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PARTIAL DIFFERENTIAL EQUATIONS

The Cauchy Problem for Parabolic Equations in Zygmund Spaces

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It is known [1, Chap. 4] that if the coefficients and the right-hand side of a linear uniformly parabolic second-order equation belong to the anisotropic Hölder space $C^{m,\alpha}\left(\bar{D}\right)$, $m = 0, 1, \ldots$, $\alpha \in (0,1)$, in the layer $D = \mathbb{R}^n \times (0,T)$, $0 < T < \infty$, and the initial function belongs to $C^{m+2,\alpha}(\mathbb{R}^n)$, then the bounded solution of the Cauchy problem belongs to the space $C^{m+2,\alpha}\left(\bar{D}\right)$. This assertion fails for the anisotropic Lipschitz spaces $C^{m,1}\left(\bar{D}\right)$. More precisely, if the right-hand side belongs to the space $C^{m,1}\left(\bar{D}\right)$, then the solution does not necessarily belong (even locally) to the space $C^{m+2,1}(D)$ [2]. For the heat equation, we have shown in [3] that the solution belongs to the slightly wider anisotropic Zygmund space $H_{m+3}\left(\bar{D}\right)$. In this result, the right-hand side and the initial function are also assumed to belong to the Zygmund spaces $H_{m+1}\left(\bar{D}\right)$ and $H_{m+3}(\mathbb{R}^n)$, respectively.

In the present paper, we generalize these results to parabolic equations with variable coefficients that belong to the same anisotropic Zygmund space as the right-hand side. We show that, in this case, the solution also belongs to the Zygmund space. Thus we have constructed a smoothness scale for solutions of the Cauchy problem, which is a complete analog of the scale of Hölder spaces for integer values of the smoothness exponent.

1. DEFINITIONS AND NOTATION

We introduce the following notation:

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \qquad |x| = \left(x_1^2 + \cdots + x_n^2\right)^{1/2}, \quad k = (k_1, \ldots, k_n), \quad |k| = k_1 + \cdots + k_n.$$

We set $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$, $\partial_{ij} = \partial_i\partial_j$, and $\partial_x^k = \partial_1^{k_1}\partial_2^{k_2}\ldots\partial_n^{k_n}$. The Cauchy problem is considered in the layer $D = \mathbb{R}^n \times (0,T)$, $0 < T < \infty$.

For a domain $\Omega \subset D$, we set $H_0(\Omega) = L_\infty(\Omega)$ with the norm $|f|_{0,\Omega} = \operatorname{vrai}\sup_{\Omega} |f|$. For positive integer a, we need the anisotropic Zygmund spaces $H_a\left(\bar{\Omega}\right)$. Let

$$\Delta_x f(x,t) = f(x + \Delta x, t) - f(x,t),$$
$$\Delta_x^2 f(x,t) = f(x + 2\Delta x, t) - 2f(x + \Delta x, t) + f(x,t).$$

The differences $\Delta_t f$ and $\Delta_t^2 f$ are defined in a similar way.

We set

$$[f]_{1,\Omega} = \sup_{\Omega} \frac{|\Delta_x^2 f(x,t)|}{|\Delta x|} + \sup_{\Omega} \frac{|\Delta_t f(x,t)|}{|\Delta t|^{1/2}}$$

and

$$\langle f\rangle_{\beta,\Omega} = \sup_{\Omega} \frac{\left|\Delta_t^\beta f(x,t)\right|}{|\Delta t|^{\beta/2}},$$

$\beta = 1, 2$. From now on, the least upper bound is taken over differences in which all points lie in the domain Ω.

We set

$$[f]_{a,\Omega}=\sum_{|k|+2s=a-1}\left[\partial_x^k\partial_t^s f\right]_{1,\Omega}$$

for integer $a\ge 2$,

$$\langle f\rangle_{a,\Omega}=\sum_{|k|+2s=a-2}\left\langle\partial_x^k\partial_t^s f\right\rangle_{2,\Omega}$$

for $a\ge 3$, and

$$|f|_{a,\Omega}=\sum_{|k|+2s\le a-1}\left|\partial_x^k\partial_t^s f\right|_{0,\Omega}+[f]_{a,\Omega}+\langle f\rangle_{a,\Omega}$$

for positive integer a. By $H_a\left(\bar\Omega\right)$ we denote the space of functions f defined in Ω such that the derivatives $\partial_x^k\partial_t^s f$, $|k|+2s<a$, exist and $|f|_{a,\Omega}$ is finite.

