ISSN 0012-2661, Differential Equations, 2008, Vol. 44, No. 10, pp. 1448–1459. © Pleiades Publishing, Ltd., 2008.
Original Russian Text © A.N. Konenkov, 2008, published in Differentsial'nye Uravneniya, 2008, Vol. 44, No. 10, pp. 1388–1398.

Solutions of Model Heat Problems in Zygmund Spaces

A. N. Konenkov
Moscow State University, Moscow, Russia

Received April 30, 2008

Abstract—We consider the first boundary value problem and the oblique derivative problem for the heat equation in the model case where the domain is a half-layer and the coefficients of the boundary operator in the oblique derivative problem are constant. Under the corresponding assumptions on the problem data, we show that the solutions belong to anisotropic Zygmund spaces, which "close" the scale of anisotropic Hölder spaces for integer values of the smoothness exponent.

DOI: 10.1134/S001226610810011X

The theory of boundary value problems in the anisotropic Hölder spaces $C^{m,\alpha}(\bar{\Omega})$, $m \geq 2$, $\alpha \in (0,1)$, for second-order linear parabolic equations has been comprehensively developed (e.g., see [1]). Solvability in the space $C^{1,\alpha}(\bar{\Omega})$ was proved in [2, 3].

For the solutions of boundary value problems for parabolic equations to be sufficiently smooth, in the closure of the domain Ω, it is necessary to require (in addition to the corresponding smoothness of the data) that certain compatibility conditions hold on the base Σ_0 of the "lateral" boundary of the domain Ω. In the anisotropic Hölder spaces, these conditions have the form [1, Chap. 4] of equality involving some combinations of derivatives of the problem data on Σ_0. These conditions follow from the fact that a smooth solution satisfies the equation and some of its differential corollaries in Ω. By continuity, these corollaries should also be valid on Σ_0. For a cylindrical domain, these conditions say that all derivatives that should be continuous of the boundary function with respect to the "time" variable t are expressed in a certain way via the right-hand side and the initial function on Σ_0.

We consider the first boundary value problem and the oblique derivative problem for the heat equation in the anisotropic Zygmund spaces $H_m(\bar{\Omega})$, $m \geq 2$, which are a "proper" (from the viewpoint of function theory) closure of the Hölder scale for integer values of the smoothness exponent. Both of these scales are special cases of the Nikol'skii spaces [4, Sec. 18].

We establish the smoothness of solutions in the model case where the domain is a half-layer D_+ and the coefficients of the boundary operator in the oblique derivative problem are constant. In this case, it turns out that, for even m in the first boundary value problem and for odd m in the oblique derivative problem, the compatibility conditions of the above-represented form are no longer sufficient for the solution to belong to $H_m(\bar{D}_+)$. In this case, we introduce an additional compatibility condition, which has a different character: some finite-difference functional of the problem data should be finite. The same functional occurs in the well-posedness estimate as an additional term as compared with the case of Hölder spaces.

For elliptic equations of arbitrary order with infinitely differentiable coefficients, boundary value problems in Zygmund spaces were considered by Triebel [5, Sec. 4.3.4]. He obtained estimates for the solutions of boundary value problems under the assumption that the uniqueness theorem holds.

In the present paper, we consider not only classical but also generalized solutions, for which the right-hand side belongs to $L_\infty(D_+)$. We show that, under appropriate conditions for other data, the solutions of these boundary value problems belong to the space

$$H_2(\bar{D}_+) \subset C^{1,\alpha}(\bar{D}_+)$$

for any $\alpha \in (0,1)$.

1. DEFINITIONS AND NOTATION

We introduce the notation

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n, \qquad x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, \qquad |x| = (|x_1|^2 + \dots + |x_n|^2)^{1/2},$$
$$P = (x, t) \in \mathbb{R}^{n+1}, \qquad |P|_1 = |x| + |t|^{1/2};$$

by $\bar{e}_i = (1, 0, \dots, 0)$ we denote the unit coordinate vectors in $\mathbb{R}^n$.

Let $k = (k_1, \dots, k_n)$ be a multi-index, $k_i \ge 0$, $i = 1, \dots, n$, $|k| = k_1 + \dots + k_n$. We set $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$, and $\partial_x^k = \partial_1^{k_1} \partial_2^{k_2} \dots \partial_n^{k_n}$; $\partial_\eta = \eta_1 \partial_1 + \eta_2 \partial_2 + \dots + \eta_n \partial_n$ is the derivative in the direction $\eta = (\eta_1, \eta_2, \dots, \eta_n)$, $\Delta = \sum_{i=1}^n \partial_i^2$, $\Delta' = \sum_{i=1}^{n-1} \partial_i^2$, and $L = \partial_t - \Delta$ is the heat operator. Let $D = \mathbb{R}^n \times (0, T)$, $0 < T < \infty$, be a layer in $\mathbb{R}^{n+1}$. We set

$$\Delta_x f(x) = \Delta_x^1 f(x) = f(x + \Delta x) - f(x), \qquad \Delta_x^2 f(x) = f(x + 2\Delta x) - 2f(x + \Delta x) + f(x).$$

In a similar way, we introduce the differences $\Delta_t^1 f$ and $\Delta_t^2 f$. In addition, we introduce the coordinate differences $\Delta_i(h) f(x) = f(x + h\bar{e}_i) - f(x)$.

We denote the Zygmund space of order $m \in \mathbb{N}$ in a domain $Q \subset \mathbb{R}^n$ by $H_m(\bar{Q})$ [5, p. 242 of the Russian translation].

We set $H_0(\Omega) = L_\infty(\Omega)$, where Ω is a domain in $\mathbb{R}^{n+1}$, with the norm $|f|_{0,\Omega} = \operatorname{vrai}\sup_{\Omega} |f|$. The definitions of anisotropic Zygmund spaces $H_m(\bar{\Omega})$ with norms $|f|_{m,\Omega}$ and $\underset{\circ}{H}_m(\bar{\Omega})$ with norms $\|f\|_{m,\Omega}$ as well as of the anisotropic Zygmund spaces $H_m^{(l)}(\Omega)$ can be found in [7].

Let $Z(x, t)$ be the fundamental solution of the heat equation,

$$Z(x, t) = \begin{cases} (4\pi t)^{-n/2} \exp\{-|x|^2/(4t)\} & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t \le 0. \end{cases} \tag{1}$$

For a function $\psi \in H_0(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$, we consider the Poisson potential

$$\Pi\psi(x, t) = \int_{\mathbb{R}^n} Z(x - y, t) \psi(y) \, dy.$$

The following assertion was proved in [6].

