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Complex Measures on Path Space: An Introduction to the Feynman Integral Applied to the Schrödinger Equation

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Abstract. A simple general approach to the construction of measures on path space is developed. It is used for the path integral representation of evolutionary equations including Feller processes, the Schrödinger equation, and dissipative Schrödinger equations. At the end of the paper we give a short guide to the immense literature on path integration sketching the main known approaches to the construction of the Feynman integral and indicating possible generalizations.

Keywords: measures on path spaces, Feynman integral, infinitely divisible distributions, Schrödinger equation, complex Markov processes and Dirichlet forms

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1. Introduction

The main objective of this paper is a methodological one. We develop systemetically a simple but general approach to the construction of measures on path space that can be used for the path integral representation of evolutionary equations including Feller processes, the Schrödinger equation (with a certain class of potentials), and complex diffusions. Unlike the usual approaches to the construction of the Feynman integral, which define the Feynman integral as some generalised functional on an appropriate space of functions (see a short review of the literature at the end of this article), the integral constructed here is defined as a genuine integral over a bona fide complex measure on a path space. The idea of such an integral seems to first appear in Maslov (1976), but our exposition will be, on the one hand, more direct and elementary, and on the other hand, more general than that of Maslov (1976) (see also Maslov and Chebotarev (1979)), and will be given in terms of more or less standard probabilistic concepts generalised to the complex case. The paper is aimed to be an elementary introduction, accessible to graduate students. At the end of the paper we give a (very short) guide to the immense mathematical literature on path integration sketching the main known approaches to the construction of the Feynman integral and indicating possible generalizations, and then conclude with some remarks on possible computational aspects, which will be dealt with in more detail in a subsequent publication.

After the original papers of Dirac and Feynman (see Dirac (1933), Dirac (1958), Feynman (1948), Feynman (1950), Feynman (1951)), integration over path space soon became a very popular and powerful tool in different domains of physics, see e.g. monographs Feynman and Hibbs (1965), Schulman (1996), Glimm and Jaffe (1977), Montvay and Münster (1994), Slavnov and Faddeev (1988) and references therein. For example, a wide discussion of the various applications of Feynman integral in quantum mechanics can be found in Schulman (1996). The significance of the Feynman integral lies in the fact that it gives explicitly the solutions to the main equation of the quantum theory, the Schrödinger equation. Therefore, the Feynman integral is as fundamental for physics as is the Schrödinger equation. One can even reformulate the whole quantum mechanics in terms of the path integral, without using the Schrödinger equation. Moreover, in many cases, specially in gauge theories (or generally in infinite dimensional models of quantum physics), dealing with Feynman integral is much more convenient than with Schrödinger equation. In fact, the scattering amplitudes of all processes of elementary particle physics are calculated usually by means of the technique of Feynman's diagrammes, which are just the graphical representations of the perturbation theory expansions based on the Feynman integral.

At the same time, the mathematical theory underlying lots of (often formal) physical calculations is far from being complete, even for the case of Schrödinger equation, i.e. for quantum mechanics (let alone quantum field theory). Therefore, the efforts of many mathematicians have been aimed at obtaining a deeper understanding of this beautiful and extremely useful object: Feynman path integration. The author hopes that this short note will contribute to this process.

2. Complex Measures on Path Space

Let $\mathscr{B}(\Omega)$ denote the class of all Borel sets of a topological space (i.e. it is the σ-algebra of sets generated by all open sets). If Ω is locally compact we denote (as usual) by $C_0(\Omega)$ the space of all continuous complex-valued functions on Ω vanishing at infinity. Equipped with the uniform norm $\|f\| = \sup_x |f(x)|$ this space is known to be a Banach space. It is also well known (Riesz-Markov theorem, see e.g. Reed and Simon (1975)) that if Ω is a locally compact space, then the set $\mathscr{M}(\Omega)$ of all finite complex regular Borel measures on Ω equipped with the norm $\|\nu\| = \sup |\int_\Omega f(x)\nu(dx)|$, where sup is taken over all functions $f \in C_0(\Omega)$ with $\|f(x)\| \leq 1$, is a Banach space, which coincides with the set of all continuous linear functionals on $C_0(\Omega)$. In other words, $\mathscr{M}(\Omega)$ is the dual space to $C_0(\Omega)$.

Remark 1. When estimating the norms of complex measures the following simple fact can be useful. For a real measure ν let $\nu = \nu^+ - \nu^-$ be its canonical decomposition into its positive and negative parts, and let the magnitude of ν be defined as the positive measure $|\nu| = \nu^+ + \nu^-$. For a real-valued measure ν the norms of the measures ν and $|\nu|$ coincide. For a complex measure ν let us define $|\nu| = |Re\nu| + |Im\nu|$. One readily finds that $\|(|\nu|)\|/2 \leq \|\nu\| \leq \|(|\nu|)\|$ for any complex measure ν.

