S_n^{\mu}(\theta) &= \frac{\cos\mu(\pi-\theta)}{(2\sin\theta/2)^{2\mu}} - \sum_{k=n+1}^{\infty} \frac{(2\mu)_k}{k!} \cos k\theta \\ &\ge \frac{1}{\sin\theta/2} \left\{ \frac{1}{2^{2\mu}} \cos\mu(\pi-\theta) \left(\sin\frac{\theta}{2} \right)^{1-2\mu} - a_{n+1} \right\}. \end{aligned} \tag{8}$$

For $n \ge 4$ we have $a_{n+1} \le 0.455\ldots$ and the function

$$\varphi(\theta) := \frac{1}{2^{2\mu}} \cos\mu(\pi-\theta) \left(\sin\frac{\theta}{2} \right)^{1-2\mu}$$

clearly increases in $\theta \in [\pi/2, \pi]$ with $\varphi(\pi/2) = 0.4763\ldots$. The desired result follows from (8).

2.2. *Some Lemmas*

In this subsection we establish a number of estimates necessary for the proof of (3) in the remaining interval. Let

$$A_k(\theta) := \int_0^{1/2} \cos(k-u)\theta \int_0^u \int_0^u \alpha(\alpha+1) \frac{dt\, ds}{(k-u+s+t)^{\alpha+2}}\, du,$$

$$B_k(\theta) := 2\int_0^{1/2} \sin u\theta \int_0^u \frac{\alpha}{(k+s)^{\alpha+1}}\, \sin k\theta\, ds\, du.$$

Lemma 1. *For $n \in \mathbf{N}$ we have*

$$\left|\sum_{k=n+1}^{\infty} A_k(\theta)\right| < \frac{1-2\mu}{8} \frac{1}{n^{2-2\mu}}, \tag{9}$$

$$\left|\sum_{k=n+1}^{\infty} B_k(\theta)\right| < \frac{\theta}{\sin\theta/2} \frac{1-2\mu}{12} \frac{1}{n^{2-2\mu}}. \tag{10}$$

Proof. Obviously

$$\left|\sum_{k=n+1}^{\infty} A_k(\theta)\right| \le \sum_{k=n+1}^{\infty} \int_0^{1/2} \int_0^u \int_0^u \alpha(\alpha+1) \frac{dt\, ds}{(k-u+s+t)^{\alpha+2}}\, du,$$

where $\alpha = 1 - 2\mu$. Then, as in [6], we observe

$$\{(s,t,u) \in \mathbf{R}^3 : 0 \le s, t \le u \le \tfrac{1}{2}\}$$
$$= \{(s,t,u) \in \mathbf{R}^3 : \max(s,t) \le u \le \tfrac{1}{2},\ 0 \le s, t \le \tfrac{1}{2}\},$$

so that

$$\begin{aligned}
&\sum_{k=n+1}^{\infty} \int_0^{1/2} \int_0^u \int_0^u \alpha(\alpha+1) \frac{dt\, ds}{(k-u+s+t)^{\alpha+2}}\, du \\
&\quad = \int_0^{1/2} \int_0^{1/2} \sum_{k=n+1}^{\infty} \int_{\max(s,t)}^{1/2} \frac{\alpha(\alpha+1)}{(k-u+s+t)^{\alpha+2}}\, du\, ds\, dt \\
&\quad \le \int_0^{1/2} \int_0^{1/2} \frac{1}{2} \sum_{k=n+1}^{\infty} \int_{\max(s,t)}^{1-\max(s,t)} \frac{\alpha(\alpha+1)}{(k-u+s+t)^{\alpha+2}}\, du\, ds\, dt \\
&\quad = 1/2 \int_0^{1/2} \int_0^{1/2} \sum_{k=n+1}^{\infty} \int_{k-1+\max(s,t)}^{k-\max(s,t)} \frac{\alpha(\alpha+1)}{(u+s+t)^{\alpha+2}}\, du\, ds\, dt \\
&\quad < \frac{1}{2} \int_0^{1/2} \int_0^{1/2} \sum_{k=n+1}^{\infty} \int_{k-1+\max(s,t)}^{k+\max(s,t)} \frac{\alpha(\alpha+1)}{(u+s+t)^{\alpha+2}}\, du\, ds\, dt \\
&\quad < \frac{1}{2} \int_0^{1/2} \int_0^{1/2} \int_{n+(1/2)(s+t)}^{\infty} \frac{\alpha(\alpha+1)}{(u+s+t)^{\alpha+2}}\, du\, ds\, dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{9(1-\alpha)}(2(n+\tfrac{3}{4})^{1-\alpha} - (n+\tfrac{3}{2})^{1-\alpha} - n^{1-\alpha}) \\
&= \frac{2n^{1-\alpha}}{9(1-\alpha)}\left(2\left(1+\frac{3}{4n}\right)^{1-\alpha} - \left(1+\frac{3}{2n}\right)^{1-\alpha} - 1\right) \\
&< \frac{\alpha}{8n^{1+\alpha}},
\end{aligned}
$$