Theorem 1. *Let $m, l \ge 0$ be integers such that $m \ge l$. Then the mapping $\Pi : \psi \to \Pi\psi$ is a bounded operator from $H_l(\mathbb{R}^n)$ to $H_m^{-l}(D)$.*

In the layer D, for a function $f \in L_\infty(D)$, we consider the volume potential

$$Vf(x, t) = \int_D Z(P - Q) f(Q) \, dQ.$$

2. SOME PROPERTIES OF FUNCTIONS IN ZYGMUND SPACES

Here we prove some properties, to be used in forthcoming considerations, of functions in the anisotropic spaces $H_m(\mathbb{R}^n)$.

Theorem 2. *Let $m \in \mathbb{N}$ and $\psi \in H_m(\mathbb{R}^n)$. Then the inequality*

$$|\Delta_{l_1}(h_1) \Delta_{l_2}(h_2) \cdots \Delta_{l_r}(h_r) \psi(x)| \le C(n, r, m) |\psi|_{m,\mathbb{R}^n} |h_1 h_2 \cdots h_r|^{m/r} \tag{2}$$

holds for the coordinate differences for $r > m$.

Proof. If one of the h_i is zero, then the assertion of the theorem is obvious.

Let $h_j \neq 0$, $j = 1, \ldots, r$. We set $\gamma = |h_1 h_2 \cdots h_r|^{1/r}$ and $u(x,t) = \Pi\psi(x,t)$. Then

$$|\Delta_{l_1}(h_1)\cdots\Delta_{l_r}(h_r)\psi(x)| \leq |\Delta_{l_1}(h_1)\cdots\Delta_{l_r}(h_r)[\psi(x) - u(x,\gamma^2)]| + |\Delta_{l_1}(h_1)\cdots\Delta_{l_r}(h_r)u(x,\gamma^2)| = I_1 + I_2.$$

Let $|h_1| \leq |h_2| \leq \cdots \leq |h_r|$. By using estimates for solutions of the Cauchy problem [6], we have

$$\begin{aligned}
I_1 &= \Bigg|\Delta_{l_m}(h_m)\cdots\Delta_{l_r}(h_r)\int_0^{h_1}\cdots\int_0^{h_{m-1}}\partial_{l_1}\cdots\partial_{l_{m-1}}[u(x+\lambda_1\bar{e}_{l_1}+\cdots+\lambda_{m-1}\bar{e}_{l_{m-1}},0) \\
&\quad - u(x+\lambda_1\bar{e}_{l_1}+\cdots+\lambda_{m-1}\bar{e}_{l_{m-1}},\gamma^2)]\,d\lambda_1\ldots d\lambda_{m-1}\Bigg| \\
&\leq C|\psi|_{m,\mathbb{R}^n}\int_0^{|h_1|}\cdots\int_0^{|h_{m-1}|}\gamma\,d\lambda_1\ldots d\lambda_{m-1} \leq C\gamma|h_1\cdots h_{m-1}||\psi|_{m,\mathbb{R}^n} \\
&\leq C|\psi|_{m,\mathbb{R}^n}\gamma^m = C|h_1\cdots h_r|^{m/r}|\psi|_{m,\mathbb{R}^n}, \\
I_2 &= \Bigg|\int_0^{h_1}\cdots\int_0^{h_r}\partial_{l_1}\cdots\partial_{l_r}u(x+\lambda_1\bar{e}_{l_1}+\cdots+\lambda_r\bar{e}_{l_r},\gamma^2)\,d\lambda_1\ldots d\lambda_r\Bigg| \\
&\leq C|\psi|_{m,\mathbb{R}^n}\int_0^{|h_1|}\cdots\int_0^{|h_r|}\gamma^{m-r}\,d\lambda_1\ldots d\lambda_r \leq C|h_1\cdots h_r|\gamma^{m-r}|\psi|_{m,\mathbb{R}^n} \\
&= C|h_1\cdots h_r|^{m/r}|\psi|_{m,\mathbb{R}^n}.
\end{aligned}$$

The proof of the theorem is complete.

In particular, it follows from this theorem that if $h = \max|h_j|$, then the rth difference of a function $\psi \in H_m(\mathbb{R}^n)$ with respect to arbitrary coordinates does not exceed Ch^m for $r > m$.

Lemma 1. *Let* $\psi \in H_3(\mathbb{R}^n)$. *Then*

$$|\Delta_i^2(h)\partial_j\psi(x) - h\Delta_j(h)\partial_i^2\psi(x)| \leq C|\psi|_{3,\mathbb{R}^n}h^2, \qquad i,j = 1,\ldots,n.$$

Proof. Without loss of generality, we assume that $h > 0$. We have

$$|\Delta_i^2(h)\partial_j\psi(x) - h\Delta_j(h)\partial_i^2\psi(x)| \leq |\Delta_i^2(h)\partial_j\psi(x) - h^{-1}\Delta_j(h)\Delta_i^2(h)\psi(x)| + |h^{-1}\Delta_j(h)\Delta_i^2(h)\psi(x) - h\Delta_j(h)\partial_i^2\psi(x)| = I_1 + I_2.$$

Both terms can be estimated with the use of Theorem 2:

$$\begin{aligned}
I_1 &\leq |\Delta_i^2(h)\partial_j\psi(x) - h^{-1}\Delta_j(h)\Delta_i^2(h)\psi(x - h\bar{e}_j/2)| \\
&\quad + h^{-1}|\Delta_j(h)\Delta_i^2(h)[\psi(x - h\bar{e}_j/2) - \psi(x)]| = J_1 + J_2, \\
J_1 &= \frac{1}{h}\Bigg|\int_{-h/2}^{h/2}\Delta_i^2(h)[\partial_j\psi(x) - \partial_j\psi(x+\alpha\bar{e}_j)]\,d\alpha\Bigg| \\
&= \frac{1}{2h}\Bigg|\int_{-h/2}^{h/2}\Delta_i^2(h)[2\partial_j\psi(x) - \partial_j\psi(x+\alpha\bar{e}_j) - \partial_j\psi(x-\alpha\bar{e}_j)]\,d\alpha\Bigg| \\
&\leq \frac{C}{h}|\psi|_{3,\mathbb{R}^n}\int_{-h/2}^{h/2}h|\alpha|d\alpha = Ch^2|\psi|_{3,\mathbb{R}^n}, \qquad J_2 \leq Ch^2|\psi|_{3,\mathbb{R}^n},
\end{aligned}$$

$$I_2 = \frac{1}{h}\left|\int_0^h\int_0^h \Delta_j(h)[\partial_i^2\psi(x+(\alpha+\beta)\bar{e}_i) - \partial_i^2\psi(x)]\,d\alpha\,d\beta\right|$$
$$\le \frac{C}{h}|\psi|_{3,\mathbb{R}^n}\int_0^h\int_0^h h^{1/2}(\alpha+\beta)^{1/2}\,d\alpha\,d\beta = Ch^2|\psi|_{3,\mathbb{R}^n}.$$

The proof of the lemma is complete.