Functions f defined in a domain $Q\subset\mathbb{R}^n$ can be treated as functions $\tilde f$ in the domain $\Omega=Q\times(0,T)\subset\mathbb{R}^{n+1}$ independent of t. This permits one to define isotropic Zygmund spaces $H_a(Q)$ with the use of the above-introduced notation; namely,

$$H_a(Q)=\left\{f:\ Q\to\mathbb{R}\mid |f|_{a,Q}=\left|\tilde f\right|_{a,\Omega}<\infty\right\}.$$

The Zygmund spaces are obtained from the corresponding Lipschitz spaces if one replaces the first-order differences in the definition by second-order differences. For example, for the function $f\in H_1([0,1])$, we have $|\Delta_x^2 f(x)|\le C|\Delta x|$. In this case, the first difference satisfies the inequality [4, p. 135]

$$|\Delta_x f(x)|\le C|\Delta x|(|\ln|\Delta x||+1). \tag{1}$$

This estimate is sharp, as shown by the example $f(x)=x\ln x\in H_1([0,1])$. On the other hand, the function $|x-1/2|\ln|x-1/2|$ satisfies inequality (1) but does not belong to the space $H_1([0,1])$. Thus functions belonging to Zygmund spaces cannot be characterized by their first modulus of continuity.

Now we define weighted anisotropic Zygmund spaces $H_a^{(b)}(D)$ in the layer D. We set

$$b_+=\max(b,0),\qquad |f|_{0,D}^{(b)}=\operatorname*{vrai\,sup}_D t^{b_+/2}|f|.$$

For positive integer a and for integer $b\ge -a$, we set

$$\begin{aligned}
[f]_{a,D}^{(b)}&=\sum_{|k|+2s=a-1}\sup_{(x,t)\in D}t^{(a+b)/2}\frac{\left|\Delta_x^2\partial_x^k\partial_t^s f(x,t)\right|}{|\Delta x|}\\
&\quad+\sum_{|k|+2s=a-1}\sup_{\substack{(x,t)\in D,\\ 0<\Delta t<T-t}}t^{(a+b)/2}\frac{\left|\Delta_t\partial_x^k\partial_t^s f(x,t)\right|}{|\Delta t|^{1/2}},\\
\langle f\rangle_{1,D}^{(b)}&=\sup_{\substack{(x,t)\in D,\\ 0<\Delta t<T-t}}t^{(1+b)/2}\frac{\left|\Delta_t f(x,t)\right|}{|\Delta t|^{1/2}},\\
\langle f\rangle_{a,D}^{(b)}&=\sum_{|k|+2s=a-2}\sup_{\substack{(x,t)\in D,\\ 0<\Delta t<(T-t)/2}}t^{(a+b)/2}\frac{\left|\Delta_t^2\partial_x^k\partial_t^s f(x,t)\right|}{|\Delta t|}\quad\text{if}\quad a\ge 2,\\
|f|_{a,D}^{(b)}&=\sum_{|k|+2s\le a-1}\left|\partial_x^k\partial_t^s f\right|_{0,D}^{(|k|+2s+b)}+[f]_{a,D}^{(b)}+\langle f\rangle_{a,D}^{(b)}\quad\text{if}\quad b\ge 0,\\
|f|_{a,D}^{(b)}&=|f|_{-b,D}+\sum_{-b<|k|+2s\le a-1}\left|\partial_x^k\partial_t^s f\right|_{0,D}^{(|k|+2s+b)}+[f]_{a,D}^{(b)}+\langle f\rangle_{a,D}^{(b)}\quad\text{if}\quad b<0.
\end{aligned}\tag{2}$$

For nonnegative integer a and integer $b \geq -a$, by $H_a^{(b)}(D)$ we denote the space of functions f defined in D such that there exist all derivatives $\partial_x^k \partial_t^s f$, $|k| + 2s < a$, and $|f|_{a,D}^{(-b)}$ is finite.

