ISSN 0012-2661, Differential Equations, 2007, Vol. 43, No. 8, pp. 1132–1141. © Pleiades Publishing, Ltd., 2007.
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PARTIAL DIFFERENTIAL EQUATIONS

Smoothness of the Double Layer Heat Potential in Zygmund Spaces

A. N. Konenkov
Moscow State University, Moscow, Russia

Received April 17, 2006

DOI: 10.1134/S0012266107080113

The solvability of boundary value problems for second-order parabolic equations in the anisotropic Hölder spaces $C^{l,\alpha}\left(\bar{\Omega}\right)$, $l \geq 2$, $0 < \alpha < 1$, is well studied (e.g., see [1, Chap. 4]). The solvability in the space $C^{1,\alpha}\left(\bar{\Omega}\right)$ was established in [2, 3].

In the present paper, we study the first boundary value problem for the heat equation in the anisotropic Zygmund spaces $H_l\left(\bar{\Omega}\right)$, which are an analog of the spaces $C^{l,\alpha}\left(\bar{\Omega}\right)$ for $\alpha = 0$. Zygmund spaces can be obtained from the corresponding Lipschitz spaces by the replacement of first-order differences in the definition of the norm by second-order differences. The solvability of the Cauchy problem in Zygmund spaces was proved in [4]. Boundary value problems for elliptic equations in Zygmund spaces were considered in [5, Subsec. 4.3.4].

We consider the model case; namely, the domain is cylindrical, its base is a half-space, and the right-hand side of the equation and the initial function are assumed to be zero. It is known that, in this case, the solution of the first boundary value problem can be represented as a double layer potential. We show that if the density of the potential belongs to H_l on the "lateral surface" of the domain, then the double layer potential belongs to $H_l\left(\bar{\Omega}\right)$, which supplements the smoothness scale of the double layer potential in the Hölder spaces $C^{l+\alpha}\left(\bar{\Omega}\right)$, $0 < \alpha < 1$ [1, 6, 7]. We also consider the double layer potential in Zygmund spaces with weights naturally coordinated with the smoothness of the density. As a corollary, we obtain estimates for the derivatives of solutions of the first boundary value problem.

1. DEFINITIONS AND NOTATION

We introduce the following notation:

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \qquad x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1},$$
$$|x| = \left(|x_1|^2 + \cdots + |x_n|^2\right)^{1/2}, \qquad P = (x,t) \in \mathbb{R}^{n+1}, \qquad |P|_1 = |x| + |t|^{1/2};$$

$\bar{e}_i$, $i = 1, \ldots, n$, are unit coordinate vectors in $\mathbb{R}^n$. Let $k = (k_1, \ldots, k_n)$ be a multiindex, $k_i \geq 0$, $i = 1, \ldots, n$, $|k| = k_1 + \cdots + k_n$. We write $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$, $\partial_x^k = \partial_1^{k_1}\partial_2^{k_2}\ldots\partial_n^{k_n}$, $\Delta = \sum_{i=1}^n \partial_i^2$, and $\Delta' = \sum_{i=1}^{n-1} \partial_i^2$; $L = \partial_t - \Delta$ is the heat operator.

In the layer $D = \mathbb{R}^n \times (0,T)$, $T \leq +\infty$, we consider a domain $\Omega \subset D$ with boundary $\partial\Omega = B_0 \cup B_T \cup \Sigma$, where B_0 is a domain lying on the plane $t = 0$, B_T is a domain lying on the plane $t = T$ ($B_T = \varnothing$ for $T = +\infty$), and Σ is the "lateral surface." For $P = (x,t) \in \bar{\Omega}$, by $d(P)$ we denote the parabolic distance from the parabolic boundary $\mathscr{P} = B_0 \cup \Sigma$; namely,

$$d(P) = \inf_{Q \in \mathscr{P} \cap \{\tau \leq t\}} |P - Q|_1.$$

We set $\Delta_x f(x) = \Delta_x^1 f(x) = f(x + \Delta x) - f(x)$ and $\Delta_x^2 f(x) = f(x + 2\Delta x) - 2f(x + \Delta x) + f(x)$. In a similar way, we define the differences $\Delta_t^1 f$ and $\Delta_t^2 f$. In addition, we introduce the coordinate differences $\Delta_i(h) f(x) = f\left(x + h\bar{e}_i\right) - f(x)$.

We set $H_0(\Omega) = L_\infty(\Omega)$ with the norm $|f|_{0,\Omega} = \operatorname{vrai}\sup_\Omega |f|$. For the definition of the Zygmund spaces $H_a\left(\bar{Q}\right)$, $a \in \mathbb{N}$, for domains $Q \subset \mathbb{R}^n$ and of parabolic Zygmund spaces $H_a\left(\bar{\Omega}\right)$ for domains $\Omega \subset D$, see [4].