3. SOME PROPERTIES OF SOLUTIONS OF THE CAUCHY PROBLEM

Let us analyze properties of solutions of the Cauchy problem in anisotropic Zygmund spaces near the hyperplane $t = 0$. Throughout the following, we deal with bounded solutions. We give an explicit description of singularities providing a logarithmic growth of the derivatives as $t \to +0$. We restrict our considerations to the cases studied below. The results are then used for deriving compatibility conditions and for proving the solvability of model boundary value problems in Zygmund spaces.

Consider the Cauchy problem

$$Lu = 0 \quad \text{in} \quad D, \qquad u|_{t=0} = \psi \in H_1(\mathbb{R}^n). \tag{3}$$

Its solution u can be represented in the form of a Poisson potential, and $u \in H_1(\bar{D})$ [6]. The function ψ can be nowhere differentiable [9]. The first difference ψ satisfies only the estimate [8, p. 119]

$$|\Delta_x\psi(x)| \le C|\Delta x|(|\ln|\Delta x|| + 1).$$

The solution $u(x,t)$ satisfies the same estimate with respect to x in $\bar{D}$. The first derivative $\partial_i u$ can be unbounded in D; it satisfies the inequality $|\partial_i u(x,t)| \le C(|\ln t| + 1)$ [6]. Both estimates are sharp. However, it turns out that the logarithmic singularity of the derivative $\partial_i u$ can be written out in closed form.

Lemma 2. *Let u be a solution of the Cauchy problem* (3). *Then*

$$|\partial_i u(x,t) - t^{-1/2}\Delta_i(t^{1/2})\psi(x)| \le C|\psi|_{1,\mathbb{R}^n}, \qquad i = 1,\dots,n, \qquad (x,t) \in D. \tag{4}$$

Proof. We have

$$|\partial_i u(x,t) - t^{-1/2}\Delta_i(t^{1/2})\psi(x)| \le |\partial_i u(x,t) - t^{-1/2}\Delta_i(t^{1/2})u(x,t)|$$
$$+ t^{-1/2}|\Delta_i(t^{1/2})[u(x,t) - \psi(x)]| = I_1 + I_2.$$

From the results in [6], for the Cauchy problem, we obtain

$$I_1 = \frac{1}{t^{1/2}}\left|\int_0^{t^{1/2}}[\partial_i u(x,t) - \partial_i u(x+\alpha\bar{e}_i,t)]\,d\alpha\right| = \frac{1}{t^{1/2}}\left|\int_0^{t^{1/2}}\int_0^{\alpha}\partial_{ii}u(x+(\alpha+\beta)\bar{e}_i,t)\,d\alpha\,d\beta\right|$$
$$\le \frac{C}{t^{1/2}}\int_0^{t^{1/2}}\int_0^{\alpha}\frac{1}{t^{1/2}}\,d\alpha\,d\beta = C,$$
$$I_2 = \frac{1}{t^{1/2}}\left|\int_0^t[\partial_\tau u(x+t^{1/2}\bar{e}_i,\tau) - \partial_\tau u(x,\tau)]\,d\tau\right|$$
$$\le \frac{1}{t^{1/2}}\int_0^t[|\partial_\tau u(x+t^{1/2}\bar{e}_i,\tau)| + |\partial_\tau u(x,\tau)|]\,d\tau \le \frac{C}{t^{1/2}}\int_0^t\frac{1}{\tau^{1/2}}\,d\tau = C.$$

The proof of the lemma is complete.

For example, if we set $n = 1$ in Lemma 2 and $\psi(x) = x \ln |x|$ in some neighborhood of zero (the function $x \ln |x|$ belongs to H_1 on any interval), then for the derivative $\partial_x u(0, t)$, we obtain the relation

$$\partial_x u(0,t) = t^{-1/2}\Delta_x(t^{1/2})\psi(0) + O(1) = (1/2)\ln t + O(1) \quad \text{for} \quad t \to +0.$$

Let the initial function ψ lie in $H_2(\mathbb{R}^n)$. Then $|\Delta_x^2 \psi(x)| \le C|\Delta x|^2(|\ln|\Delta x|| + 1)$. By Theorem 1, the same inequality holds for the second difference of the solution $\Delta_x^2 u(x,t)$ in $\bar{D}$, and the absolute value of the difference $u(x,t) - \psi(x)$ does not exceed $Ct(|\ln t| + 1)$.

For $h \in \mathbb{R}$, consider the finite-difference operator

$$\Lambda_h u(x,t) = \sum_{i=1}^{n} \Delta_i^2(h) u(x,t).$$

Lemma 3. *Let u be a solution of the Cauchy problem $Lu = 0$ in D, and let $u|_{t=0} = \psi \in H_2(\mathbb{R}^n)$. Then*

$$|u(x,t) - \psi(x) - \Lambda_{t^{1/2}}\psi(x)| \le C|\psi|_{2,\mathbb{R}^n}\, t, \qquad (x,t) \in D. \tag{5}$$

Proof. We have

$$u(x,t) - \psi(x) - \Lambda_{t^{1/2}}\psi(x) = \int_0^t [\partial_\tau u(x,\tau) - \partial_t u(x,t)]\, d\tau + [t\partial_t u(x,t) - \Lambda_{t^{1/2}} u(x,t)]$$
$$+ \Lambda_{t^{1/2}}[u(x,t) - \psi(x)] = I_1 + I_2 + I_3.$$

By using Theorems 1 and 2, we obtain

$$|I_1| = \left| \int_0^t \int_\tau^t \partial_{\lambda\lambda} u(x,\lambda)\, d\lambda\, d\tau \right| \le C \int_0^t \int_\tau^t \lambda^{-1}\, d\lambda\, d\tau = Ct,$$

$$|I_2| = \left| \int_0^{t^{1/2}} \int_0^{t^{1/2}} \sum_{i=1}^n [\partial_{ii} u(x,t) - \partial_{ii} u(x + (\alpha+\beta)\bar{e}_i, t)]\, d\alpha\, d\beta \right|$$
$$\le C \int_0^{t^{1/2}} \int_0^{t^{1/2}} |\alpha + \beta| t^{-1/2}\, d\alpha\, d\beta = Ct,$$

$$|I_3| = \left| \int_0^t \Lambda_{t^{1/2}} \partial_\tau u(x,\tau)\, d\tau \right|$$
$$\le \int_0^t \sum_{i=1}^n [|\partial_\tau u(x + t^{1/2}\bar{e}_i, \tau) - \partial_\tau u(x,\tau)| + |\partial_\tau u(x + 2t^{1/2}\bar{e}_i, \tau) - \partial_\tau u(x + t^{1/2}\bar{e}_i, \tau)|]\, d\tau$$
$$\le C \int_0^t t^{1/2}\tau^{-1/2} d\tau = Ct.$$

The proof of the lemma is complete.