2. THE CAUCHY PROBLEM

In the layer D, we consider a second-order parabolic operator

$$Lu = u_t - a_{ij}(x,t)\partial_{ij}u - b_i(x,t)\partial_i u - c(x,t)u \tag{3}$$

whose coefficients satisfy the uniform parabolicity condition

$$(\exists \lambda > 0) \quad \lambda|\xi|^2 \leq a_{ij}(x,t)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2, \qquad (x,t) \in \bar{D}, \qquad \xi \in \mathbb{R}^n, \tag{4}$$

and the conditions

$$a_{ij}, b_i, c \in H_m\left(\bar{D}\right), \qquad |a_{ij}|_{m,D}, |b_i|_{m,D}, |c|_{m,D} \leq A \tag{5}$$

for some positive integer m.

Consider the Cauchy problem

$$Lu = f \quad \text{in} \quad D, \qquad u|_{t=0} = \psi. \tag{6}$$

Theorem 1. *Let m be a positive integer, let the coefficients of the operator* (3) *satisfy conditions* (4) *and* (5), *and let $f \in H_m\left(\bar{D}\right)$ and $\psi \in H_{m+2}(\mathbb{R}^n)$. Then the bounded solution u of the Cauchy problem* (6) *belongs to $H_{m+2}\left(\bar{D}\right)$; moreover,*

$$|u|_{m+2,D} \leq C(n,m,T,\lambda,A)\left(|f|_{m,D} + |\psi|_{m+2,\mathbb{R}^n}\right).$$

Proof. Consider the case $m = 1$. First, suppose that the lower-order coefficients are lacking in the operator: $Lu = u_t - a_{ij}(x,t)\partial_{ij}u$. Let us prove the assertion of the theorem in the layer $D_\tau = \mathbb{R}^n \times (0,\tau)$, where τ is a sufficiently small positive number to be specified below. The assertion of the theorem in the entire layer D is then obtained by a "step-by-step argument."

If a function $u(x,t)$ is a solution of the Cauchy problem (6) in D_τ, then the function $\tilde{u}(x,t) = u\left(\tau^{1/2}x, \tau t\right)$ is a solution of the problem

$$\partial_t \tilde{u} - \tilde{a}_{ij}\partial_{ij}\tilde{u} = \tilde{f} \quad \text{in} \quad D_1, \qquad \tilde{u}|_{t=0} = \tilde{\psi},$$

where $\tilde{a}_{ij}(x,t) = a_{ij}\left(\tau^{1/2}x, \tau t\right)$, $\tilde{f}(x,t) = \tau^{-1} f\left(\tau^{1/2}x, \tau t\right)$, and $\tilde{\psi}(x) = \psi\left(\tau^{1/2}x\right)$. Furthermore, $[\tilde{a}_{ij}]_{1,D_1} \leq \tau^{1/2}A$. Moreover, the space $H_1(\mathbb{R}^n)$ is continuously embedded in the Hölder space $C^{1/2}(\mathbb{R}^n)$ [5, p. 274]; consequently,

$$\begin{aligned}
[\tilde{a}_{ij}(\cdot,t)]_{1/2,\mathbb{R}^n} &= \sup_{x \in \mathbb{R}^n,\ |\Delta x| > 0} \frac{|\Delta_x \tilde{a}_{ij}(x,t)|}{|\Delta x|^{1/2}} = \tau^{1/4}\left[a_{ij}(\cdot,\tau t)\right]_{1/2,\mathbb{R}^n} \\
&\leq C\tau^{1/4}\left|a_{ij}\right|_{1,D_\tau} \leq CA\tau^{1/4}, \quad 0 \leq t \leq 1.
\end{aligned}$$