In the space $H_a\left(\bar{\Omega}\right)$, we consider the subspace $\mathring{H}_a\left(\bar{\Omega}\right) = \left\{f \in H_a\left(\bar{\Omega}\right) : \ |f(x,t)| \le Ct^a\right\}$ with the norm

$$\|f\|_{a,\Omega} = |f|_{a,\Omega} + \sup_{(x,t)\in\Omega} t^{-a/2}|f(x,t)|.$$

In the domain Ω, we introduce weighted anisotropic Zygmund spaces $H_a^{(b)}(\Omega)$. Set $b_+ = \max(b,0)$ and $|f|_{0,\Omega}^{(b)} = \operatorname{vrai}\sup_\Omega \left(d^{-b_+}+1\right)^{-1}|f|$. For positive integers a and integers $b \ge -a$, we set

$$[f]_{a,\Omega}^{(b)} = \sum_{|k|+2s=a-1} \sup\left(d_{\min}^{-(a+b)}+1\right)^{-1} \frac{\left|\Delta_x^2 \partial_x^k \partial_t^s f(x,t)\right|}{|\Delta x|}$$
$$+ \sum_{|k|+2s=a-1} \sup\left(d_{\min}^{-(a+b)}+1\right)^{-1} \frac{\left|\Delta_t \partial_x^k \partial_t^s f(x,t)\right|}{|\Delta t|^{1/2}},$$
$$\langle f\rangle_{1,\Omega}^{(b)} = \left(d_{\min}^{-(1+b)}+1\right)^{-1} \frac{|\Delta_t f(x,t)|}{|\Delta t|^{1/2}},$$
$$\langle f\rangle_{a,\Omega}^{(b)} = \sum_{|k|+2s=a-2} \sup\left(d_{\min}^{-(a+b)}+1\right)^{-1} \frac{\left|\Delta_t^2 \partial_x^k \partial_t^s f(x,t)\right|}{|\Delta t|} \quad \text{if} \quad a \ge 2,$$
$$|f|_{a,\Omega}^{(b)} = \begin{cases} \sum_{|k|+2s\le a-1} \left|\partial_x^k \partial_t^s f\right|_{0,\Omega}^{(|k|+2s+b)} + [f]_{a,\Omega}^{(b)} + \langle f\rangle_{a,\Omega}^{(b)} & \text{if} \quad b \ge 0 \\ |f|_{-b,\Omega} + \sum_{-b<|k|+2s\le a-1} \left|\partial_x^k \partial_t^s f\right|_{0,\Omega}^{(|k|+2s+b)} + [f]_{a,\Omega}^{(b)} + \langle f\rangle_{a,\Omega}^{(b)} & \text{if} \quad b < 0. \end{cases}$$

Here $d_{\min}$ stands for the minimum value of $d(P)$ for the points occurring in the differences. Here and throughout the following, we assume that if the lower summation index is less than the upper one, then the corresponding sum is absent.

For nonnegative integer a and integer $b \ge -a$, by $H_a^{(b)}(\Omega)$ we denote the space of functions f that are defined in Ω, have all derivatives $\partial_x^k \partial_t^s f$, where $|k|+2s < a$, and have a finite value of $|f|_{a,\Omega}^{(b)}$.

Let $Z(x,t)$ be the fundamental solution of the heat equation,

$$Z(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\left\{-|x|^2/(4t)\right\} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \le 0. \end{cases} \tag{1}$$

For Z, we have the inequality [1, p. 428]

$$\begin{aligned} &(\exists C_{k,s}, \quad c_{k,s} > 0, \quad \forall t > 0, \quad \forall x \in \mathbb{R}) \\ &\left|\partial_t^s \partial_x^k Z(x,t)\right| \le C_{k,s} t^{-(|k|+2s+n)/2} \exp\left\{-c_{k,s}|x|^2/t\right\}, \qquad |k| \ge 0, \qquad s \ge 0. \end{aligned} \tag{2}$$

For $h > 0$ and $x \ge 0$, we also use the inequality

$$\int_0^h\!\!\int_0^h (x+\alpha+\beta)^{-1} d\alpha\, d\beta \le \int_0^h\!\!\int_0^h (\alpha+\beta)^{-1} d\alpha\, d\beta = (2\ln 2)h. \tag{3}$$

For the function $\psi \in H_0\left(\mathbb{R}^n\right) = L_\infty\left(\mathbb{R}^n\right)$, we consider the Poisson potential

$$\Pi\psi(x,t) = \int_{\mathbb{R}^n} Z(x-y,t)\psi(y)dy. \tag{4}$$

We have the following assertion [4].

Theorem 1. *Let $m, l \geq 0$ be integers, $m \geq l$. Then the mapping $\Pi : \psi \to \Pi\psi$ is a bounded operator from the space $H_l(\mathbb{R}^n)$ to $H_m^{-l}(D)$.*

2. SOME PROPERTIES OF FUNCTIONS IN THE ZYGMUND SPACE WITH ONE VARIABLE

If $\theta \in H_1([0,T])$, then the first difference can be estimated only as [8, p. 119]

$$|\theta(t) - \theta(\tau)| \leq C|t-\tau|(|\ln|t-\tau|| + 1).$$

To derive sharp estimates for the double layer potential in Zygmund spaces, we need logarithm-free inequalities for θ. To this end, for a fixed $t \in (0,T]$, we construct a representation of the function $\theta(\tau)$ for $0 \leq \tau \leq t$ in the form of a sum of two terms, one of which admits a "good" estimate and the other is sufficiently smooth on $[0,t)$.