We say that a map ν from $\mathscr{R}^d \times \mathscr{B}(\mathscr{R}^d)$ into $\mathfrak{C}$ is a *complex transition kernel*, if for every x, the map $A \mapsto \nu(x, A)$ is a (finite complex) measure on $\mathscr{R}^d$, and for every $A \in \mathscr{B}(\mathscr{R}^d)$, the map $x \mapsto \nu(x, A)$ is $\mathscr{B}$-measurable. A *complex transition function* (abbreviated CTF) on $\mathscr{R}^d$ is a family $\nu_{s,t}, 0 \leq s \leq t$, of complex transition kernels such that $\nu_{s,s}(x, dy) = \delta(y - x)$ for all s, x where δ is the Dirac measure in x, and such that for every positive numbers $0 \leq s \leq t \leq v$, the Chapman-Kolmogorov equation

$$\int \nu_{s,t}(x, dy)\nu_{t,v}(y, A) = \nu_{s,v}(x, A) \tag{1}$$

is satisfied. A CTF will be called *regular*, if there exists a positive constant K such that for all x and $s \leq t$, the norm $\|\nu_{s,t}(x, \cdot)\|$ of the measure $A \mapsto \nu_{s,t}(x, A)$ does not exceed $\exp\{K(t - s)\}$.

Notice that if all measures in a CTF are probability measures, then the CTF is automatically regular, because all probability measures have the unit norm. A CTF is said to be homogeneous in time (respectively in space), if $\nu_{s,t}(x, A)$ depends on s, t only through the difference $t - s$ (resp. depends on x, A only through the difference $A - x$). If a CTF is homogeneous in time (resp. in space) it is natural to denote $\nu_{0,t}$ by ν_t (resp. $\nu_{s,t}(0, A)$ by $\nu_{s,t}(A)$) and to write the Chapman-Kolmogorov equation in the form

$$\int \nu_s(x, dy)\nu_t(y, A) = \nu_{s+t}(x, A)$$

(time homogeneity) or

$$\int \nu_{s,t}(dy)\nu_{t,v}(A - y) = \nu_{s,v}(A)$$

(space homogeneity) respectively.

It turns out that CTFs appear naturally in the theory of evolutionary equations. Let us say that a two-parameter family $U(s, t), 0 \leq s \leq t$, of continuous linear maps in $C_0(\mathscr{R}^d)$ is a *propagator in* $C_0(\mathscr{R}^d)$, if (i) $U(t, t)$ is the unit operator for all t, (ii) $U(s, t)U(t, v) = U(s, v)$ for all $s \leq t \leq v$, (iii) $U(s, t)$ is strongly continuous in both s and t. We say that the propagator $U(s, t)$ is *regular* if there exists a constant $K > 0$ such that $\|U(s, t)\| \leq \exp\{K(t - s)\}$ for all $s \leq t$.

PROPOSITION 1 *If U (s,t) is a propagator in* $C_0(\mathscr{R}^d)$, *then there exists a CTF* ν *such that*

$$(U(s, t)f)(x) = \int \nu_{s,t}(x, dy)f(y) \tag{2}$$

for all $f \in C_0(\mathscr{R}^d)$. *If U (s,t) is regular, then* ν *is regular. In particular, if* T_t *is a strongly continuous semigroup of bounded linear operators in* $C_0(\mathscr{R})^d$, *then there exists a time-homogeneous CTF* ν *such that*

$$T_t f(x) = \int \nu_t(x, dy)f(y).$$

Proof: It is straightforward. In fact, the existence of the measure $\nu_{s,t}(x, \cdot)$ such that (2) is satisfied follows from the Riesz-Markov theorem, and the evolution identity $U(s, t)U(t, v) = U(s, v)$ is equivalent to the Chapman-Kolmogorov equation. Since $\int \nu_{s,t}(x, dy)f(y)$ is continuous for all $f \in C_0(\mathscr{R}^d)$, it follows by the monotone convergence theorem (and the fact that each complex measure is a linear combination of four positive measures) that $\nu_{s,t}(x,A)$ is a Borel function of x. Since $\|\nu_{s,t}(x,\cdot)\| = \|U(s,t)\|$, the conditions of the regularity of the propagator and the corresponding CTF are equivalent. ■

Now we construct a measure on the path space corresponding to each regular propagator (or CTF), introducing first some (rather standard) notations. Let $\dot{\mathcal{R}}_d$ denote the one point compactification of the Euclidean space $\mathcal{R}^d$ (i.e. $\dot{\mathcal{R}}_d = \mathcal{R}^d \cup \infty$ and is homeomorphic to the sphere S^d). Let $\dot{\mathcal{R}}_d^{[s,t]}$ denote the infinite product of $[s,t]$ copies of $\dot{\mathcal{R}}_d$, i.e. it is the set of all functions from $[s,t]$ to $\dot{\mathcal{R}}_d$, the path space. As usual, we equip this set with the product topology, in which it is a compact space (Tikhonov's theorem, see e.g, Reed and Simen (1972)). Let $Cyl_{[s,t]}^k$ denote the set of functions on $\dot{\mathcal{R}}_d^{[s,t]}$ having the form

$$\phi^f_{t_1,\dots t_k}(y(\cdot)) = f(y(t_1),\dots,y(t_k))$$

for some bounded complex Borel function f on $(\dot{\mathcal{R}}^d)^k$ and some points $t_1 < t_2 < \dots < t_k$ from the interval $[s,t]$. The union $Cyl_{[s,t]} = \cup_{k\in\mathcal{N}} Cyl_{[s,t]}^k$ is called the set of cylindrical functions (or functionals) on $\dot{\mathcal{R}}_d^{[s,t]}$. It follows from the Stone-Weierstrasse theorem (see e.g. (Reed and Simon, (1975)) that the linear span of all continuous cylindrical functions is dense in the space $C(\dot{\mathcal{R}}_d^{[s,t]})$ of all complex continuous functions on $\dot{\mathcal{R}}_d^{[s,t]}$. Any CTF ν defines a family of linear functionals $\nu_{s,t}^x, x\in\mathcal{R}^d$, on $Cyl_{[s,t]}$ by the formula

$$\nu_{s,t}^x(\phi^f_{t_1\dots t_k}) = \int f(y(t_1),\dots,y(t_k))\nu_{s,t_1}(x,dy_1) \qquad \nu_{t_1,t_2}(y_1,dy_2)\dots\nu_{t_{k-1},t_k}(y_{k-1},dy_k)\nu_{t_k,t}(y_k,dy). \tag{3}$$