Therefore, to prove the inclusion $u \in H_3\left(\bar{D}_\tau\right)$, it suffices to justify the desired assertion in the layer D_1 assuming that the coefficients a_{ij} satisfy the uniform parabolicity condition (4) and vary slowly:

$$\left[a_{ij}\right]_{1,D_1} \leq \tau^{1/2}A, \qquad \left[a_{ij}(\cdot,t)\right]_{1/2,\mathbb{R}^n} \leq CA\tau^{1/4}, \quad 0 \leq t \leq 1. \tag{7}$$

Let us show that, for sufficiently small τ, the conditions $a_{ij} \in H_1\left(\bar{D}_1\right)$, together with conditions (4) and (7), imply the estimate

$$|u|_{3,D_1} \leq C(n,\lambda,A,\tau)\left(|Lu|_{1,D_1} + |u(\cdot,0)|_{3,\mathbb{R}^n}\right) \tag{8}$$

for any function $u \in H_3\left(\bar{D}_1\right)$. Then the solvability of the Cauchy problem in the space $H_3\left(\bar{D}_1\right)$ can be obtained from the solvability of the Cauchy problem for the heat equation in $H_3\left(\bar{D}_1\right)$ [3] by the method of continuation with respect to a parameter.

First, let us prove some auxiliary assertions.

Lemma 1. *Let $m \in \mathbb{N}$ and $f, g \in H_m\left(\bar{D}\right)$. Then $fg \in H_m\left(\bar{D}\right)$, and the estimate*

$$|fg|_{m,D} \le C|f|_{m,D}|g|_{m,D}$$

is valid.

Proof. The inequality

$$|h_1 h_2|_{l,\mathbb{R}^n} \le C\,|h_1|_{l,\mathbb{R}^n}\,|h_2|_{l,\mathbb{R}^n} \tag{9}$$

was proved in [6, p. 120 of the Russian translation] for functions h_1 and h_2 belonging to the isotropic Zygmund space $H_l\left(\mathbb{R}^n\right)$, $l \in \mathbb{N}$. The functions f and g can be continued to functions $\tilde{f}, \tilde{g} \in H_m\left(\mathbb{R}^{n+1}\right)$ such that $|\tilde{f}|_{m,\mathbb{R}^{n+1}} \le C|f|_{m,D}$ and $|\tilde{g}|_{m,\mathbb{R}^{n+1}} \le C|g|_{m,D}$ [5]. By using the estimate (9) for the derivatives $\tilde{f}\tilde{g}$ separately with respect to space and time variables [and by also using the estimate (9) for half-integer l, i.e., for Hölder spaces, in the latter case], we obtain the assertion of the lemma.

Let $\zeta \in C^\infty\left(\mathbb{R}^n\right)$, $0 \le \zeta(x) \le 1$, be a function such that $\zeta(x) = 1$ for $|x| \le 1$ and $\zeta(x) = 0$ for $|x| \ge 2$. We set $\zeta_R^\xi(x) = \zeta\left(R^{-1}(x-\xi)\right)$ for $R \ge 1$ and $\xi \in \mathbb{R}^n$ and $\zeta_R^Y(x,t) = \zeta_R^\xi(x)$ for $Y = (\xi, \chi) \in \bar{D}_1$, and by $G_R(Y) = \{(x,t) \in \mathbb{R}^{n+1} \mid |x-\xi| < R,\ 0 < t < 1\}$ we denote the cylinder in D_1 with axis passing through the point Y and with radius R.

Lemma 2. *Let $m \in \mathbb{N}$, $f \in H_m\left(\bar{D}_1\right)$, and $R \ge 1$. Then*

$$|f|_{m,D_1} \le 4 \sup_{Y \in D_1} \left|\zeta_R^Y f\right|_{m,D_1}.$$

Proof. First, let us verify the inequality

$$|f|_{m,D_1} \le 4 \sup_{Y \in D_1} |f|_{m,G_R(Y)}. \tag{10}$$

Indeed, if $2s + k \le m-1$, then $\left|\partial_t^s \partial_x^k f\right|_{0,D_1} = \sup_{Y \in D_1}\left|\partial_t^s \partial_x^k f\right|_{0,G_R(Y)}$. If $2s + k = m-1$ and the points of the difference $\Delta_x^2 \partial_t^s \partial_x^k f$ lie in some ball with radius 1 and center $Y \in D$, then $\left|\Delta_x^2 \partial_t^s \partial_x^k f\right| \le |\Delta x|\left|\partial_t^s \partial_x^k f\right|_{1,G_R(Y)}$. But if the difference step satisfies $|\Delta x| > 1$, then

$$\left|\Delta_x^2 \partial_t^s \partial_x^k f\right| \le 4\left|\partial_t^s \partial_x^k f\right|_{0,D_1} \le 4|\Delta x| \sup_{Y \in D_1} |f|_{0,G_R(Y)}.$$