where the last inequality is obtained using the formula

$$2F(x) - F(2x) - F(0) = -x^2 F''(\xi), \qquad 0 < \xi < 2x,$$

with $F(x) = (1+x)^{1-\alpha}$ and $x = 3/(4n)$. This proves (9).

Now

$$
\begin{aligned}
\left|\sum_{k=n+1}^{\infty} B_k(\theta)\right| &\le \frac{2}{\sin\theta/2}\int_0^{1/2} \sin u\theta \int_0^u \frac{\alpha}{(n+1+s)^{\alpha+1}}\, ds\, du \\
&= \frac{2}{\sin\theta/2}\int_0^{1/2} \sin u\theta \left(\frac{1}{(n+1)^\alpha} - \frac{1}{(u+n+1)^\alpha}\right) du \\
&\le \frac{2\theta}{\sin\theta/2}\int_0^{1/2} u \left(\frac{1}{(n+1)^\alpha} - \frac{1}{(u+n+1)^\alpha}\right) du \\
&< \frac{\alpha\theta}{12\sin(\theta/2)n^{\alpha+1}},
\end{aligned}
$$

which is (10). ■

Lemma 2. *For $n \in \mathbf{N}$, $x \in (0,1)$, let*

$$p_n := \frac{(x)_n}{n!\, n^{x-1}}, \qquad q_n := np_n, \qquad \text{and} \qquad r_n := n\left(\frac{1}{\Gamma(x)} - p_n\right).$$

Then, for all $n \in \mathbf{N}$, we have

$$p_n < p_{n+1} < \frac{1}{\Gamma(x)}, \qquad x \in (0,1), \tag{11}$$

$$q_{n+1} - q_n < q_{n+2} - q_{n+1} < \frac{1}{\Gamma(x)}, \qquad x \in [0.5, 0.7], \tag{12}$$

$$0 < r_n < r_{n+1} < \frac{x(1-x)}{2}\frac{1}{\Gamma(x)}, \qquad x \in [0.5, 0.7]. \tag{13}$$

Proof. Inequality $p_n < p_{n+1}$, for $n \in \mathbf{N}$ and $0 < x < 1$, reduces to

$$\left(\frac{n}{n+1}\right)^{1-x} < \frac{x+n}{n+1},$$

and this is a consequence of the elementary inequality $a^r < 1 + r(a-1)$, valid for $0 < a, r < 1$. Now (11) follows from the well-known limit $\lim_{n\to\infty} p_n(x) = 1/\Gamma(x)$.