Due to the Chapman-Kolmogorov equation, this definition is correct, i.e. if one considers an element from $Cyl_{[s,t]}^k$ as an element from $Cyl_{[s,t]}^{k+1}$ (any function of k variables $y_1,\dots,y_k$ can be considered as a function of $k+1$ variables $y_1,\dots,y_{k+1}$, which does not depend on y_{k+1}), then the two corresponding formulae (3) will be consistent.

PROPOSITION 2. *If the propagator U (s,t) in $C_0(\mathcal{R}^d)$ is regular and ν is its corresponding CTF, then the functional (3) is bounded. Hence, it can be extended by continuity to a unique bounded linear functional ν^x on $C(\dot{\mathcal{R}}_d^{[s,t]})$, and consequently there exists a (regular) complex Borel measure $D_x^{s,t}$ on the path space $\dot{\mathcal{R}}_d^{[s,t]}$ such that*

$$\nu_{s,t}^x(F) = \int F(y(\cdot))D_x^{s,t}y(\cdot) \tag{4}$$

for all $F\in C(\dot{\mathcal{R}}_d^{[s,t]})$. In particular,

$$(U(s,t)f)(x) = \int f(y(t))D_x^{s,t}y(\cdot).$$

Proof: It is a direct consequence of the Riesz-Markov theorem, because the regularity of CTF implies that the norm of the functional $\nu_{s,t}^x$ does not exceed $\exp\{K(t-s)\}$. ■

Let us say that a CTF is *normalized,* if $\nu_{s,t}^x(\mathcal{R}^d) = 1$ for all s, t and x. Clearly, in that case, the measures D_x from (4) are consistent in the sense that if $s \le t \le v$ and $F\in Cyl_{[s,t]}^k$, then $\int F(y(\cdot))D_x^{s,t}y(\cdot) = \int F(y(\cdot))D_x^{s,v}y(\cdot)$.

Since all CTF consisting of probability measures are regular, an important result of probability theory that each Feller semigroup corresponds to a Markov process (see e.g. Revuz and Yor (1991)), is a direct consequence of Proposition 2. Let us notice also the following simple fact that can be used (see examples below) in proving the regularity of a semigroup.

PROPOSITION 3 *Let B and A be linear operators in $C_0(\mathscr{R}^d)$ such that A is bounded and B is the generator of a strongly continuous regular semigroup $T_t = \exp\{tB\}$. Then $A + B$ is the generator of the regular semigroup $\tilde{T}_t = \exp\{t(A + B)\}$.*

Proof: Follows directly from the fact that $\tilde{T}_t$ can be presented as the convergent (in the sense of the norm) series of standard perturbation theory (see e.g. Maslov, (1972) or Reed and Simmon, (1975))

$$\tilde{T}_t = T_t + \int_0^t T_{t-s}AT_s ds + \int_0^t ds \int_0^s d\tau T_{t-s}AT_{s-\tau}AT_\tau + \dots \qquad \blacksquare$$

Before giving examples, let us discuss in greater detail a particular case of the spatially homogeneous CTFs, and their connection with infinitely divisible characteristic functions. Let $\mathscr{F}(\mathscr{R}^d)$ denote the Banach space of Fourier transforms of elements of $\mathscr{M}(\mathscr{R}^d)$, i.e. the space of (automatically continuous) functions on $\mathscr{R}^d$ of the form $f_\mu(p) = \int e^{-ipx}\mu(dx)$ for some $\mu \in \mathscr{M}(\mathscr{R}^d)$, with the induced norm $\|f_\mu\| = \|\mu|$. Since $\mathscr{M}(\mathscr{R}^d)$ is a Banach algebra with convolution as the multiplication, it follows that $\mathscr{F}(\mathscr{R}^d)$ is also a Banach algebra with respect to the standard (pointwise) multiplication. We say that an element $f \in \mathscr{F}(\mathscr{R}^d)$ is *infinitely divisible* if there exists a family $(f_t, t \geq 0,)$ of elements of $\mathscr{F}(\mathscr{R}^d)$ such that $f_0 = 1, f_1 = f$, and $f_{t+s} = f_t f_s$ for all positive *s, t.* Clearly if f is infinitely divisible, then it has no zeros and a continuous function $g = \log f$ is well defined (and is unique up to an imaginary shift). Moreover, the family f_t has the form $f_t = \exp\{tg\}$ and is defined uniquely up to a multiplier of the form $e^{2\pi ikt}, k \in \mathscr{N}$. Let us say that a continuous function g on $\mathscr{R}^d$ is a *complex characteristic exponent* (abbreviated CCE), if e^g is an infinitely divisible element of $\mathscr{F}(\mathscr{R}^d)$, or equivalently, if e^{tg} belongs to $\mathscr{F}(\mathscr{R}^d)$ for all $t > 0$.