The differences with respect to t can be considered in a similar but simpler way. The proof of inequality (10) is complete. It remains to note that $|f|_{m,G_R(Y)} \le |\zeta_R^Y f|_{m,D_1}$, since $\zeta_R^Y = 1$ in $G_R(Y)$. The proof of the lemma is complete.

Lemma 3. *Let $f, g \in H_1\left(\bar{D}_1\right)$, $R \ge 1$, and $Y \in D_1$. Then*

$$|fg|_{1,G_R(Y)} \le C|f|_{1,G_{2R}(Y)}|g|_{1,G_{2R}(Y)}.$$

Proof. For the difference with respect to t, we have

$$\begin{aligned}|\Delta_t[f(x,t)g(x,t)]| &\le |f(x,t)\Delta_t g(x,t)| + |g(x,t+\Delta t)\Delta_t f(x,t)| \\ &\le 2|\Delta t|^{1/2}|f|_{1,G_R(Y)}|g|_{1,G_R(Y)}\end{aligned}$$

for $(x,t), (x,t+\Delta t) \in G_R(Y)$. The difference with respect to x satisfies the inequality

$$\left|\Delta_x^2[f(x,t)g(x,t)]\right| \le 4|\Delta x||f|_{0,G_R(Y)}|g|_{0,G_R(Y)}$$

for $|\Delta x| \geq 1$. If $|\Delta x| < 1$, then, by using the estimate (1) and the representation

$$\Delta_x^2[f(x,t)g(x,t)] = f(x+\Delta x,t)\Delta_x^2 g(x,t) + g(x+\Delta x,t)\Delta_x^2 f(x,t) + \Delta_x f(x,t)\Delta_x g(x,t) + \Delta_x f(x+\Delta x,t)\Delta_x g(x+\Delta x,t),$$

we obtain the inequality

$$\begin{aligned}\left|\Delta_x^2[f(x,t)g(x,t)]\right| &\leq |\Delta x||f|_{0,G_R(Y)}[g]_{1,G_R(Y)} + |\Delta x||g|_{0,G_R(Y)}[f]_{1,G_R(Y)} \\ &\quad + C|\Delta x|^2(|\ln|\Delta x|| + 1)^2|f|_{1,G_{2R}(Y)}|g|_{1,G_{2R}(Y)} \\ &\leq C|\Delta x||f|_{1,G_{2R}(Y)}|g|_{1,G_{2R}(Y)}.\end{aligned}$$

The proof of the lemma is complete.

Let us now prove inequality (8). We fix $u \in H_3(\bar{D}_1)$, $Y = (\xi,\chi) \in D_1$, $R \geq 1$, and let ζ_R^Y be the function occurring in Lemma 2. By $L(Y) = \partial_t - a_{ij}(Y)\partial_{ij}$ we denote the parabolic operator with coefficients "frozen" at the point Y. By using estimates for solutions of the Cauchy problem in the Zygmund space $H_3(\bar{D}_1)$ for the heat equation [3], we obtain

$$\begin{aligned}\left|\zeta_R^Y u\right|_{3,D_1} &\leq C\left(\left|\zeta_R^\xi u(\cdot,0)\right|_{3,\mathbb{R}^n} + \left|L(Y)\left(\zeta_R^Y u\right)\right|_{1,D_1}\right) \\ &\leq C\left(\left|\zeta_R^\xi u(\cdot,0)\right|_{3,\mathbb{R}^n} + \left|\zeta_R^Y Lu\right|_{1,D_1} + \left|\zeta_R^Y(L-L(Y))u\right|_{1,D_1} + R^{-1}|u|_{3,D_1}\right) \\ &\leq C\left(|u(\cdot,0)|_{3,\mathbb{R}^n} + |Lu|_{1,D_1} + \sum_{i,j=1}^n \left|\zeta_R^Y\left(a_{ij} - a_{ij}(Y)\right)\partial_{ij}u\right|_{1,D_1} + R^{-1}|u|_{3,D_1}\right).\end{aligned} \tag{11}$$