Our next aim is to prove

$$\lim_{n\to\infty} r_n = \frac{x(1-x)}{2}\frac{1}{\Gamma(x)}, \tag{14}$$

which, by the way, also implies

$$\lim_{n\to\infty}(q_{n+1}-q_n) = \frac{1}{\Gamma(x)}\ .$$

Since

$$\begin{aligned} p_n &= \frac{x(x+1)\dots(x+n-1)}{1.2\dots(n-1)}n^{-x} \\ &= x\exp\left\{x\left(\sum_{k=1}^{n-1}\frac{1}{k}-\log n\right)\right\}\prod_{k=1}^{n-1}\left\{\left(1+\frac{x}{k}\right)e^{-x/k}\right\}, \end{aligned}$$

and

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{k=1}^{\infty}\left\{\left(1+\frac{x}{k}\right)e^{-x/k}\right\},$$

where γ is Euler's constant (see [1, p. 3]), we find

$$\begin{aligned} \log\frac{1}{\Gamma(x)} - \log p_n &= -x\left(\sum_{k=1}^{n-1}\frac{1}{k}-\log n-\gamma\right) \\ &\quad + \sum_{k=n}^{\infty}\left\{\log\left(1+\frac{x}{k}\right)-\frac{x}{k}\right\}. \end{aligned}$$

Using the asymptotic formulas

$$\sum_{k=1}^{n-1}\frac{1}{k}-\log n-\gamma = -\frac{1}{2n}-\sum_{k=1}^{m-1}\frac{B_{2k}}{2kn^{2k}}-\frac{B_{2m}}{2mn^{2m}}\,u, \qquad 0<u<1,$$

where B_{2k} are the Bernoulli numbers (see [1, p. 627]), and

$$\sum_{k=n}^{\infty}\left\{\log\left(1+\frac{x}{k}\right)-\frac{x}{k}\right\} = -\frac{x^2}{2n}+O\left(\frac{1}{n^2}\right)$$

we obtain

$$\log\frac{1}{\Gamma(x)}-\log p_n = \frac{x(1-x)}{2}\frac{1}{n}+O\left(\frac{1}{n^2}\right),$$

which, together with the representation

$$r_n = p_n n\left\{\exp\left(\log\frac{1}{\Gamma(x)}-\log p_n\right)-1\right\},$$

gives (14).

To prove (12) it remains to show that $q_{n+2} - 2q_{n+1} + q_n > 0$ or, equivalently,

$$d_n d_{n+1} - 2d_n + 1 > 0, \tag{15}$$

where

$$d_n = \left(1 + \frac{x}{n}\right)\left(1 + \frac{1}{n}\right)^{1-x}.$$

An expansion gives

$$d_n = 1 + \frac{1}{n} + \tfrac{1}{2}x(1-x)\frac{1}{n^2} + \sum_{k=3}^{\infty}(x\sigma_{k-1} + \sigma_k)\frac{1}{n^k},$$

with

$$\sigma_k := \binom{1-x}{k} = (1-x)\frac{(-1)^{k+1}}{k!}(x)_{k-1}.$$

For $0.5 \le x \le 0.7$ and $n \ge 2$ fixed one easily deduces that the sequences

$$\tau_k = (-1)^{k+1}(x\sigma_{k-1} + \sigma_k)n^{-k}, \qquad k \ge 5,$$

are positive and monotonically decreasing. This implies

$$\sum_{k=3}^{4}(x\sigma_{k-1} + \sigma_k)\frac{1}{n^k} < d_n - 1 - \frac{1}{n} - \tfrac{1}{2}x(1-x)\frac{1}{n^2} < \sum_{k=3}^{5}(x\sigma_{k-1} + \sigma_k)\frac{1}{n^k}.$$

Then, a routine calculation yields (15) for $n \ge 2$. The case $n = 1$ is straightforward.

The monotonicity $r_n < r_{n+1}$ follows immediately from (12), so that (13) is a consequence of (14). ∎

Lemma 3. *Let*

$$\Delta_n := \frac{1}{n^{1-2\mu}}\left(\frac{1}{\Gamma(2\mu)} - \frac{(2\mu)_n}{n!\,n^{2\mu-1}}\right). \tag{16}$$

Then

$$\left|\sum_{k=n+1}^{\infty}\Delta_k \cos k\theta\right| < \frac{\mu(1-2\mu)}{\Gamma(2\mu)}\frac{1}{n^{1-2\mu}}$$

holds for $\pi/n \le \theta \le \pi/2$.