Remark 2. In probability theory, a characteristic function f of a probability measure is said to be infinitely divisible, if $f^{1/n}$ is again a characteristic function of a probability measure for all integer n. From this definition it follows (see e.g. Feller (1971) that f has no zeros. For the case of the Fourier transforms of the complex measures this would not be so. For example, the function $(1 + e^{ip})/2$, which is the characteristic function of a two-point probability measure on $\mathscr{R}$, is not infinitely divisible in the sense of probability theory (because it has zeros), but at the same time, the function $(1 + e^{ip})^{1/n}$ although it is not the characteristic function of a positive measure, is the Fourier transform of a finite complex measure (which is, in fact, real, but not necessarily positive). In fact, since the function $(1 + y)^{1/n}$ can be expanded in a power series in y which is uniformly convergent in the closed disk $|y| \leq 1$, it follows that the function $(1 + e^{ip})^{1/n}$ can be expanded in a uniformly convergent Fourier series, and therefore it belongs to $\mathscr{F}(\mathscr{R}^d)$, because $\mathscr{F}(\mathscr{R}^d)$ is a Banach algebra. That is why, in the complex case, a stronger definition of infinite divisibility (given above) is more appropriate.

Remark 3. It seems that one can not expect to have an explicit representation for infinitely divisible elements of $\mathscr{F}(\mathscr{R}^d)$ (such as the Lévy-Khintchine representation for the infinitely divisible characteristic functions of probability measures), because the elements of $\mathscr{F}(\mathscr{R}^d)$ cannot be described by an algebraic property. One knows only that these functions are dense in the space of all continuous functions, and some Sobolev-type inclusion theorems for them are available (see Maslov (1987). On the other hand, the condition of regularity is rather restrictive and it would be very interesting to describe explicitly the whole class of regular CCE. Some examples of regular CCE are given below.

It follows from the definitions that the set of space and time homogeneous CTFs $\nu_t(dx)$ is in one-to-one correspondence with CCE g, in such a way that for any positive t the function e^{tg} is the Fourier transform of the transition measure $\nu_t(dx)$.

PROPOSITION 4 *If g is a CCE, then the solution to the Cauchy problem*

$$\frac{\partial u}{\partial t} = g\left(\frac{1}{i}\frac{\partial}{\partial x}\right)u \tag{5}$$

defines a strongly continuous and spatially homogeneous semigroup T_t of bounded linear operators in $C_0(\mathscr{R}^d)$ (i.e. $(T_t u_0)(x)$ is the solution to Equation (5) with the initial function u_0). Conversely, each such semigroup is the solution to the Cauchy problem of an equation of type (5) with some CCE g.

Proof: This is straightforward. Since (5) is a pseudo-differential equation, it follows that the Fourier transform $\tilde{u}(t,p)$ of the function $u(t,s)$ satisfies the ordinary differential equation

$$\frac{\partial \tilde{u}}{\partial t}(t,p) = g(p)\tilde{u}(t,p),$$

whose solution is $\tilde{u}_0(p)\exp\{tg(p)\}$. Since e^{tg} is the Fourier transform of the complex transition measure $\nu_t(dx)$, it follows that the solution to the Cauchy problem of Equation (5) is given by the formula $(T_t u_0)(x) = \int u_0(y)\nu_t(dy - x)$, which is as required. ■

Clearly the CTF corresponding to a CCE g can be made normalized by an appropriate shift of the CCE g.

We say that a CCE is *regular,* if Equation (5) defines a regular semigroup T_t.

If the transition measures ν_t are absolutely continuous with respect to Lebesgue measure, their densities are Fourier transforms of functions e^{tg}. Denoting these densities by $u_G(t,-x)$, one readily sees that the function $u_G(t, x - x_0)$ is the Green function of the Cauchy problem to Equation (5).

Example 1. Consider now the standard example, showing the difficulties of the rigorous definitions of the Feynman path integral, namely, the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}G\Delta u, \quad u = u(t,x), \tag{6}$$

where G is a complex constant with $ReG \geq 0$. The case $G = i$ (respectively $G = 1$) corresponds to the Schrödinger equation describing a free quantum particle (respectively the simplest heat equation). Equation (6) is of form (5) with $g = -Gp^2/2$. The Green function of this equation is known to be (and is obtained by simple calculations)

$$(2\pi G t)^{-d/2} \exp\left\{ \frac{(x-x_0)^2}{2tG} \right\},$$

and one readily sees that if $G = i$ the solution T_t to the Cauchy problem of Equation (6) does not define a bounded operator in $C_0(\mathscr{R}^d)$. If $ImG > 0$ and $ReG > 0$, the operators T_t form a strongly continuous group of bounded operators in $C_0(\mathscr{R}^d)$ with the norms $\|T_t\| = (|G|/ReG)^{d/2}$. Hence, this semigroup T_t is regular only if G is real (in this case the corresponding measure on the path space is the standard Wiener measure). Hence, only for real G, can the solution to Equation (6) be represented in the form of the path integral from Proposition 2 (this seems to be first noted in Cameron (1960). However, consider the Fourier transform of Equation (6) (in physical language, Equation (6) in momentum (or p-) representation), namely the equation

$$\frac{\partial \psi}{\partial t} = -\frac{1}{2} G p^2 \psi \quad \psi = \psi(t,p). \tag{7}$$

From the point of view of the L_2-theory (and therefore, from the point of view of quantum mechanics) Equations (6) and (7) are equivalent, but from the point of view of pointwise continuity they are not. In fact, one readily sees that for all G with a non-negative real part, the solution T_t to the Cauchy problem for Equation (7) defines a strongly continuous semigroup of bounded operators in $C_0(\mathscr{R}^d)$ with the norms $\|T_t\| \leq 1$, i.e. a regular semigroup, which can be thus defined by means of the path integral from Proposition 2.