Remark. Under the assumptions of Lemma 3, the derivative $\partial_t u$ can be unbounded; it satisfies the inequality $|\partial_t u(x,t)| \le C(|\ln t| + 1)$ in D. It follows from the estimates for I_2 and I_3 that the logarithmic singularity of $\partial_t u$ for $\psi \in H_2(\mathbb{R}^n)$ can be written out in closed form as well,

$$|\partial_t u(x,t) - t^{-1}\Lambda_{t^{1/2}}\psi(x)| \le C|\psi|_{2,\mathbb{R}^n}.$$

Now consider solutions of the Cauchy problem for the equation $Lu = f$ in D with the homogeneous initial condition $u|_{t=0} = 0$.

Lemma 4. *Let* $f \in H_1(D)$. *Then*

$$|\partial_i u(x,t) - t^{1/2}\Delta_i(t^{1/2})f(x,0)| \le C|f|_{1,D}t, \qquad i=1,\dots,n, \qquad (x,t)\in D.$$

Proof. We have

$$\begin{aligned}\partial_i u(x,t) - t^{1/2}\Delta_i(t^{1/2})f(x,0) &= \partial_i Vf(x,t) - t^{1/2}\Delta_i(t^{1/2})f(x,0)\\ &= (\partial_i Vf(x,t) - t\partial_i\Pi[f(\cdot,0)](x,t)) + t(\partial_i\Pi[f(\cdot,0)](x,t) - t^{-1/2}\Delta_i(t^{1/2})f(x,0))\\ &= I_1 + I_2.\end{aligned}$$

Let us estimate

$$\begin{aligned}I_1 &= \int_0^t (\partial_i\Pi[f(\cdot,\tau)](x,t-\tau) - \partial_i\Pi[f(\cdot,0)](x,t))\,d\tau\\ &= \int_0^t \partial_i\Pi[f(\cdot,\tau)-f(\cdot,0)](x,t-\tau)\,d\tau\\ &\quad+ \int_0^t (\partial_i\Pi[f(\cdot,0)](x,t-\tau) - \partial_i\Pi[f(\cdot,0)](x,t))\,d\tau = J_1 + J_2.\end{aligned}$$

By Theorem 1,

$$|J_1| \le C\int_0^t (t-\tau)^{-1/2}\tau^{1/2}\,d\tau = Ct$$

and

$$|J_2| = \left|\int_0^t\int_\tau^t \partial_i\partial_\lambda\Pi[f(\cdot,0)](x,\lambda)\,d\lambda\,d\tau\right| \le C\int_0^t\int_\tau^t \lambda^{-1}\,d\lambda\,d\tau = Ct.$$

The term I_2 satisfies the desired estimate by Lemma 2. The proof of the lemma is complete.

Lemma 5. *Let* $f \in H_2(D)$. *Then*

$$|\Delta u(x,t) - \Lambda_{t^{1/2}}f(x,0)| \le C|f|_{2,D}t.$$

Proof. Set $v(x,t) = \Pi[f(\cdot,0)](x,t)$. We have

$$\begin{aligned}|\Delta u(x,t) - \Lambda_{t^{1/2}}f(x,0)| &= |\Delta Vf(x,t) - \Lambda_{t^{1/2}}f(x,0)|\\ &\le |\Delta Vf(x,t) - (v(x,t)-f(x,0))| + |v(x,t)-f(x,0)-\Lambda_{t^{1/2}}f(x,0)|.\end{aligned}$$

The second term was considered in Lemma 3, and for the first term, we have

$$\begin{aligned}|\Delta Vf(x,t) - (v(x,t)-f(x,0))| &= \left|\int_0^t \sum_{i=1}^n \partial_i\Pi[\partial_i f(\cdot,\tau) - \partial_i f(\cdot,0)](x,t-\tau)\,d\tau\right|\\ &\le C\int_0^t (t-\tau)^{-1/2}\sum_{i=1}^n |\partial_i f(\cdot,\tau)-\partial_i f(\cdot,0)|_{0,\mathbb{R}^n}\,d\tau \le C\int_0^t (t-\tau)^{-1/2}\tau^{1/2}\,d\tau = Ct.\end{aligned}$$

The proof of the lemma is complete.

Corollary. *The inequality*

$$|\Delta_t(t)\partial_t u(x,0) - \Lambda_{t^{1/2}} f(x,0) - \Delta_t(t)f(x,0)| \le C|f|_{2,D}t \tag{6}$$

holds for $f \in H_2(D)$.

Note that the finite-difference operators occurring in Lemmas 2–5 can have other stencils in view of Theorem 2. For example, the estimate (4) remains valid under the replacement of $t^{-1/2}\Delta_i(t^{1/2})\psi(x)$ by $t^{-1/2}\Delta_i(t^{1/2})\psi(x+\bar{c}t^{1/2})$ for a given vector $\bar{c} \in \mathbb{R}^n$, since the difference between these expressions can be represented as a sum of second coordinate differences and

$$|t^{-1/2}\Delta_i(t^{1/2})[\psi(x) - \psi(x+\bar{c}t^{1/2})]| \le C|\psi|_{1,\mathbb{R}^n}$$

by virtue of (2). In addition, one can replace the operator $\Lambda_h\psi(x)$ by the central difference operator

$$\bar{\Lambda}_h\psi(x) = \sum_{i=1}^{n} \Delta_i^2(h)\psi(x - h\bar{e}_i),$$

which replaces the Laplace operator most "naturally."

4. THE FIRST BOUNDARY VALUE PROBLEM

In the domain $D_+ = D \cap \{x_n > 0\}$ with the lateral surface $\Sigma = \bar{D} \cap \{x_n = 0\}$, consider the problem

$$Lu = f \quad \text{in} \quad D_+, \qquad u|_\Sigma = \varphi, \qquad u|_{t=0} = \psi. \tag{7}$$

A generalized solution of this problem is defined as a bounded function $u \in C(\bar{D}_+)$ that satisfies the relation

$$\int\limits_{D_+} u(P)L^*\eta(P)\,dP = \int\limits_{D_+} f(P)\eta(P)\,dP \tag{8}$$

for any function $\eta \in C_0^\infty(\mathbb{R}^{n+1})$, $\operatorname{supp}\eta \subset D_+$, where $L^* = -\partial_t - \Delta$ is the adjoint of the heat operator, and satisfies the initial and boundary conditions.

Following [1, p. 363], we set $u^{(0)}(x) = \psi(x)$ and $u^{(l+1)}(x) = \Delta u^{(l)}(x) + \partial_t^l f(x,0)$ for $l \ge 0$, or, in closed form, $u^{(l)}(x) = \Delta^l\psi(x) + \sum_{j=0}^{l-1} \partial_t^j \Delta^{l-j-1} f(x,0)$.