Proof. It follows from (11) that Δ_n is a positive and strictly decreasing sequence, so that

$$\left|\sum_{k=n+1}^{\infty}\Delta_k \cos k\theta\right| \le \frac{\Delta_{n+1}}{\sin\theta/2}. \tag{17}$$

It is clear that, for $\pi/n \le \theta \le \pi/2$, we have

$$\sin\frac{\theta}{2} \ge \sin\frac{\pi}{2n} \ge \frac{1}{n}.$$

This and (13) (for $x = 2\mu = 0.69155 < 0.7$) imply

$$\begin{aligned}\frac{\Delta_{n+1}}{\sin\theta/2} &\le n\Delta_{n+1} < (n+1)\Delta_{n+1}\\ &= \frac{1}{(n+1)^{1-2\mu}}(n+1)\left(\frac{1}{\Gamma(2\mu)} - \frac{(2\mu)_{n+1}}{(n+1)!(n+1)^{2\mu-1}}\right)\\ &< \frac{\mu(1-2\mu)}{\Gamma(2\mu)}\frac{1}{n^{1-2\mu}}.\end{aligned}$$

Together with (17) this completes the proof of Lemma 3. ∎

Lemma 4. *Let*

$$\begin{aligned}g(\theta) &:= \frac{1}{2^{2\mu}}\cos\mu(\pi-\theta)\left\{\frac{1}{(\sin\theta/2)^{2\mu}} - \frac{1}{(\theta/2)^{2\mu}}\right\},\\ h(\theta) &:= \frac{1}{\theta^{2\mu}}\left(\cos\mu(\pi-\theta) - \frac{\theta/2}{\sin\theta/2}\cos\mu\pi\right).\end{aligned}$$

Then, for $\pi/n \le \theta \le \pi/2$, *we have*

$$\begin{aligned}g(\theta) &> \left(\frac{\mu}{12}\pi^{2-2\mu}\cos\mu\pi\right)\frac{1}{n^{2-2\mu}} - \left(\frac{\mu}{960}\pi^{4-2\mu}\cos\mu\pi\right)\frac{1}{n^{4-2\mu}},\\ h(\theta) &> (\mu\pi^{1-2\mu}\sin\mu\pi)\frac{1}{n^{1-2\mu}} - \left\{\left(\frac{\mu^2}{2}+\frac{1}{24}\right)\pi^{2-2\mu}\cos\mu\pi\right\}\frac{1}{n^{2-2\mu}}\\ &\quad -\left(\frac{\mu^3}{6}\pi^{3-2\mu}\sin\mu\pi\right)\frac{1}{n^{3-2\mu}}\\ &\quad -\left\{\left(\frac{1}{16\sin 1} - \frac{7}{96}\right)\pi^{4-2\mu}\cos\mu\pi\right\}\frac{1}{n^{4-2\mu}}.\end{aligned}$$

Proof. Using the mean-value theorem and the Taylor series of $\sin x$ we easily get that

$$(18)\qquad \frac{1}{(\sin\theta/2)^{2\mu}} - \frac{1}{(\theta/2)^{2\mu}} > 2\mu\left(\frac{1}{6}\left(\frac{\theta}{2}\right)^{2-2\mu} - \frac{1}{120}\left(\frac{\theta}{2}\right)^{4-2\mu}\right) =: \omega(\theta), \quad \text{say.}$$

Since $\omega(\theta)$ is a positive and strictly increasing function, we have $\omega(\theta) \ge \omega(\pi/n)$. Using this together with $\cos\mu(\pi-\theta) > \cos\mu\pi$ and (18) we verify the inequality for g.