Lemma 1. *Let $\theta \in H_1([0,T])$ and $|\theta(t)| \leq Kt$ on $[0,T]$. Then the representation $\theta(\tau) = \theta_1(\tau) + \theta_2(\tau)$ is valid for any fixed $t \in (0,T]$ and for arbitrary $\tau \in [0,t]$, where $\theta_1 \in C([0,t])$ and $\theta_2 \in C^2([0,t))$; moreover,*

$$|\theta_1(\tau)| \leq C|\theta|_{1,[0,t]}(t-\tau), \tag{5}$$

$$|\theta_2(0)| \leq C|\theta|_{1,[0,t]}t, \tag{6}$$

$$|\theta_2'(0)| \leq CK, \tag{7}$$

$$|\theta_2''(\tau)| \leq C|\theta|_{1,[0,t]}(t-\tau)^{-1}. \tag{8}$$

The constants occurring in the estimates are independent of θ, t, τ, and T.

Proof. We construct the desired representation in closed form. We fix $t \in (0,T]$ and consider the function

$$\bar{\theta}_t(\tau) = \bar{\theta}(\tau) = \frac{2}{(t-\tau)^2}\int_\tau^t (\lambda - \tau)\left[2\theta\left(\frac{\tau+\lambda}{2}\right) - \theta(\lambda)\right]d\lambda, \qquad 0 < \tau < t.$$

We have

$$\begin{aligned}\left|\theta(\tau) - \bar{\theta}(\tau)\right| &= \frac{2}{(t-\tau)^2}\left|\int_\tau^t (\lambda-\tau)\left[\theta(\tau) - 2\theta\left(\frac{\tau+\lambda}{2}\right) + \theta(\lambda)\right]d\lambda\right| \\ &\leq \frac{2|\theta|_{1,[0,t]}}{(t-\tau)^2}\int_\tau^t (\lambda-\tau)^2 d\lambda = C|\theta|_{1,[0,t]}(t-\tau).\end{aligned} \tag{9}$$

One can assume that $\bar{\theta}(t) = \theta(t)$ and $\bar{\theta} \in C([0,t])$. By writing $\bar{\theta}$ in the form

$$\bar{\theta}(\tau) = \frac{2}{(t-\tau)^2}\left[2\int_{2\tau}^{t+\tau}(\lambda - 2\tau)\theta\left(\frac{\lambda}{2}\right)d\lambda - \int_\tau^t (\lambda-\tau)\theta(\lambda)d\lambda\right]$$

and by performing the differentiation, we obtain

$$\begin{aligned}\bar{\theta}'(\tau) &= \frac{4}{(t-\tau)^3}\int_\tau^t (\lambda-\tau)\left[2\theta\left(\frac{\tau+\lambda}{2}\right) - \theta(\lambda)\right]d\lambda \\ &\quad + \frac{2}{(t-\tau)^2}\left[-4\int_{2\tau}^{t+\tau}\theta\left(\frac{\lambda}{2}\right)d\lambda + 2(t-\tau)\theta\left(\frac{t+\tau}{2}\right) + \int_\tau^t \theta(\lambda)d\lambda\right],\end{aligned}$$

or, after simplification,

$$\bar{\theta}'(\tau) = \frac{2}{(t-\tau)^3} \int_\tau^t \left[(t-2\lambda+\tau)\theta(\lambda) + 2(t-\tau)\theta\left(\frac{t+\tau}{2}\right) - 4(t-\lambda)\theta\left(\frac{\lambda+\tau}{2}\right) \right] d\lambda.$$

The function $\bar{\theta}'$ does not necessarily have the derivative. We represent it as a sum of two terms, $\bar{\theta}'(\tau) = \mu_1(\tau) + \mu_2(\tau)$, one of which is bounded and the other can be differentiated once more on $[0,t)$. It is the second term that will be taken for the derivative of the desired function $\theta_2(\tau)$. More precisely, denote the last integrand by $\varkappa$, and let

$$\varkappa_1 = (t-\tau)\left[\left(\theta(\lambda) - 2\theta\left(\frac{\lambda+\tau}{2}\right) + \theta(\tau) \right) - \left(\theta(t) - 2\theta\left(\frac{t+\tau}{2}\right) + \theta(\tau) \right) \right],$$

$$\varkappa_2 = \varkappa - \varkappa_1 = (t-\tau)\theta(t) - 2(\lambda-\tau)\theta(\lambda) - 2(t-2\lambda+\tau)\theta\left(\frac{\lambda+\tau}{2}\right).$$

Set

$$\mu_i(\tau) = \frac{2}{(t-\tau)^3} \int_\tau^t \varkappa_i d\lambda, \qquad i=1,2, \qquad \tilde{\theta}(\tau) = -\int_\tau^t \mu_1(\lambda)d\lambda, \qquad \theta_2(\tau) = \theta(t) - \int_\tau^t \mu_2(\lambda)d\lambda,$$

and $\theta_1(\tau) = \theta(\tau) - \theta_2(\tau)$. Then, by construction, $\bar{\theta}(\tau) = \tilde{\theta}(\tau) + \theta_2(\tau)$ and $\theta_1(\tau) = \theta(\tau) - \bar{\theta}(\tau) + \tilde{\theta}(\tau)$.