In what follows, when discussing compatibility conditions, for simplicity, we use the following notation. For a function φ defined on Σ, we set $\bar{\Delta}_t\varphi = \Delta_t(t)\varphi(x',0) = \varphi(x',t) - \varphi(x',0)$. For a function g defined on the bottom base $B_0 = \bar{D}_+ \cap \{t = 0\}$ and for $h > 0$, we set $\Lambda_h g = \Lambda_h g(x',0)$.

For $\sigma > 0$, we say that the first boundary value problem satisfies the compatibility conditions of order σ if

(i) for $\sigma \notin \mathbb{N}$, one has $\partial_t^l\varphi|_{\Sigma_0} = u^{(l)}|_{\Sigma_0}$, $l = 0, 1, \ldots, [\sigma]$, where $[\sigma]$ is the integer part of σ;

(ii) for $\sigma \in \mathbb{N}$, one has $\partial_t^l\varphi|_{\Sigma_0} = u^{(l)}|_{\Sigma_0}$, $l = 0, 1, \ldots, \sigma - 1$, and, in addition,

$$K_1 = \sup_{0<t\le T} t^{-1}|\bar{\Delta}_t\varphi - \Lambda_{t^{1/2}}\psi|_{0,\Sigma_0} < \infty \quad \text{if} \quad \sigma = 1,$$

and

$$K_\sigma = \sup_{0<t\le T} t^{-1}|\bar{\Delta}_t\partial_t^{\sigma-1}\varphi - \Lambda_{t^{1/2}}u^{(\sigma-1)} - \bar{\Delta}_t\partial_t^{\sigma-2}f|_{0,\Sigma_0} < \infty \quad \text{if} \quad \sigma \ge 2.$$

For noninteger σ, we set $K_\sigma = 0$. We refer to K_σ as the *compatibility constant.*

Theorem 3. *Let* $m \ge 0$ *be an integer, let* $f \in H_m(\bar{D}_+)$, $\varphi \in H_{m+2}(\Sigma)$, *and* $\psi \in H_{m+2}(\bar{\mathbb{R}}^n_+)$, *and let the compatibility conditions of order* $(m+2)/2$ *be satisfied. Then there exists a unique solution* $u \in C(\bar{D}_+)$ *of the first boundary value problem* (7) (*a generalized solution for* $m = 0$); *moreover,* $u \in H_{m+2}(\bar{D}_+)$, *and*

$$|u|_{m+2,D_+} \le C(|f|_{m,D_+} + |\varphi|_{m+2,\Sigma} + |\psi|_{m+2,\mathbb{R}^n_+} + K_{(m+2)/2}). \tag{9}$$

Proof. Let $m = 0$. Let us prove the existence. The function $\psi \in H_2(\bar{\mathbb{R}}^n_+)$ can be extended as a bounded function $\tilde{\psi} \in H_2(\mathbb{R}^n)$. We also extend $f \in L_\infty(D_+)$ by zero to $f \in L_\infty(D)$. Therefore [6], $V\tilde{f} \in H_2(D)$, and by Theorem 1, $\Pi\tilde{\psi} \in H_2(D)$. The volume potential $V\tilde{f}$ is a generalized solution of the heat equation $Lv = f$ in D [10, p. 273]. By using these potentials, one can reduce the problem to the first boundary value problem with zero right-hand side and initial condition,

$$L\tilde{u} = 0 \quad \text{in} \quad D_+, \qquad \tilde{u}|_\Sigma = \tilde{\varphi}, \qquad \tilde{u}|_{t=0} = 0, \tag{10}$$

where $\tilde{\varphi} = \varphi - (V\tilde{f} + \Pi\tilde{\psi})|_\Sigma \in H_2(\Sigma)$. Let us show that if the first-order compatibility condition is satisfied, then $\tilde{\varphi} \in \overset{\circ}{H}_2(\Sigma)$, i.e., $|\tilde{\varphi}(x', t)| \le Kt$; moreover,

$$K \le C(|f|_{0,D_+} + |\psi|_{2,\mathbb{R}^n_+} + K_1).$$

Since, by assumption, $\varphi|_{\Sigma_0} = \psi|_{\Sigma_0}$, we have

$$\begin{aligned} |\tilde{\varphi}(x', t)| \le |\Pi\tilde{\psi}(x', 0, t) - \psi(x', 0) - \Lambda_{t^{1/2}}\psi(x', 0)| + |V\tilde{f}(x', 0, t)| \\ + |\bar{\Delta}_t\varphi(x', 0) - \Lambda_{t^{1/2}}\psi(x', 0)| \le Ct|\psi|_{2,\mathbb{R}^n_+} + t|f|_{0,D_+} + tK_1. \end{aligned}$$

The first term was considered in Lemma 3, and the estimate for the third term is given by the first-order compatibility condition.

The solution of problem (10) can be written out in closed form with the use of a double-layer potential: $\tilde{u} = 2W\tilde{\varphi}$. Therefore, $\tilde{u} \in H_2(\bar{D}_+)$ and $\|\tilde{u}\|_{2,\mathbb{R}_n^+} \le C\|\tilde{\varphi}\|_{2,\mathbb{R}_n^+}$, which implies the existence of a generalized solution of the original problem in $H_2(\bar{D}_+)$ with the desired estimate.

The uniqueness of the solution of problem (10) follows from the fact that a generalized solution of the homogeneous heat equation is locally an infinitely differentiable function satisfying the equation pointwise; therefore, $\tilde{u} \in C(\bar{D}_+) \cap C^{2,1}_{x,t}(D_+)$ is a classical solution of problem (10). The uniqueness of such a solution is well known.

Let $m \ge 1$. In this case, the right-hand side satisfies the Hölder condition; consequently, u is a classical solution of the first boundary value problem in $\bar{D}_+$. The functions $f \in H_m(\bar{D}_+)$ and $\psi \in H_{m+2}(\bar{\mathbb{R}}^n_+)$ can be continued as bounded functions $\tilde{f} \in H_m(\bar{D})$ and $\tilde{\psi} \in H_{m+2}(\mathbb{R}^n)$, respectively [4, p. 263]. From theorems on the smoothness of a volume potential and a Poisson potential [6], we find that the above-defined function $\tilde{u}$ is a solution of problem (10) with $\tilde{\varphi} \in H_{m+2}(\Sigma)$; moreover,

$$|\tilde{\varphi}|_{m+2,\Sigma} \le C(|f|_{m,D_+} + |\varphi|_{m+2,\Sigma} + |\psi|_{m+2,\mathbb{R}^n_+}).$$

Let us show that $\tilde{\varphi} \in H_{\circ m+2}(\Sigma)$. Consider the following two cases.