Example 2. Consider a more general example of complex diffusion (or complex Schrödinger equation), namely the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}(G\Delta - \beta x^2)u, \quad u = u(t,x), \tag{8}$$

where both G and β are supposed to have non-negative real and imaginary parts and to have non-vanishing magnitudes. It is not difficult to write down explicitly the Green function of this equation. It has the form

$$u_G(t,x,x_0) = C^d \exp\left\{ -\frac{\omega}{2}(x^2 + x_0^2) + \lambda x x_0 \right\}$$

with

$$\omega = \sqrt{\frac{\beta}{G}} \coth(\sqrt{\beta G t}), \quad \lambda = \sqrt{\frac{\beta}{G}} (\sinh(\sqrt{\beta G t}))^{-1},$$
$$C = \left(\frac{\beta}{G}\right)^{1/4} (2\pi \sinh(\sqrt{\beta G t}))^{-1/2}.$$

The norm of the integral operator T_t in $C_0(\mathscr{R}^d)$ with the kernel u_G can easily be calculated, and it equals $(|C|\sqrt{2\pi/Re\omega})^d$. One finds then that the norms of T_t are uniformly bounded for small t only if $ReG > 0$. In that case,

$$Re\omega = \frac{1}{t}(1 + O(t^2))Re\frac{1}{G}, \quad |C| = (2\pi t|G|)^{-1/2}(1 + O(t^2)), \quad \|T_t\| = \left(\frac{|G|}{ReG}\right)^{d/2}.$$

Hence, for all β (under the assumptions given above), Equation (8) defines a regular semigroup exactly when G is a positive real number. Notice that Equation (8) is the Fourier transform of the Schrödinger equation describing a quantum oscillator with a complex frequency.

The above example shows that it is not straightforward to characterize the class of regular CCE. Clearly, if a CCE is given by the Lévy-Khintchine formula (i.e. it defines a transition function consisting of probability measures), then this CCE is regular. Another class is given by the following result obtained essentially in Maslov (1987), Maslov Chebotarev (1979) (in a more concrete framework).

PROPOSITION 5 *Any element of $\mathscr{F}(\mathscr{R}^d)$ is a regular CCE. Moreover, the corresponding measure $D_x^{[0,t]}$ on the path space from Proposition 2 is concentrated on the set of piecewise-constant paths in $\tilde{\mathscr{R}}_d^{[0,t]}$ with a finite number of jumps. In other words, D_x is the measure of a jump-process.*

Proof: Let $g = g_\mu$ and let $\tilde{g} = g_{|\mu|}$. The function $\exp\{tg\}$ is the Fourier transform of the measure $\delta_0 + t\mu + \frac{t^2}{2}\mu * \mu + \ldots$ which can be denoted by $\exp^*(t\mu)$ (it is equal to the sum of the standard exponential series, but with the convolution of measures instead of the standard multiplication). Clearly $\|\exp^*(t\mu)\| \leq \|\exp^*(t|\mu|)\|$, and both these series are convergent series in the Banach algebra $\mathscr{M}(\mathscr{R}^d)$. Therefore $\|e^{gt}\| \leq \|e^{\tilde{g}t}\| \leq \exp\{t\|(|\mu|)\|\}$, and consequently g is a regular CCE. Moreover, the same estimate shows that the measure on the path space corresponding to the CCE g is absolutely continuous with respect to the measure on the path space corresponding to the CCE $\tilde{g}$. But the latter is the probability measure of a compound Poisson process, because $\tilde{g}$ is given by the Lévy-Khintchine formula with a finite Lévy-measure, and it is well known in the theory of stochastic processes (see e.g. Feller, (1971) that such measures are concentrated on piecewise-constant paths. ■

Remark 4. The measure $D_x^{0,t}$ from Proposition 5 can be realised as the measure of the process $X_x^u = x + Y_1 + Y_2 + \ldots + Y_{N(u)}, 0 \leq u \leq t$, where $\{Y_n, n \in \mathscr{N}\}$ are a sequence of ''i.i.d.'' measurable functions taking values in $\mathscr{R}^d$ with common complex-valued distribution $\mu(A) =$ (measure of the event $Y_n \in A$) and N is an independent Poisson process.

Therefore, we have two different classes (essentially different, because they obviously are not disjoint) of regular CCE: those given by the Lévy-Khintchine formula, and those given by Proposition 5. The following statement allows one to combine these regular CCE.