Let us show that the functions θ_1 and θ_2 thus defined satisfy the desired conditions. We have

$$\left|\tilde{\theta}'(\tau)\right| = |\mu_1(\tau)| \le \frac{C|\theta|_{1,[0,t]}}{(t-\tau)^3} \int_\tau^t (t-\tau)[(\lambda-\tau)+(t-\tau)]d\lambda = C|\theta|_{1,[0,t]},$$

whence we obtain $|\tilde{\theta}(\tau)| \le C(t-\tau)|\theta|_{1,[0,t]}$. This estimate, together with (9), implies (5) and (6):

$$|\theta_1(\tau)| \le \left|\theta(\tau) - \bar{\theta}(\tau)\right| + \left|\tilde{\theta}(\tau)\right| \le C|\theta|_{1,[0,t]}(t-\tau),$$

$$|\theta_2(0)| \le |\theta(0)| + |\theta_1(0)| \le C|\theta|_{1,[0,t]}t.$$

The validity of inequality (7) readily follows from the formula

$$\theta_2'(0) = \mu_2(0) = \frac{2}{t^3} \int_0^t \left[t\theta(t) - 2\lambda\theta(\lambda) - 2(t-2\lambda)\theta\left(\frac{\lambda}{2}\right) \right] d\lambda$$

and the estimate $|\theta(t)| \le Kt$.

Let us prove the estimate (8). Arguing as in the derivation of the function $\bar{\theta}'(\tau)$, we obtain the relations

$$\theta_2''(\tau) = \frac{1}{(t-\tau)^4} \int_\tau^t \Bigg[2\theta(t)(t-\tau) - 4\Bigg\{ 6(t-\lambda)\theta\left(\frac{\lambda+\tau}{2}\right)$$

$$- (t-3\lambda+2\tau)\theta(\lambda) - (t-\tau)\left(2\theta(\tau) + \theta\left(\frac{t+\tau}{2}\right)\right)\Bigg\}\Bigg] d\lambda$$

$$= \frac{12}{(t-\tau)^4} \int_\tau^t (t-\lambda)\left[\theta(\lambda) - 2\theta\left(\frac{\lambda+\tau}{2}\right) + \theta(\tau) \right] d\lambda$$

$$-\frac{2}{(t-\tau)^4}\int_\tau^t\left[2(t-3\lambda+2\tau)\theta(\tau)+2(t-\tau)\theta\left(\frac{t+\tau}{2}\right)-(t-\tau)\theta(t)\right]d\lambda$$
$$-\frac{4}{(t-\tau)^3}\int_\tau^t\left[2\theta(\lambda)-2\theta\left(\frac{t+\tau}{2}\right)\right]d\lambda=I_1+I_2+I_3.$$

This, together with the inequalities

$$|I_1|\le\frac{C|\theta|_{1,[0,t]}}{(t-\tau)^4}\int_\tau^t(t-\lambda)(\lambda-\tau)d\lambda=\frac{C|\theta|_{1,[0,t]}}{t-\tau},$$
$$|I_2|=\frac{2}{(t-\tau)^2}\left|\theta(\tau)-2\theta\left(\frac{t+\tau}{2}\right)+\theta(t)\right|\le\frac{C|\theta|_{1,[0,t]}}{t-\tau},$$
$$|I_3|=\frac{4}{(t-\tau)^3}\left|\int_\tau^t\left[\theta(\lambda)-2\theta\left(\frac{t+\tau}{2}\right)+\theta(t+\tau-\lambda)\right]d\lambda\right|\le\frac{C|\theta|_{1,[0,t]}}{t-\tau},$$

implies the desired estimate (8) and completes the proof of the lemma.

3. DOUBLE LAYER POTENTIAL

In the domain $D_+=D\cap\{x_n>0\}$ with "lateral surface" $\Sigma=\bar{D}\cap\{x_n=0\}$, for the density φ, we consider the double layer potential

$$W\varphi(x,t)=-\int_\Sigma\partial_nZ\left(x'-y',x_n,t-\tau\right)\varphi\left(y',\tau\right)dy'\,d\tau.$$

Theorem 2. *Let $m\ge l\ge0$ be integers. Then the mapping $W:\varphi\to W\varphi$ is a bounded operator from the space $\mathring{H}_l(\Sigma)$ to $H_m^{-l}(D_+)$ and from $\underset{\circ}{H}_l(\Sigma)$ to $\underset{\circ}{H}_l(\bar{D}_+)$.*

Proof. Let $n\ge2$. The case $n=1$ can be reduced to the two-dimensional case if one considers the density $\varphi\left(x_1,t\right)=\varphi(t)$ independent of the space variable. By setting [see (4)]

$$u\left(x',t,\tau\right)=\Pi[\varphi(\cdot,\tau)]\left(x',t-\tau\right),$$

one can represent $W\varphi$ in the form

$$W\varphi(x,t)=-\int_0^t\partial_nZ\left(x_n,t-\tau\right)u\left(x',t,\tau\right)d\tau.$$

Here $Z\left(x_n,t\right)$ is the fundamental solution of the heat equation with one space variable. This representation permits one to use the properties of the Poisson potential for the analysis of the double layer potential.