The estimate for the first two terms on the right-hand side in inequality (11) has been obtained with the use of inequality (9) and Lemma 1, respectively, and the last term estimates all terms in $L(Y)\left(\zeta_R^Y u\right)$ containing derivatives of the function ζ_R^Y.

By applying Lemma 3 to the third term, we obtain

$$\begin{aligned}\left|\zeta_R^Y\left(a_{ij} - a_{ij}(Y)\right)\partial_{ij}u\right|_{1,D_1} &= \left|\zeta_R^Y\left(a_{ij} - a_{ij}(Y)\right)\partial_{ij}u\right|_{1,G_{4R}(Y)} \\ &\leq C\left|\zeta_R^Y\right|_{1,G_{8R}(Y)}\left|(a_{ij} - a_{ij}(Y))\right|_{1,G_{16R}(Y)}\left|\partial_{ij}u\right|_{1,G_{16R}(Y)} \\ &\leq C\left|\zeta_R^Y\right|_{1,D_1}\left|(a_{ij} - a_{ij}(Y))\right|_{1,G_{16R}(Y)}|u|_{3,D_1}.\end{aligned}$$

The norm $|\zeta_R^Y|_{1,D_1}$ is bounded above by a constant independent of $R \geq 1$ and Y. By using condition (7), we obtain

$$\begin{aligned}&\left|(a_{ij} - a_{ij}(Y))\right|_{0,G_{16R}(Y)} \leq C\left[a_{ij}(\cdot,\chi)\right]_{1/2,\mathbb{R}^n}\tau^{1/4}R^{1/2} + [a_{ij}]_{1,D_1} \leq CA\left(\tau^{1/4}R^{1/2} + \tau^{1/2}\right), \\ &\left[(a_{ij} - a_{ij}(Y))\right]_{1,G_{16R}(Y)} = [a_{ij}]_{1,G_{16R}(Y)} \leq [a_{ij}]_{1,D_1} \leq A\tau^{1/2}.\end{aligned}$$

Thus we have estimated the third term in (11), and inequality (11) acquires the form

$$\left|\zeta_R^Y u\right|_{3,D_1} \leq C\left(|u(\cdot,0)|_{3,\mathbb{R}^n} + |Lu|_{1,D_1} + \left(\tau^{1/4}R^{1/2} + \tau^{1/2} + R^{-1}\right)|u|_{3,D_1}\right),$$

where C is independent of R, Y, and τ. By using Lemma 2, we obtain

$$|u|_{3,D_1} \leq C\left(|u(\cdot,0)|_{3,\mathbb{R}^n} + |Lu|_{1,D_1} + \left(\tau^{1/4}R^{1/2} + \tau^{1/2} + R^{-1}\right)|u|_{3,D_1}\right).$$

By choosing first R large enough and then τ small enough to ensure that the coefficient of $|u|_{3,D_1}$ on the right-hand side in the last inequality is equal to $1/2$, we obtain inequality (8).

It remains to prove the assertion of the theorem for $m = 1$ for the case in which the operator L contains lower-order terms $b_i, c \in H_1(\bar{D})$. We take an $\alpha \in (0,1)$. Since $H_l(\bar{D}) \subset C^{l-1,\alpha}(\bar{D})$, $l \in \mathbb{N}$