As for h, we use the Taylor expansion

$$\frac{x}{\sin x} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^{2n}-2}{(2n)!}B_{2n}x^{2n} = 1 + \tfrac{1}{6}x^2 + \tfrac{7}{360}x^4 + \cdots$$

for $|x| < \pi$, where B_{2n} are the Bernoulli numbers satisfying $(-1)^{n+1}B_{2n} > 0, n = 0, 1, 2, \ldots$. For $0 < \theta < \pi/2$, we have

$$\begin{aligned}\frac{\theta/2}{\sin\theta/2} &= 1 + \frac{\theta^2}{24} + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{2^{2n}-2}{(2n)!}B_{2n}\left(\frac{\theta}{2}\right)^{2n}\\ &< 1 + \frac{\theta^2}{24} + \frac{\theta^4}{16}\sum_{n=2}^{\infty}(-1)^{n+1}\frac{2^{2n}-2}{(2n)!}B_{2n}\\ &= 1 + \frac{\theta^2}{24} + \frac{\theta^4}{16}\left(\frac{1}{\sin 1} - \frac{7}{6}\right).\end{aligned}$$

This and the elementary inequalities,

$$\sin x > x - \frac{x^3}{3!}, \qquad \cos x > 1 - \frac{x^2}{2!},$$

yield

$$\begin{aligned}h(\theta) \;\geq\; &(\mu\sin\mu\pi)\theta^{1-2\mu} - \left\{\left(\frac{\mu^2}{2} + \frac{1}{24}\right)\cos\mu\pi\right\}\theta^{2-2\mu}\\ &-\left(\frac{\mu^3}{6}\sin\mu\pi\right)\theta^{3-2\mu} - \left\{\left(\frac{1}{16\sin 1} - \frac{7}{96}\right)\cos\mu\pi\right\}\theta^{4-2\mu}\\ =\; &\Omega(\theta), \quad \text{say}.\end{aligned}$$

An elementary calculation shows that $\Omega(\theta)$ is a concave function on $[\pi/n,\ \pi/2]$, so that

$$\Omega(\theta) \geq \min\{\Omega(\pi/2),\ \Omega(\pi/n)\} = \Omega(\pi/n).$$

This completes the proof. ∎

2.3. *The Case* $\pi/n \leq \theta \leq \pi/2$

We make use of the abbreviations introduced in the last subsection, and also write

$$\rho(\theta) := \frac{\theta/2}{\sin\theta/2}, \qquad f(\theta) := \frac{\cos\mu(\pi-\theta)}{(2\sin\theta/2)^{2\mu}}.$$

Then

$$S_n^{\mu}(\theta) = f(\theta) + \sum_{k=n+1}^{\infty}\Delta_k\cos k\theta - \frac{1}{\Gamma(2\mu)}\sum_{k=n+1}^{\infty}\frac{\cos k\theta}{k^{1-2\mu}}.$$

As above we set $\alpha := 1 - 2\mu$ and obtain, from [12, V, 2.29],

$$\begin{aligned}\sum_{k=1}^{n}\frac{\cos k\theta}{k^{\alpha}} \;=\; &-\rho(\theta)\Bigg\{\theta^{\alpha-1}\left[\int_0^{\theta/2}\frac{\cos t}{t^{\alpha}}\,dt - \int_0^{(n+1/2)\theta}\frac{\cos t}{t^{\alpha}}\,dt\right]\\ &+\sum_{k=1}^{n}\int_{k-1/2}^{k+1/2}\left(\frac{1}{u^{\alpha}} - \frac{1}{k^{\alpha}}\right)\cos u\theta\,du\Bigg\}.\end{aligned}$$

As in [6] we find

$$\int_{k-1/2}^{k+1/2} \left(\frac{1}{u^\alpha} - \frac{1}{k^\alpha} \right) \cos u\theta \, du = A_k(\theta) + B_k(\theta),$$

with $A_k(\theta)$, $B_k(\theta)$ as in Lemma 1. Since

$$\int_0^\infty \frac{\cos t}{t^\alpha} \, dt = \Gamma(1-\alpha) \, \sin \frac{\pi\alpha}{2}, \qquad 0 < \alpha < 1,$$