In the study of double layer potentials, we repeatedly use the estimate

$$\int_0^t\frac{1}{(t-\tau)^{\alpha/2}}\exp\left\{-\frac{x_n^2}{t-\tau}\right\}d\tau\le\frac{C_\alpha}{|x_n|^{(\alpha-2)}},\qquad\alpha>2,\qquad x_n\ne0.$$

Let $\varphi \in H_0(\Sigma) = \underset{\circ}{H}_0(\Sigma)$. Then, by using Theorem 1, we obtain

$$\left|\partial_x^k W\varphi(x,t)\right| = \left|\int_0^t \partial_n^{k_n+1} Z\left(x_n, t-\tau\right) \partial_{x'}^{k'} u\left(x', t, \tau\right) d\tau\right| \le C\int_0^t \frac{1}{(t-\tau)^{(|k|+2)/2}} \exp\left\{-\frac{cx_n^2}{t-\tau}\right\} d\tau \le \frac{C}{x_n^{|k|}},$$

which implies the assertion of the theorem for $l = 0$. Here and throughout the following, we assume that the norm $\|\varphi\|_{l,\Sigma}$ is included in the constant: $C = C_0\|\varphi\|_{l,\Sigma}$.

Let $l = 1$. Since the potential $W\varphi$ satisfies the heat equation in D_+, it follows that the assertion of the theorem for $l = 1$ can be reduced to the estimates

$$|W\varphi(x,t)| \le Ct^{1/2}, \tag{10}$$

$$\left|\partial_x^k W\varphi(x,t)\right| \le Cx_n^{-(|k|-1)}, \qquad |k| \ge 2, \tag{11}$$

$$\left|\Delta_i^2 W\varphi(x,t)\right| \le C\left|\Delta x_i\right|, \qquad i = 1, 2, \ldots, n, \tag{12}$$

$$|\Delta_t W\varphi(x,t)| \le C|\Delta t|^{1/2}. \tag{13}$$

The first inequality follows form the fact that $|u\left(x', t, \tau\right)| \le |\varphi(\cdot, \tau)|_{0,\mathbb{R}^{n-1}} \le C\tau^{1/2}$:

$$|W\varphi(x,t)| \le C\int_0^t \left|\partial_n Z\left(x_n, t-\tau\right)\right| \tau^{1/2} d\tau \le Ct^{1/2}\int_0^t \left|\partial_n Z\left(x_n, t-\tau\right)\right| d\tau \le Ct^{1/2}.$$

Let us prove the estimate (11). Set

$$v\left(x', t, \tau\right) = \Pi[\varphi(\cdot, t)]\left(x', t-\tau\right), \qquad w\left(x', t, \tau\right) = \Pi[\varphi(\cdot, \tau) - \varphi(\cdot, t)]\left(x', t-\tau\right).$$

Then $u = v + w$. By Theorem 1, for v, we have

$$\left|\partial_\tau^s \partial_{x'}^{k'} v\left(x', t, \tau\right)\right| \le C_{s,k'}|\varphi|_{1,\Sigma}(t-\tau)^{-(2s+|k'|-1)/2}, \qquad 2s + |k'| \ge 2;$$

moreover,

$$\left|\partial_t^s \partial_{x'}^{k'} v\left(x', t, 0\right)\right| \le C_{s,k'} t^{-(2s+|k'|)/2}|\varphi(\cdot, t)|_{0,\mathbb{R}^{n-1}} \le C_{s,k'} t^{-(2s+|k'|-1)/2}\|\varphi\|_{1,\Sigma}, \qquad 2s + |k'| \ge 0.$$

Likewise, for w, we obtain

$$\left|\partial_{x'}^{k'} w\left(x', t, \tau\right)\right| \le C_{k'}(t-\tau)^{-|k'|/2}|\varphi(\cdot, \tau) - \varphi(\cdot, t)|_{0,\mathbb{R}^{n-1}} \le C_{k'}(t-\tau)^{-(|k'|-1)/2}, \qquad |k'| \ge 0.$$

By assumption, $|k| \ge 2$. Consider the following two cases.

1. Let $|k'| \le 1$. Then $k_n \ge 1$ and

$$\partial_x^k W\varphi(x,t) = -\int_0^t \partial_n^{k_n+1} Z\left(x_n, t-\tau\right)\left[\partial_{x'}^{k'} v\left(x', t, \tau\right) + \partial_{x'}^{k'} w\left(x', t, \tau\right)\right] d\tau = J_1 + J_2.$$