1. $(m+2)/2 \notin \mathbb{N}$. Then, by virtue of the compatibility conditions,

$$\partial_t^l \tilde{\varphi}|_{\Sigma_0} = 0, \qquad l = 0, 1, \ldots, (m+1)/2.$$

In addition,

$$\begin{aligned} |\partial_t^{(m+1)/2}\tilde{\varphi}(x', t)| = |\partial_t^{(m+1)/2}\tilde{\varphi}(x', t) - \partial_t^{(m+1)/2}\tilde{\varphi}(x', 0)| \\ \le Ct^{1/2}(|f|_{m,D_+} + |\varphi|_{m+2,\Sigma} + |\psi|_{m+2,\mathbb{R}^n_+}). \end{aligned}$$

Consequently,

$$|\tilde{\varphi}(x', t)| \le Ct^{(m+2)/2}[|f|_{m,D_+} + |\varphi|_{m+2,\Sigma} + |\psi|_{m+2,\mathbb{R}^n_+}],$$

and the theorem on the smoothness of a double-layer potential [7] implies the desired estimate for $\tilde{u} = 2W\tilde{\varphi}$.

2. $(m+2)/2 \in \mathbb{N}$. Then $\partial_t^l \tilde{\varphi}|_{\Sigma_0} = 0$, $l = 0, 1, \ldots, m/2$. By $\tilde{u}^{(l)}(x)$, $x \in \mathbb{R}^n$, we denote the function obtained by the replacement of the functions f and ψ in the definition of $u^{(l)}$ by $\tilde{f}$ and $\tilde{\psi}$, respectively. Note that $\tilde{u}^{(l)}(x) = u^{(l)}(x)$ for $x_n \ge 0$.

For the function $g \in H_3(\bar{D})$, the derivative $\partial_t Vg$ of the volume potential is a solution of the Cauchy problem $Lv = \partial_t g$, $v|_{t=0} = g|_{t=0}$. Therefore, $\partial_t Vg = V[\partial_t g] + \Pi[g(\cdot, 0)]$ in D. Let us differentiate the potential $V\tilde{f}$ l times, $l = 0, 1, \ldots, m/2-1$, with respect to t by successively using this formula. With the use of the relation $\partial_t \Pi g = \Pi \Delta g$, we obtain

$$\partial_t^l V\tilde{f} = V[\partial_t^l \tilde{f}] + \Pi\left[\sum_{j=0}^{l-1} \partial_t^{l-j-1} \Delta^j \tilde{f}(\cdot, 0)\right] = V[\partial_t^l \tilde{f}] + \Pi[\tilde{u}^{(l)} - \Delta^l \tilde{\psi}], \qquad 0 \le l \le \frac{m}{2} - 1. \tag{11}$$

Since $\Delta \tilde{u}^{((m-2)/2)}(x) = \tilde{u}^{(m/2)}(x) - \partial_t^{m/2-1} \tilde{f}(x, 0)$, we have

$$\partial_t^{m/2} V\tilde{f} = \partial_t \partial_t^{m/2-1} V\tilde{f} = \partial_t V[\partial_t^{m/2-1}\tilde{f}] + \Pi[\tilde{u}^{(m/2)} - \Delta^{m/2}\tilde{\psi} - \partial_t^{m/2-1}\tilde{f}(\cdot, 0)].$$

For $\tilde{\varphi}$, we now have

$$\begin{aligned}
|\partial_t^{m/2}\tilde{\varphi}(x', t)| &= |\partial_t^{m/2}\varphi(x', t) - \partial_t^{m/2} V\tilde{f}(x', 0, t) - \partial_t^{m/2}\Pi\tilde{\psi}(x', 0, t)| \\
&= |\partial_t^{m/2}\varphi(x', t) - \partial_t V[\partial_t^{m/2-1}\tilde{f}](x', 0, t) - \Pi[\tilde{u}^{(m/2)} - \partial_t^{m/2-1}\tilde{f}(\cdot, 0)](x', 0, t)| \\
&\le |\bar{\Delta}_t \partial_t V[\partial_t^{m/2-1}\tilde{f}](x', 0, 0) - \Lambda_{t^{1/2}}\partial_t^{m/2-1} f(x', 0, 0) - \bar{\Delta}_t \partial_t^{m/2-1} f(x', 0, 0)| \\
&\quad + |\Pi[\tilde{u}^{(m/2)} - \partial_t^{m/2-1}\tilde{f}(\cdot, 0)](x', 0, t) - (u^{(m/2)}(x', 0) - \partial_t^{m/2-1} f(x', 0, 0)) \\
&\quad - \Lambda_{t^{1/2}}[u^{(m/2)}(x', 0) - \partial_t^{m/2-1} f(x', 0, 0)]| \\
&\quad + |\bar{\Delta}_t \partial_t^{m/2}\varphi(x', 0) - \Lambda_{t^{1/2}} u^{(m/2)}(x', 0) - \bar{\Delta}_t \partial_t^{m/2-1} f(x', 0, 0)| \\
&\le Ct(|f|_{m, D_+} + |\psi|_{m+2, \mathbb{R}_+^n} + K_{(m+2)/2}).
\end{aligned}$$

The first term satisfies the desired estimate by virtue of (6), and so does the second term by Lemma 3. Therefore, $\tilde{\varphi} \in \underset{\circ}{H}_{m+2}(\Sigma)$, $u = -2W\tilde{\varphi} \in H_{m+2}(\bar{D}_+)$, and

$$|\tilde{u}|_{m+2, D_+} \le C\|\tilde{\varphi}\|_{m+2, \Sigma} \le C(|f|_{m, D_+} + |\varphi|_{m+2, \Sigma} + |\psi|_{m+2, \mathbb{R}_+^n} + K_{(m+2)/2}).$$

The proof of the theorem is complete.

5. OBLIQUE DERIVATIVE PROBLEM

For the first-order boundary operator $M = \partial_\beta + \beta_0$, where $\beta = (\beta_1, \ldots, \beta_{n-1}, \beta_n)$, $\beta_n = 1$, in D_+, we consider the oblique derivative problem

$$Lu = f \quad \text{in} \quad D_+, \qquad Mu|_\Sigma = \varphi, \qquad u|_{t=0} = \psi. \tag{12}$$

A generalized solution of problem (12) is defined as a bounded function $u \in C_{x,t}^{1,0}(\bar{D}_+)$ that is a generalized solution of the equation $Lu = f$ in D_+ [see (8)] and satisfies the initial and boundary conditions.