PROPOSITION 6 *The class of regular CCE is a convex cone.*

Proof: One needs to prove only that the sum $g_1 + g_2$ of regular CCE is also a regular CCE. This follows from a simple observation that if the functions $\exp\{tg_j\}, j = 1, 2$ are the Fourier transforms of the transition measures $\nu_t^j(dx)$, then the function $\exp\{t(g_1 + g_2)\}$ is the Fourier transform of the convolution of ν_t^1 and ν_t^2. ■

Example 3. Consider the Schrödinger equation

$$\frac{\partial \psi}{\partial t} = \left(\frac{i}{2m}\Delta - iV(x)\right)\psi \tag{9}$$

with a potential V from $\mathscr{F}(\mathscr{R}^d)$, i.e. $V = V_\mu$ with $\mu \in (\mathscr{R}^d)$ and $m > 0$ denoting, as usual, the mass of a quantum particle. The Fourier transform of this equation has the form

$$\frac{\partial \phi}{\partial t} = \left(-\frac{i}{2m}p^2 - iV\left(\frac{1}{i}\frac{\partial}{\partial p}\right)\right)\phi. \tag{10}$$

The operator $V(\frac{1}{i}\frac{\partial}{\partial p})$ is a bounded operator in $C_0(\mathscr{R}^d)$ with norm $\|\mu\|$, whose action is given by

$$V\left(\frac{1}{i}\frac{\partial}{\partial p}\right)\phi = \int \phi(x-y)\mu(dy).$$

Therefore, using Proposition 3 (with B being the multiplication operator on ip^2 and A being $V(\frac{1}{i}\frac{\partial}{\partial p})$), one concludes that the solution to the Cauchy problem of Equation (10) is a regular semigroup in $C_0(\mathscr{R}^d)$, and therefore it can be written as the path integral from Proposition 2.

Remark 5. It is interesting to observe that the action of a regular semigroup T_t can be extended by continuity to an action on the space of all bounded continuous functions on $\mathscr{R}^d$ and the representation in terms of the path integral is preserved by this extension. In particular, one can take the exponential function e^{ipx} as an initial function to the Cauchy problem of Equation (10), with the corresponding solution being the Green function of Equation (9) in p-representation.

It is possible to describe solutions to Equation (10) in terms of certain integrals with respect to the measures defined by the complex jump process corresponding to the equation

$$\frac{\partial \phi}{\partial t} = -iV\left(\frac{1}{i}\frac{\partial}{\partial p}\right)\phi. \tag{11}$$

PROPOSITION 7 *Maslov, (1987), Maslov and Chebotarev (1979). The solution to the Cauchy problem of Equation (10) can be written in the form of a complex Feynman-Kac formula*

$$\phi(t,p) = \int \exp\left\{-\frac{i}{2m}\int_0^t q(\tau)^2 d\tau\right\}\phi_0(q(t))D_p^{[0,t]}q(\cdot), \tag{12}$$

where D_p is the measure of the jump process corresponding to Equation (11).

Proof: Notice first that the integral under the exponential in the last formula is well defined, because almost all paths are piecewise-constant. Let T_t be the regular semigroup corresponding to Equation (11). By the Trotter formula (see e.g. Reed and Simon (1975), Nelson (1964), the solution to the Cauchy problem of Equation (10) can be written in the form

$$\phi(t,p) = \lim_{n\to\infty}\left(\exp\left\{-\frac{it}{2nm}p^2\right\}T_{t/n}\right)^n\phi_0,$$

i.e.

$$\phi(t,p) = \lim_{n\to\infty}\int \exp\left\{-\frac{it}{2nm}(q_1^2+\ldots+q_n^2)\right\}\phi_0(q_n)$$
$$\times\,\nu_{t/n}(p,dq_1)\nu_{t/n}(q_1,dq_2)\ldots\nu_{t/n}(q_{n-1},dq_n).$$

Using (3), we can rewrite this as

$$\phi(t,p) = \lim_{n\to\infty}\nu_t^p(F_n) = \lim_{n\to\infty}\int F_n(q(\cdot))D_p^t q(\cdot),$$

where F_n is the cylindrical function

$$F_n(q(\cdot)) = \exp\left\{-\frac{it}{2nm}(q(t/n)^2+q(2t/n)^2+\ldots+q(t)^2)\right\}\phi_0(q(t)).$$

By the dominated convergence theorem this implies (12).

In the same way as Equation (9), one can consider an essentially more general equation to obtain the following result.

PROPOSITION 8 *Let f be any continuous real function and let V be as in Example 3. Then the equation*

$$\frac{\partial\psi}{\partial t} = \left(\frac{i}{2}f(|\Delta|) - iV(x)\right)\psi$$

taken in its p-representation has the form

$$\frac{\partial\phi}{\partial t} = \left(\frac{i}{2}f(p^2) - iV\left(\frac{1}{i}\frac{\partial}{\partial p}\right)\right)\phi,$$

which defines a regular semigroup in $C_0(\mathscr{R}^d)$. Moreover, the solution of the Cauchy problem to the latter equation can be written by means of the complex Feynman-Kac formula

$$\phi(t,p) = \int \exp\left\{\frac{i}{2m}\int_0^t f(q(\tau)^2)d\tau\right\}\phi_0(q(t))D_p^{[0,t]}q(\cdot), \tag{13}$$

where D_p is the measure of the jump process corresponding to Equation (11).