Let us estimate the integrals J_1 and J_2:

$$|J_1| = \left| \int_0^t \left[\partial_\tau \partial_n^{k_n-1} Z\left(x_n, t-\tau\right)\right] \partial_{x'}^{k'} v\left(x', t, \tau\right) d\tau \right|$$

$$= \left| \partial_n^{k_n-1} Z\left(x_n, t\right) \partial_{x'}^{k'} v\left(x', t, 0\right) + \int_0^t \partial_n^{k_n-1} Z\left(x_n, t-\tau\right) \partial_\tau \partial_{x'}^{k'} v\left(x', t, \tau\right) d\tau \right|$$

$$\le \frac{C}{t^{(|k|-1)/2}} \exp\left\{-\frac{c x_n^2}{t}\right\} + C \int_0^t \frac{1}{(t-\tau)^{(|k|+1)/2}} \exp\left\{-\frac{c x_n^2}{t-\tau}\right\} d\tau \le \frac{C}{x_n^{(|k|-1)}},$$

$$|J_2| \le C \int_0^t \frac{1}{(t-\tau)^{(|k|+1)/2}} \exp\left\{-\frac{c x_n^2}{t-\tau}\right\} d\tau \le \frac{C}{x_n^{(|k|-1)}}.$$

Consequently, the estimate (11) is valid in this case.

2. Let $|k'| \ge 2$. Then

$$\left|\partial_x^k W\varphi(x,t)\right| = \left| \int_0^t \partial_n^{k_n+1} Z\left(x_n, t-\tau\right) \partial_{x'}^{k'} u\left(x', t, \tau\right) d\tau \right|$$

$$\le C \int_0^t \frac{1}{(t-\tau)^{(|k|+1)/2}} \exp\left\{-\frac{c x_n^2}{t-\tau}\right\} d\tau \le \frac{C}{x_n^{(|k|-1)}}.$$

The proof of the estimate (11) is complete.

Let us prove the estimate (12). We have

$$\left|\Delta_i^2 W\varphi(x,t)\right| = \left|W\left[\Delta_i^2 \varphi\right](x,t)\right| \le C \left|\Delta_i^2 \varphi\right|_{0,\Sigma} \le C\left|\Delta x_i\right|$$

for $i = 1, 2, \ldots, n-1$; if $i = n$ and $\Delta x_n > 0$, then, by using inequality (3), we obtain

$$\left|\Delta_n^2 W\varphi(x,t)\right| = \left| \int_0^{\Delta x_n} \int_0^{\Delta x_n} \partial_{nn} W\varphi\left(x + (\alpha+\beta)\bar{e}_n, t\right) d\alpha\, d\beta \right|$$

$$\le C \int_0^{\Delta x_n} \int_0^{\Delta x_n} \frac{d\alpha\, d\beta}{x_n + \alpha + \beta} \le C \Delta x_n.$$

Now let us prove the estimate (13). Throughout the following, we assume that $\Delta t > 0$. We have

$$\Delta_t W\varphi(x,t) = \Delta_n^2\left(|\Delta t|^{1/2}\right) W\varphi(x, t+\Delta t) - \Delta_n^2\left(|\Delta t|^{1/2}\right) W\varphi(x,t)$$
$$+ 2\Delta_t W\varphi\left(x + \bar{e}_n |\Delta t|^{1/2}, t\right) - \Delta_t W\varphi\left(x + 2\bar{e}_n |\Delta t|^{1/2}, t\right).$$

The first two terms have been estimated above, and the last two ones can be estimated in a same way:

$$\left|\Delta_t W\varphi\left(x + j\bar{e}_n |\Delta t|^{1/2}, t\right)\right| = \left| \int_0^{\Delta t} \partial_t W\varphi\left(x + j\bar{e}_n |\Delta t|^{1/2}, t+\lambda\right) d\lambda \right|$$

$$\le C \int_0^{\Delta t} \frac{d\lambda}{x_n + j|\Delta t|^{1/2}} \le C|\Delta t|^{1/2}, \qquad j = 1, 2.$$

The proof of the theorem for $l = 1$ is complete.

Now let $l = 2$. Since $\partial_i \varphi \in \overset{\circ}{H}_1(\Sigma)$ and $\partial_x^k \partial_i W\varphi = \partial_x^k W[\partial_i \varphi]$ in D_+ for $i = 1, 2, \ldots, n-1$, we see that it remains to prove the estimates

$$|W\varphi(x,t)| \le Ct, \tag{14}$$

$$\left|\partial_n^j W\varphi(x,t)\right| \le Cx_n^{-(j-2)}, \qquad j \ge 3, \tag{15}$$

$$\left|\Delta_i^2 \partial_n W\varphi(x,t)\right| \le C\left|\Delta x_i\right|, \qquad i = 1, 2, \ldots, n, \tag{16}$$

$$|\Delta_t \partial_n W\varphi(x,t)| \le C|\Delta t|^{1/2}, \tag{17}$$

$$\left|\Delta_t^2 W\varphi(x,t)\right| \le C|\Delta t|. \tag{18}$$

Inequality (14) can be proved by analogy with (10).