For $h \in \mathbb{R}$, we introduce the finite-difference operator

$$M_h u(x, t) = \sum_{i=1}^n \beta_i \Delta_i(h) u(x, t).$$

For $\sigma > 0$, we say that the oblique derivative problem satisfies compatibility conditions of order σ if

(i) for $\sigma \notin \mathbb{N}$, one has $\partial_t^l \varphi|_{\Sigma_0} = Mu^{(l)}|_{\Sigma_0}$, $l = 0, 1, \ldots, [\sigma]$;

(ii) for $\sigma \in \mathbb{N}$, one has $\partial_t^l \varphi|_{\Sigma_0} = Mu^{(l)}|_{\Sigma_0}$, $l = 0, 1, \ldots, \sigma - 1$, and

$$K_\sigma = \sup_{0<t\le T} t^{-1} |\bar{\Delta}_t \partial_t^{\sigma-1}\varphi - t^{1/2} M_{t^{1/2}} u^{(\sigma)}|_{0, \Sigma_0} < \infty.$$

For noninteger σ, we adopt the convention that the compatibility constant K_σ is zero.

Theorem 4. *Let $m \geq 0$ be an integer, let $f \in H_m(\bar{D}_+)$, let $\varphi \in H_{m+1}(\Sigma)$, let $\psi \in H_{m+2}(\mathbb{R}^n_+)$, and let compatibility conditions of order $(m+1)/2$ be satisfied. Then there exists a unique solution $u \in C^{1,0}_{x,t}(\bar{D}_+)$ of the oblique derivative problem* (12)*; moreover, $u \in H_{m+2}(\bar{D}_+)$ and*

$$|u|_{m+2,D_+} \leq C(|f|_{m,D_+} + |\varphi|_{m+1,\Sigma} + |\psi|_{m+2,\mathbb{R}^n_+} + K_{(m+1)/2}). \tag{13}$$

Proof. The functions $f \in H_m(\bar{D}_+)$ and $\psi \in H_{m+2}(\bar{\mathbb{R}}^n_+)$ can be extended to bounded functions $\tilde{f} \in H_m(\bar{D})$ and $\tilde{\psi} \in H_{m+2}(\mathbb{R}^n)$, respectively [4, p. 263]. The theorems on the smoothness of a volume potential and a Poisson potential [6] imply that $V\tilde{f} \in H_{m+2}(\bar{D}_+)$ and $\Pi\tilde{\psi} \in H_{m+2}(\bar{D}_+)$, and by using these potentials, one can reduce problem (12) to a problem with zero right-hand side and initial condition,

$$L\tilde{u} = 0 \quad \text{in} \quad D_+, \qquad M\tilde{u}|_\Sigma = \tilde{\varphi}, \qquad \tilde{u}|_{t=0} = 0, \tag{14}$$

where $\tilde{\varphi} = \varphi - M(V\tilde{f} + \Pi\tilde{\psi})|_\Sigma \in H_{m+1}(\Sigma)$ and $|\tilde{\varphi}|_{m+1,\Sigma} \leq C(|f|_{m,D_+} + |\varphi|_{m+1,\Sigma} + |\psi|_{m+2,R^n_+})$. The uniqueness of a bounded solution of such a problem is well known.

Let us show that the inclusion

$$\tilde{\varphi} \in \underset{\circ}{H}_{m+1}(\Sigma) \tag{15}$$

holds under compatibility conditions of order $(m+1)/2$; i.e., $|\tilde{\varphi}(x',t)| \leq Kt^{(m+1)/2}$; moreover, $K \leq C(|f|_{m,D_+} + |\varphi|_{m+1,\Sigma} + |\psi|_{m+2,\mathbb{R}^n_+} + K_{(m+1)/2})$.

Consider the following two cases.

1. $(m+1)/2 \notin \mathbb{N}$. Then, by virtue of the compatibility conditions, $\partial^l_t\tilde{\varphi}|_{\Sigma_0} = 0$, $l = 0, 1, \ldots, m/2$. Moreover,

$$|\partial^{m/2}_t\tilde{\varphi}(x',t)| = |\partial^{m/2}_t\tilde{\varphi}(x',t) - \partial^{m/2}_t\tilde{\varphi}(x',0)| \leq Ct^{1/2}(|f|_{m,D_+} + |\varphi|_{m+1,\Sigma} + |\psi|_{m+2,\mathbb{R}^n_+}).$$

Consequently,

$$|\tilde{\varphi}(x',t)| \leq Ct^{(m+1)/2}(|f|_{m,D_+} + |\varphi|_{m+1,\Sigma} + |\psi|_{m+2,\mathbb{R}^n_+}).$$

2. $(m+1)/2 \in \mathbb{N}$. Then $\partial^l_t\tilde{\varphi}|_{\Sigma_0} = 0$, $l = 0, 1, \ldots, (m-1)/2$. By using formula (11), on Σ, we have

$$\begin{aligned}
\partial^{(m-1)/2}_t\tilde{\varphi} &= \partial^{(m-1)/2}_t\varphi - M\partial^{(m-1)/2}_t V\tilde{f} - M\partial^{(m-1)/2}_t\Pi\tilde{\psi} \\
&= \partial^{(m-1)/2}_t\varphi - MV[\partial^{(m-1)/2}_t\tilde{f}] - M\Pi[\tilde{u}^{((m-1)/2)} - \Delta^{(m-1)/2}\tilde{\psi}] - M\Pi[\Delta^{(m-1)/2}\tilde{\psi}] \\
&= \partial^{(m-1)/2}_t\varphi - MV[\partial^{(m-1)/2}_t\tilde{f}] - \Pi[M\tilde{u}^{((m-1)/2)}].
\end{aligned}$$

By taking into account the relations

$$\begin{gathered}
\partial^{(m-1)/2}_t\varphi|_{\Sigma_0} = u^{((m-1)/2)}|_{\Sigma_0}, \qquad u^{((m+1)/2)}(x',0) = \Delta u^{((m-1)/2)}(x',0) + \partial^{(m-1)/2}_t f(x',0,0), \\
\bar{\Delta}_t\Pi[M\tilde{u}^{(m-1)}](x',0,0) = \Pi[M\tilde{u}^{(m-1)}](x',0,t) - u^{(m-1)}(x',0),
\end{gathered}$$

for $\tilde{\varphi}$, we obtain

$$\begin{aligned}
|\partial^{(m-1)/2}_t\tilde{\varphi}(x',t)| &\leq |\Lambda_{t^{1/2}}(M-\beta_0)u^{((m-1)/2)}(x',0) - t^{1/2}M_{t^{1/2}}\Delta u^{((m-1)/2)}(x',0)| \\
&\quad + |\bar{\Delta}_t\Pi[M\tilde{u}^{((m-1)/2)}](x',0) - \Lambda_{t^{1/2}}Mu^{((m-1)/2)}(x',0)| \\
&\quad + |(M-\beta_0)V[\partial^{(m-1)/2}_t\tilde{f}](x',0,t) - t^{1/2}M_{t^{1/2}}\partial^{(m-1)/2}_t f(x',0,0)| \\
&\quad + |\beta_0\Lambda_{t^{1/2}}u^{((m-1)/2)}(x',0)| + |\beta_0 V[\partial^{(m-1)/2}_t\tilde{f}](x',0,t)| \\
&\quad + |\bar{\Delta}_t\partial^{(m-1)/2}_t\varphi(x',0) - t^{1/2}M_{t^{1/2}}u^{(m+1)/2}(x',0)| \\
&\leq Ct(|f|_{m,D_+} + |\varphi|_{m+1,\Sigma} + |\psi|_{m+2,\mathbb{R}^n_+} + K_{(m+1)/2}).
\end{aligned}$$

The desired inequality for the first three terms follows from Lemmas 1, 3, and 4, respectively. This completes the proof of the inclusion (15).