(see, e.g., [12, p. 190]), it follows from the above that

$$-\sum_{k=n+1}^{\infty} \frac{\cos k\theta}{k^\alpha} \; = \; \rho(\theta)\theta^{\alpha-1} \left(\int_0^{(n+1/2)\theta} \frac{\cos t}{t^\alpha} \, dt - \Gamma(1-\alpha) \, \sin \frac{\pi\alpha}{2} \right)$$
$$+ \rho(\theta) \left(\sum_{k=n+1}^{\infty} A_k(\theta) + \sum_{k=n+1}^{\infty} B_k(\theta) \right).$$

From [12, V. 2.29] we also have

$$\int_0^{(n+1/2)\theta} \frac{\cos t}{t^\alpha} \, dt \geq 0 \qquad \text{for} \quad \alpha \geq \alpha_0.$$

Hence

$$-\sum_{k=n+1}^{\infty} \frac{\cos k\theta}{k^\alpha} \geq \rho(\theta) \left(-\theta^{\alpha-1} \, \Gamma(1-\alpha) \, \sin \frac{\pi\alpha}{2} + \sum_{k=n+1}^{\infty} A_k(\theta) + \sum_{k=n+1}^{\infty} B_k(\theta) \right).$$

Using g, h from Lemma 4 we write

$$f(\theta) - \frac{\rho(\theta) \cos \mu\pi}{\theta^{2\mu}} = g(\theta) + h(\theta)$$

so that, putting everything together, we arrive at

$$S_n^\mu(\theta) \; \geq \; g(\theta) + h(\theta) + \sum_{k=n+1}^{\infty} \Delta_k \cos k\theta$$
$$+ \frac{\rho(\theta)}{\Gamma(2\mu)} \left(\sum_{k=n+1}^{\infty} A_k(\theta) + \sum_{k=n+1}^{\infty} B_k(\theta) \right).$$

Using $\rho(\theta) \leq \pi\sqrt{2}/4$ and applying the estimates derived in the lemmas above, this leads to

$$n^{1-2\mu} \, S_n^\mu(\theta) > C_0(\mu) - C_1(\mu)\frac{1}{n} - C_2(\mu)\frac{1}{n^2} - C_3(\mu)\frac{1}{n^3},$$

with

$$C_0 = \mu\pi^{1-2\mu}\sin\mu\pi - \frac{\mu(1-2\mu)}{\Gamma(2\mu)} = 0.35423\ldots,$$

$$C_1 = \frac{\pi\sqrt{2}}{4}\,\frac{1-2\mu}{\Gamma(2\mu)}\left(\frac{1}{8}+\frac{\pi\sqrt{2}}{24}\right) + \left(\frac{\mu^2}{2}-\frac{\mu}{12}+\frac{1}{24}\right)\pi^{2-2\mu}\cos\mu\pi = 0.23228\ldots,$$

$$C_2 = \frac{\mu^3}{6}\pi^{3-2\mu}\sin\mu\pi = 0.08566\ldots,$$

$$C_3 = \left(\frac{\mu}{960}+\frac{1}{16\sin 1}-\frac{7}{96}\right)\pi^{4-2\mu}\cos\mu\pi = 0.03532\ldots.$$

This clearly implies $S_n^{\mu}(\theta) > 0$, for $n \geq 3$ and $\pi/n \leq \theta \leq \pi/2$ and thus completes the proof of (3). ■

3. Proof of Theorem 1 and Related Considerations

The following result is due to Brickman et al. [4], see [7, Corollary 2.2] for a different proof.