Let us prove the estimate (15). We fix $(x,t) \in D_+$ and set $\theta(\tau) = u(x', t, \tau)$. First, we show that $\theta \in \overset{\circ}{H}_1([0,t])$ and $\|\theta\|_{1,[0,t]} \le C\|\varphi\|_{2,\Sigma}$, where C is independent of t. Without loss of generality, we assume that $\Delta\tau > 0$. Then $\Delta\tau \le (t-\tau)/2$ and

$$\Delta_\tau^2 \theta(\tau) = \Delta_\tau^2 \Pi\left[\varphi(\cdot, s)\right](x', t-\tau)\big|_{s=\tau+\Delta\tau} + \Pi\left[\Delta_\tau^2 \varphi(\cdot, \tau)\right](x', t-\tau-\Delta\tau)$$
$$+ \Delta_\tau \Pi\left[\Delta_\tau \varphi(\cdot, s)\right](x', t-\tau-\Delta\tau)\big|_{s=\tau+\Delta\tau} + \Delta_\tau \Pi\left[\Delta_\tau \varphi(\cdot, s)\right](x', t-\tau)\big|_{s=\tau}.$$

The desired inequality for the first two terms readily follows from Theorem 1, and the last two terms can be estimated with the use of this theorem as follows:

$$\left|\Delta_\tau \Pi\left[\Delta_\tau \varphi(\cdot, s)\right](x', t-\tau-\Delta\tau)\big|_{s=\tau+\Delta\tau}\right|$$
$$= \left|\int_0^{\Delta\tau} \partial_t \Pi\left[\Delta_\tau \varphi(\cdot, \tau+\Delta\tau)\right](x', t-\tau-\Delta\tau+\lambda)\, d\lambda\right|$$
$$= \left|\int_0^{\Delta\tau} \sum_{i=1}^n \partial_i \Pi\left[\Delta_\tau \partial_i \varphi(\cdot, \tau+\Delta\tau)\right](x', t-\tau-\Delta\tau+\lambda)\, d\lambda\right|$$
$$\le C \int_0^{\Delta\tau} \frac{|\Delta\tau|^{1/2}}{(t-\tau-\Delta\tau+\lambda)^{1/2}} d\lambda \le C|\Delta\tau|^{1/2} \int_0^{\Delta\tau} \frac{d\lambda}{\lambda^{1/2}} \le C|\Delta\tau|.$$

Furthermore, $|\theta(\tau)| \le |\varphi(\cdot, \tau)|_{0,\mathbb{R}^{n-1}} \le K\tau$. Let us now rewrite $\partial_n^j W\varphi$, $j \ge 3$, in the form

$$\partial_n^j W\varphi(x,t) = -\int_0^t \partial_n^{j+1} Z(x_n, t-\tau)\left[\theta_1(\tau) + \theta_2(\tau)\right] d\tau = S_1 + S_2,$$

where $\theta = \theta_1 + \theta_2$ is the representation of θ in accordance with Lemma 1. This, together with the estimates

$$|S_1| \le C \int_0^t \frac{1}{(t-\tau)^{j/2}} \exp\left\{-\frac{cx_n^2}{t-\tau}\right\} d\tau \le \frac{C}{x_n^{(j-2)}},$$

$$|S_2| = \left|\int_0^t \left[\partial_\tau^2 \partial_n^{j-3} Z(x_n, t-\tau)\right] \theta_2(\tau) d\tau\right|$$
$$= \left|\partial_n^{j-1} Z(x_n, t)\, \theta_2(0) + \partial_n^{j-3} Z(x_n, t)\, \theta_2'(0) + \int_0^t \partial_n^{j-3} Z(x_n, t-\tau)\, \theta_2''(\tau) d\tau\right|$$
$$\le \frac{C}{t^{(j-2)/2}} \exp\left\{-\frac{cx_n^2}{t}\right\} + C \int_0^t \frac{1}{(t-\tau)^{j/2}} \exp\left\{-\frac{cx_n^2}{t-\tau}\right\} d\tau \le \frac{C}{x_n^{(j-2)}},$$

implies the desired estimate (15).

Let us prove the estimate (16). We assume that $\Delta x_i > 0$ and first consider the case $i = n$. Then

$$\left|\Delta_n^2 \partial_n W\varphi(x,t)\right| = \left|\int_0^{\Delta x_n}\int_0^{\Delta x_n} \partial_n^3 W\varphi\left(x + (\alpha+\beta)\bar{e}_n, t\right) d\alpha\, d\beta\right|$$
$$\le C\int_0^{\Delta x_n}\int_0^{\Delta x_n} \frac{d\alpha\, d\beta}{x_n + \alpha + \beta} \le C\Delta x_n.$$

For $i = 1, 2, \ldots, n-1$, we have

$$\Delta_i^2 \partial_n W\varphi(x,t) = \Delta_n^2\left(\Delta x_i\right)\partial_n W\varphi(x,t) - 2\Delta_n^2\left(\Delta x_i\right)\partial_n W\varphi\left(x + \Delta x_i \bar{e}_i, t\right)$$
$$+ \Delta_n^2\left(\Delta x_i\right)\partial_n W\varphi\left(x + 2\Delta x_i \bar{e}_i, t\right) + 2\Delta_i^2 \partial_n W\varphi\left(x + \Delta x_i \bar{e}_n, t\right)$$
$$- \Delta_i^2 \partial_n W\varphi\left(x + 2\Delta x_i \bar{e}_n, t\right).$$

The first three terms have already been considered, and the last two terms can be estimated in a same way:

$$\left|\Delta_i^2 \partial_n W\varphi\left(x + j\Delta x_i \bar{e}_n, t\right)\right| = \left|\int_0^{\Delta x_i}\int_0^{\Delta x_i} \partial_i^2 \partial_n W\varphi\left(x + (\alpha+\beta)\bar{e}_i + j\Delta x_i \bar{e}_n, t\right) d\alpha\, d\beta\right|$$
$$\le C\int_0^{\Delta x_i}\int_0^{\Delta x_i} \frac{d\alpha\, d\beta}{x_n + j\Delta x_i} \le C\Delta x_i, \qquad j = 1, 2.$$

The derivation of the estimate (17) reproduces the proof of (13) word for word with $W\varphi$ replaced by $\partial_n W\varphi$.