Let us show that the solvability of the oblique derivative problem (14) is a consequence of the solvability of the Cauchy problem and the first boundary value problem for the heat equation. We construct the desired solution in closed form. First, note that the solvability of the Cauchy problem

$$Lu = f \in H_m(\bar{D}), \qquad u|_{t=0} = \psi \in H_{m+2}(\mathbb{R}^n)$$

in the space $H_{m+2}(\bar{D})$ [6] implies its solvability for the operator $\mathcal{L}v = \partial_t v - a_{ij}\partial_{ij}v - b_i\partial_i v - cv$ with constant coefficients.

On Σ, we introduce the parabolic operator

$$L'v = \partial_t v - \left(\Delta' v + \sum_{i,j=1}^{n-1} \beta_i\beta_j\partial_{ij}v + 2\beta_0\sum_{i=1}^{n-1}\beta_i\partial_i v + \beta_0^2 v\right)$$

with parabolicity constant (that is, the least eigenvalue of the leading coefficient matrix) $\lambda = 1$. Consider the Cauchy problem

$$L'v = \tilde{\varphi} \quad \text{in} \quad \Sigma, \qquad v|_{\Sigma_0} = 0.$$

Its solution satisfies the relations $v \in H_{m+3}(\Sigma)$ [6] and

$$|v|_{m+3,\Sigma} \le C|\tilde{\varphi}|_{m+1,\Sigma} \le C(|f|_{m,D_+} + |\varphi|_{m+1,\Sigma} + |\psi|_{m+2,\mathbb{R}^n_+}).$$

Moreover,

$$|v(x',t)| \le Ct|\tilde{\varphi}|_{0,\Sigma\cap\{\tau\le t\}} \le Ct^{m+3}(|f|_{m,D_+} + |\varphi|_{m+1,\Sigma} + |\psi|_{m+2,\mathbb{R}^n_+} + K_{(m+1)/2}).$$

Consequently, $v \in \underset{\circ}{H}_{m+3}(\Sigma)$ and

$$\|v\|_{m+3,\Sigma} \le C(|f|_{m,D_+} + |\varphi|_{m+1,\Sigma} + |\psi|_{m+2,\mathbb{R}^n_+} + K_{(m+1)/2}). \tag{16}$$

Now let w be a solution of the first boundary value problem

$$Lw = 0 \quad \text{in} \quad D_+, \qquad w|_\Sigma = v, \qquad w|_{t=0} = 0.$$

The compatibility conditions of order $(m+3)/2$ are satisfied with compatibility constant that does not exceed the right-hand side of inequality (16). It follows from Theorem 3 that $w \in \underset{\circ}{H}_{m+3}(\bar{D}_+)$ and

$$\|w\|_{m+3,D_+} \le C\|v\|_{m+3,\Sigma} \le C(|f|_{m,D_+} + |\varphi|_{m+1,\Sigma} + |\psi|_{m+2,\mathbb{R}^n_+} + K_{(m+1)/2}).$$

Now consider the first-order operator $\tilde{M} = \partial_{\tilde{\beta}} - \beta_0$, where $\tilde{\beta} = (-\beta', 1)$, and set

$$\tilde{u} = \tilde{M}w \in \underset{\circ}{H}_{m+2}(\bar{D}_+).$$

Let us show that $\tilde{u}$ is the desired solution of problem (14). By construction, $\tilde{u}$ satisfies the equation $L\tilde{u} = 0$ in D_+ and the initial condition $\tilde{u}|_{t=0} = 0$. For $M\tilde{u}$, we have

$$M\tilde{u} = M\tilde{M}w = \partial_{nn}w - \sum_{i,j=1}^{n-1}\beta_i\beta_j\partial_{ij}w - 2\beta_0\sum_{i=1}^{n-1}\beta_i\partial_i w - \beta_0^2 w.$$

Since $\partial_{nn}w|_\Sigma = \partial_t v - \Delta' v$, it follows that

$$M\tilde{u}|_\Sigma = \partial_t v - \left(\Delta' v + \sum_{i,j=1}^{n-1}\beta_i\beta_j\partial_{ij}v + 2\beta_0\sum_{i=1}^{n-1}\beta_i\partial_i v + \beta_0^2 v\right) = L'v = \tilde{\varphi}.$$

The proof of the theorem is complete.

ACKNOWLEDGMENTS

The author is grateful to E.I. Moiseev and E.A. Baderko for discussion of the results.

REFERENCES

1. Ladyzhenskaya, O.A., Solonnikov, V.A., and Ural'tseva, N.N., *Lineinye i kvazilineinye uravneniya parabolicheskogo tipa* (Linear and Quasilinear Equations of Parabolic Type), Moscow: Nauka, 1968.
2. Baderko, E.A., *Differ. Uravn.*, 1992, vol. 28, no. 1, pp. 17–23.
3. Baderko, E., *Math. Methods Appl. Sci.*, 1997, vol. 20, pp. 449–459.
4. Besov, O.V., Il'in, V.P., and Nikol'skii, S.M., *Integral'nye predstavleniya funktsii i teoremy vlozheniya* (Integral Representation of Functions and Embedding Theorems), Moscow: Nauka, 1996.
5. Triebel, H., *Theory of Function Spaces*, Basel: Birkhäuser, 1983. Translated under the title *Teoriya funktsional'nykh prostranstv*, Moscow: Mir, 1986.
6. Konenkov, A.N., *Differ. Uravn.*, 2005, vol. 41, no. 6, pp. 820–831.
7. Konenkov, A.N., *Differ. Uravn.*, 2007, vol. 43, no. 8, pp. 1106–1115.
8. Timan, A.F., *Teoriya priblizheniya funktsii deistvitel'nogo peremennogo* (Theory of Approximations to Functions of a Real Variable), Moscow: Gosudarstv. Izdat. Fiz.-Mat. Lit., 1960.
9. Novikov, I.Ya. and Stechkin, S.B., *Uspekhi Mat. Nauk*, 1998, vol. 53, no. 6, pp. 53–128.
10. Vladimirov, V.S., *Uravneniya matematicheskoi fiziki* (Equations of Mathematical Physics), Moscow: Nauka, 1987.