Remark 6 Similarly one obtains the following result for the anharmonic oscillator. Let

$H = -\Delta + x^2$ be the generator of a quantum oscillator, let f be a continuous real function, and let V be a complex-valued function from L^2 such that the coefficients c_k, $k = 0, 1, \dots,$ of its expansions with respect to Hermite functions are absolutely summable, i.e. $\sum_k |c_k| < \infty$. Then the equation

$$\frac{\partial \psi}{\partial t} = (if(H) - iV(x))\psi,$$

defines a regular semigroup, if considered in the spectral representation of the harmonic oscillator, and a similar Feynman-Kac formula holds. Furthermore, due to Proposition 3, this can be generalized easily to the following situation, which includes all Schrödinger equations, namely to the case of the equation

$$\frac{\partial \phi}{\partial t} = i(A - B)\phi,$$

where A is selfadjoint operator, for which therefore exists (according to spectral theory) a unitary transformation U such that UAU^{-1} is the multiplication operator in some $L^2(X, d\mu)$, where X is locally compact, and B is such that UBU^{-1} is a bounded operator in $C_0(X)$.

Finally, we consider an example of a different kind, which can be called complex diffusion. Equations of this kind, or more precisely, their non-homogeneous or even stochastic generalizations, have recently become popular in the theory of continuous quantum measurement and in quantum optics, see e.g. Belavkin *et al.* (1995) or Quantum and Semiclassical Optics and references therein.

PROPOSITION 9. *Consider the complex Schrödinger equation for an anharmonic oscillator with a complex frequency*

$$\frac{\partial \psi}{\partial t} = \frac{1}{2}(\beta\Delta - Gx^2 - iV(x))\phi, \tag{14}$$

where $V = V_\mu$ is an element of $\mathcal{F}(\mathcal{R}^d)$, and $ReG \geq 0$. The Fourier transform of this equation (Equation (14) in the p-representation) has the form

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\left(G\Delta - \beta p^2 - iV\left(\frac{1}{i}\frac{\partial}{\partial p}\right)\right)\phi, \tag{15}$$

and the Cauchy problem of Equation (15) defines a regular semigroup of operators in $C_0(\mathcal{R}^d)$, and thus can be presented as the path integral from Proposition 2.

Proof: If $V = 0$, Equation (15) was considered in the Example 2 above. We complete the proof in the same way as for Example 3, namely, by means of Proposition 3. ■

Remark 7. We observe that formula (3) defines a family of finite complex distributions on the path space, which gives rise to a finite complex measure on this path space (under the regularity assumptions). Therefore, this family of measures can be called a complex Markov process. Unlike the case of the standard Markov processes, the generator, say A, of the corresponding semigroup T_t is not self-adjoint, and the corresponding bilinear

''Dirichlet form'' (Av, v) is complex. Such forms present a natural generalization of the real Dirichlet form, which became very popular recently (see e.g. Ma and Röckner (1992), Jacob (1996), Hoh and Jacob (1996) and references therein). In the complex situation, only some particular special cases have so far been investigated, see Albeverio and Ugolini (1997).

3. A Review of Main Approaches to the Construction of Path Integral

We discussed one of the possible approaches to the construction of the rigorous theory of the Feynman integral. Let us now review other known approaches indicating the main ideas and giving references.

1. Analytic continuation, or complex rotation. This is one of the earliest approaches to the mathematical theory of Feynman integrals (see e.g. Cameron (1960), Nelson (1964), Jonson (1982)). In this approach one considers first one of the main parameters, say the mass m, in the standard Schrödinger Equation (9) to be imaginary, i.e. of the form $m = i\tilde{m}$, with $\tilde{m} > 0$. In this case, Equation (9) is a diffusion equation (with a complex source), whose solution can be therefore written in terms of an integral over the standard Wiener measure (using the Feynman-Kac formula). This define a function of m for imaginary m. The analytic continuation of this function (if it exists) can be naturally called the Feynman integral. Equivalently one can carry out the analytical continuation in time. A similar, but slightly different approach is obtained by the idea of rotation in configuration space. Namely, changing the variables x to $y = \sqrt{ix}$ in Equation (9) leads to the equation

$$\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2m}\Delta - iV(-\sqrt{iy}) \right)\psi,$$

which is again of diffusion type and can be thus treated by means of the Feynman-Kac formula and the Wiener measure. However, to use this approach one needs certain (quite restrictive) analytic assumptions on V.

2. Parceval equality. This approach was first systematically developed in Albeverio and Hoehg-Krohn (1976). Let h be a complex constant with a non-negative real part and let L be a complex matrix such that $1 + L$ is non-degenerate with a positive real part. To see the motivation for the main definition given below, suppose first that a function g has the form

$$g(x) = g_\mu(x) = \int_{\mathcal{R}^d} e^{-ipx}\hat{g}(p)dp = \int_{\mathcal{R}^d} e^{-ipx}\mu(dp)$$

with some $\hat{g}$ from the Schwarz space S, where we denoted by μ the finite measure on $\mathcal{R}^d$ with the density $\hat{g}(p)$. Then, g also belongs to the Schwarz space and moreover, due to the Parceval equality,

$$\int \exp\left\{ -\frac{1}{2h}((1+L)x, x) \right\} g(x)dx = (2\pi h)^{d/2}(\det(1+L))^{-1/2}$$

$$\int \exp\left\{-\frac{h}{2}((1+L)^{-1}p,p)\right\}\mu(dp), \tag{16}$$

where both sides of this equation are well defined as Riemann integrals. Suppose, more generally, that