Let us prove the estimate (18). We have

$$\Delta_t^2 W\varphi(x,t) = \Delta_n^3\left(|\Delta t|^{1/2}\right) W\varphi(x, t + 2\Delta t) - 2\Delta_n^3\left(|\Delta t|^{1/2}\right) W\varphi(x, t + \Delta t)$$
$$+ \Delta_n^3\left(|\Delta t|^{1/2}\right) W\varphi(x, t + 2\Delta t) + 3\Delta_t^2 W\varphi\left(x + |\Delta t|^{1/2}\bar{e}_n, t\right)$$
$$- 3\Delta_t^2 W\varphi\left(x + 2|\Delta t|^{1/2}\bar{e}_n, t\right) + \Delta_t^2 W\varphi\left(x + 3|\Delta t|^{1/2}\bar{e}_n, t\right).$$

For the first three terms, we obtain the desired inequality with regard of (16):

$$\left|\Delta_n^3\left(|\Delta t|^{1/2}\right) W\varphi(x, t + j\Delta t)\right| = \left|\int_0^{|\Delta t|^{1/2}} \Delta_n^2\left(|\Delta t|^{1/2}\right)\partial_n W\varphi\left(x + \alpha\bar{e}_n, t + j\Delta t\right) d\alpha\right|$$
$$\le C\int_0^{|\Delta t|^{1/2}} |\Delta t|^{1/2} d\alpha = C|\Delta t|, \qquad j = 0, 1, 2,$$

and for the last three terms, it follows from (15):

$$\left|\Delta_t^2 W\varphi\left(x + j|\Delta t|^{1/2}\bar{e}_n, t\right)\right| = \left|\int_0^{\Delta t}\int_0^{\Delta t} \partial_t^2 W\varphi\left(x + j|\Delta t|^{1/2}\bar{e}_n, t + \alpha + \beta\right) d\alpha\, d\beta\right|$$
$$\le C\int_0^{\Delta t}\int_0^{\Delta t} \frac{d\alpha\, d\beta}{\left(x_n + j|\Delta t|^{1/2}\right)^2} \le C|\Delta t|, \qquad j = 1, 2, 3,$$

which completes the proof of the theorem for the case $l = 2$.

If $l \geq 3$, then the theorem can be reduced to the above-considered cases, since $\partial_i W\varphi = W[\partial_i\varphi]$, $i = 1, 2, \ldots, n-1$, $\partial_t W\varphi = W[\partial_t\varphi]$, and $\partial_n^2 W\varphi = (\partial_t - \Delta') W\varphi = W[(\partial_t - \Delta')\varphi]$. The proof of the theorem is complete.

4. THE FIRST BOUNDARY VALUE PROBLEM

In the domain D_+, consider the first boundary value problem for the heat equation

$$Lu = 0 \quad \text{in} \quad D_+, \qquad u|_\Sigma = \varphi, \qquad u|_{t=0} = 0. \tag{19}$$

The solution of this problem can be represented as a double layer potential. This, in conjunction with Theorem 2, implies the following assertion.

Theorem 3. *Let l be a positive integer, and let $\varphi \in \underset{\circ}{H}_l(\Sigma)$. Then the solution u of the first boundary value problem* (19) *belongs to $\underset{\circ}{H}_l\left(\bar{D}_+\right)$ and also to the weighted spaces $H_m^{-l}(D_+)$ for integer $m \geq l$; moreover,*

$$\|u\|_{l,D_+} \leq C(n,l)\|\varphi\|_{l,\Sigma}, \qquad |u|_{m,D_+}^{(-l)} \leq C(n,l,m)\|\varphi\|_{l,\Sigma}.$$

Note that the proof of Theorem 2 implies the more accurate estimates

$$\left|\partial_t^s \partial_x^k u(x,t)\right| \leq C_{2s,|k|} x_n^{-(2s+|k|-l)} \|\varphi\|_{l,\Sigma}, \quad (x,t) \in D_+,$$

for the derivatives $\partial_t^s \partial_x^k u$ of order $2s + |k| > l$ of the solution; i.e., the derivatives can grow as some power of the distance from the "lateral surface" of the domain Σ rather than from the parabolic boundary $\mathscr{P}$, as is assumed in the definition of the spaces $H_m^{-l}(D_+)$.

ACKNOWLEDGMENTS

The author is grateful to E.I. Moiseev and E.A. Baderko for discussion of the results.

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