[5, p. 274], where $C^{l-1,\alpha}\left(\bar{D}\right)$ are parabolic Hölder spaces, it follows that problem (6) is a Cauchy problem with coefficients and right-hand side in $C^{0,\alpha}\left(\bar{D}\right)$ and with the initial function in $C^{2,\alpha}\left(\mathbb{R}^n\right)$. Consequently [1, Chap. 4], $u \in C^{2,\alpha}\left(\bar{D}\right)$; moreover, the norm of this solution in the space $C^{2,\alpha}\left(\bar{D}\right)$ admits the estimate

$$|u|_{2+\alpha,D} \le C\left(|f|_{\alpha,D} + |\psi|_{2+\alpha,\mathbb{R}^n}\right) \le C\left(|f|_{1,D} + |\psi|_{3,\mathbb{R}^n}\right).$$

We set $L_0 u = u_t - a_{ij}\partial_{ij} u$. Then

$$L_0 u = f + b_i \partial_i u + cu = F \in H_1\left(\bar{D}\right), \qquad |F|_{1,D} \le C\left(|f|_{1,D} + |\psi|_{3,\mathbb{R}^n}\right).$$

By virtue of preceding considerations, $u \in H_3\left(\bar{D}\right)$ and

$$|u|_{3,D} \le C\left(|f|_{1,D} + |\psi|_{3,\mathbb{R}^n}\right).$$

The proof of the theorem for $m = 1$ is complete.

For $m \ge 2$, the proof goes by induction. Let the desired assertion be valid for some $m \ge 1$, and let the assumptions of the theorem with m replaced by $m+1$ hold. Then the derivatives $\partial_i u$, $i = 1, \ldots, n$, of the solution of problem (6) are the solutions of the Cauchy problems

$$Lv = F_i \quad \text{in} \quad D, \qquad v|_{t=0} = \partial_i \psi,$$

where the right-hand sides $F_i = \partial_i f + L\partial_i u - \partial_i(Lu)$ contain spatial derivatives of u of order ≤ 2, and, by the induction assumption,

$$F_i \in H_m\left(\bar{D}\right), \qquad |F_i|_{m,D} \le C\left(|\partial_i f|_{m,D} + |f|_{m,D} + |\partial_i \psi|_{m+2,\mathbb{R}^n}\right).$$

This, together with the induction assumption, implies that

$$\partial_i u \in H_{m+2}\left(\bar{D}\right), \qquad |\partial_i u|_{m+2,D} \le C\left(|f|_{m+1,D} + |\psi|_{m+3,D}\right).$$

Since the derivatives with respect to t can be expressed via derivatives with respect to x and the derivatives of the right-hand side, we have the desired assertion. The proof of the theorem is complete.

Now consider the Cauchy problem with less smooth data. The right-hand side of the equation is assumed to satisfy the Zygmund condition locally, and the weighted space containing it is chosen so as to ensure that the solution belongs to the spaces $H_1\left(\bar{D}\right)$ or $H_2\left(\bar{D}\right)$, which are analogs of the Hölder parabolic spaces $C^{\alpha}\left(\bar{D}\right)$ and $C^{1+\alpha}\left(\bar{D}\right)$ for $\alpha = 1$. In this case, the highest derivatives of the solution belong to the same space as the right-hand side.

Theorem 2. *Suppose that* $l = 0, 1$, $f \in H_1^{(l)}(D)$, $\psi \in H_{2-l}\left(\mathbb{R}^n\right)$, *and the coefficients of the operator* L *satisfy conditions* (4) *and* (5) *for* $m = 1$. *Then the bounded solution* u *of the Cauchy problem* (6) *belongs to the space* $H_3^{(l-2)}(D)$, *and*

$$|u|_{3,D}^{(l-2)} \le C(n, m, T, \lambda, A)\left(|f|_{1,D}^{(l)} + |\psi|_{2-l,\mathbb{R}^n}\right).$$

In particular [see (2)], we have the inclusion $u \in H_{2-l}\left(\bar{D}\right)$.

This theorem was proved in [3] for the heat equation. The derivation of the inequality

$$|u|_{3,D}^{(l-2)} \le C\left(|Lu|_{1,D}^{(l)} + |u(\cdot, 0)|_{2-l,\mathbb{R}^n}\right)$$

for an arbitrary function $u \in H_3^{(2-l)}(D)$ can be performed in the same way as in Theorem 1. Then, by using the method of continuation with respect to a parameter, we obtain the assertion of the theorem.

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