Lemma 5. *Let $f \in \mathcal{S}_\lambda$. Then there exists a probability measure β on $[0, 2\pi)$ such that*

$$f(z) = \int_0^{2\pi} \frac{z}{(1-e^{-i\varphi}z)^{2-2\lambda}}\, d\beta(\varphi).$$

Proof of Theorem 1. It is an immediate consequence of this that, for any $f \in \mathcal{S}_\lambda$, $\lambda \geq \lambda_0$, and $z = e^{i\chi}$, we have

$$\begin{aligned}\operatorname{Re} s_n(f/z, e^{i\chi}) &= \int_0^{2\pi} \operatorname{Re} s_n\left(\frac{1}{(1-z)^{2-2\lambda}}, e^{i(\chi-\varphi)}\right) d\beta(\varphi)\\ &= \int_0^{2\pi} S_n^{1-\lambda}(\chi-\varphi)\, d\beta(\varphi) > 0,\end{aligned}$$

which, by the minimum principle, implies the whole result. ■

The actual starting point of the considerations in the present paper was the following theorem from [8]. Here we use the standard notation of $\prec$ for the subordination of analytic functions in the unit disk (see [8] for further details).

Lemma 6. *For $\frac{1}{2} \leq \lambda < 1$ and $f \in \mathcal{S}_\lambda$ we have*

$$\frac{s_n(f/z, z)}{f} \prec \frac{1}{f_\lambda}, \qquad n \in \mathbf{N},$$

with $f_\lambda(z) = z(1-z)^{2\lambda-2}$.

Note that Lemma 6 immediately implies (1). Numerical tests give very strong evidence for the following related conjecture:

Conjecture 1. *Let* $\lambda_0 \leq \lambda < 1$. *Then*

$$\sqrt{1-z}\, s_n(f_\lambda/z, z) \prec \sqrt{\frac{1+z}{1-z}}, \qquad n \in \mathbf{N}. \tag{19}$$

It is not hard to see that Conjecture 1 contains and strengthens (3) and that (19) cannot hold for any smaller λ. A slight modification of this conjecture, similar to the one used in [9], then gives the equivalent form

$$\sqrt{\frac{1-z}{1+z}}\; s_{2n+1}\left(\frac{1+z}{(1-z^2)^{2-2\lambda}}, z\right) \prec \sqrt{\frac{1-z}{1+z}}, \qquad n \in \mathbf{N}, \tag{20}$$

which is very reminiscent on the version of the famous Vietoris theorem discussed in [9]. Actually, Vietoris' theorem is equivalent to the case $\lambda = \frac{3}{4}$, and with $2n+1$ replaced by n. However, numerical experiments show that this replacement is not valid for any $\lambda < \frac{3}{4}$. We hope to come back to this problem in a forthcoming paper.

We mention in passing that Conjecture 1, once established, has a much stronger consequence, similar to the passage from (3) to Lemma 6.

Conjecture 2. *Let* $\lambda_0 \leq \lambda < 1$ *and* $f \in \mathcal{S}_\lambda$. *Furthermore, let* $F(z) := z\,{}_2F_1(\frac{1}{2}, 1, 2 - 2\lambda, z)$, *where* ${}_2F_1$ *denotes the hypergeometric function. Then*

$$\left|\arg\left(\frac{s_n(f,z)}{(F * f)(z)}\right)\right| \leq \frac{\pi}{4}, \qquad z \in \mathbf{D}. \tag{21}$$

Here $*$ *stands for the Hadamard product between analytic functions.*

Although (21), for $\lambda = \frac{3}{4}$ and $f = f_\lambda$, is essentially Lemma 6 (and that same λ), it is not clear what a sensible passage from Conjecture 2 to Lemma 6 as a whole might be. Much more study in that direction is required. In any case, Conjecture 2 contains and strengthens Theorem 1 (this is because $|\arg(F * f)(z)| \leq \pi/4$ in $\mathbf{D}$).

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S. Koumandos
Department of Mathematics and Statistics
The University of Cyprus
P.O. Box 20537
1678 Nicosia
Cyprus
skoumand@ucy.ac.cy

St. Ruscheweyh
Mathematisches Institut
Universität Würzburg
97074 Würzburg
Germany
ruscheweyh@mathematik.uni-wuerzburg.de