$$g(x) = g_\mu(x) = \int_{\mathscr{R}^d} e^{-ipx}\mu(dp)$$

for any finite Borel complex measure μ (not necessarily with a density). In this case, though the l.h.s. of (16) may not be well defined in the sense of Riemann or Lebesgue, the r.h.s. is still well defined and can be therefore considered as some sort of the regularization of the (possibly divergent) integral on the l.h.s. of (16). In other words, in this case, the integral on the l.h.s. of (16) can be naturally defined by the r.h.s. expression of this equation. In order to get in (16) an expression not depending on the dimension (which one needs to pass succesfully to the infinite dimensional limit), one needs to normalize (or, in physical language, renormalize) this integral by the multiplier $(2\pi h)^{-d/2}$. This leads to the following definition (Albeverio and Hoehg-Krohn (1976)), which can be now given directly in the infinite dimensional setting. Let H be a real separable Hilbert space, let

$$g(x) = g_\mu(x) = \int_H e^{-ipx}\mu(dp) \tag{17}$$

be a Fourier transform of a finite complex Borel measure μ in H and let L be a selfadjoint trace class operator in H such that $1+L$ is an isomorphism of H with a non-negative real part. Define the (normalized) Fresnel integral

$$\int_H^* \exp\left\{-\frac{1}{2h}((1+L)x,x)\right\}g(x)Dx = (\det(1+L))^{-1/2}$$
$$\int_H \exp\left\{-\frac{h}{2}((1+L)^{-1}p,p)\right\}\mu(dp). \tag{18}$$

This definition can be applied to give a rigorous path integral representation for the solutions of Schrödinger Equation (9) with potentials V from $\mathscr{F}(\mathscr{R}^d)$. For this application one takes as the Hilbert space H in (18) the space H_t (sometimes called the Cameron-Martin space) of continuous curves $\gamma : [0,t]\mapsto\mathscr{R}^d$ such that $\gamma(t) = 0$ and the derivative $\dot{\gamma}$ of γ (in the sense of distributions) belongs to $L_2([0,t])$, the scalar product in H_t being defined as

$$(\gamma_1,\gamma_2) = \int_0^t \dot{\gamma}_1(s)\dot{\gamma}_2(s)ds.$$

The definition of the normalized (infinite dimensional) Fresnel integral (18) can be generalized in various ways. The most advanced definition in this direction was given in Cartier and DeWitt-Morette (1993), where the (infinite dimensional) differential D_Θ, z was defined (in a sense, axiomatically) by the formula

$$\int_{\Phi} \Theta(\phi, J) D_{\Theta}, z^{\phi} = Z(J),$$

where Φ and Φ' are two Banach spaces and $\Theta : \Phi \times \Phi' \mapsto \mathscr{C}, Z : \Phi \mapsto \mathscr{C}$ are two given maps.

3. *Discrete approximations* (see e.g. Truman (1978), Truman (1979), Elworthy and Truman (1984) One says that a Borel measurable complex valued function f on $\mathscr{R}^d$ is $\mathscr{F}_h$-integrable, if the limit

$$\lim_{\varepsilon \to 0} (2\pi h)^{-d/2} \int_{\mathscr{R}^d} \exp\left\{ -\frac{1}{2h} |x|^2 \right\} f(x) \psi(\epsilon x) dx \tag{19}$$

exists for any $\psi \in S(\mathscr{R}^d) : \psi(0) = 1$, and is ψ-independent. The limit (19) is then called the *normalised Fresnel integral* (abbreviated NFI) of f (with parameter h) and will be denoted by

$$\int_{\mathscr{R}^d}^{*} \exp\left\{ -\frac{1}{2h} x^2 \right\} f(x) Dx.$$

Let H be a separable real Hilbert space. A Borel measurable complex-valued function f on H is called $\mathscr{F}_h - integrable$ iff for any increasing sequence $\{P_n\}$ of finite dimensional orthogonal projections in H, which is strongly convergent to the identity operator, the limit

$$\lim_{n \to \infty} \int_{P_n H}^{*} \exp\left\{ -\frac{1}{2h} x^2 \right\} f(x) dx$$

exists and its value is independent of the choice of the sequence $\{P_n\}$. In such a case, their common value denoted by

$$\int_{H}^{*} \exp\left\{ -\frac{1}{2h} |x|^2 \right\} f(x) Dx \tag{20}$$

is called the NFI of f (with parameter h). In the case of positive h (resp. purely imaginary h), the NFI (20) is called the normalised Gaussian (resp. oscillatory) integral. Not surprisingly, it turns out that this definition leads to the same formula (18), which was the starting point for the definition of Albeverio and Hoehg-Krohn, (1976).

4. *Path integral as a symbol for perturbation theory.* This approach was systematically developed in Salvnov and Faddeev (1998). Here one considers the path integral simply as a convenient concise symbol, which encodes the rules of perturbation theory in a compact form. From this point of view, one can develope rigorously (at least for Gaussian type integrals, which are important for quantum field theory) a technique of calculations and transformations, which contains all combinatorial aspects of the method of Feynman's diagrams.

5. *Path interal from the point of view of white noise calculus.* In this approach, see T. Hida *et al*. (1993) path integral is considered as a distribution in Hida's infinite dimensional calculus, called also the white noise analysis, which is a calculus on the dual S' to the Schwarz space $S(\mathscr{R}^d)$.

Various extensions of the approaches described above and their applications are developed in many papers, see e.g. Albeverio and Brzezniak (1993), Albeverio *et al*. (1995), Albeverio *et al*. (1979), Albeverio and Hoegh-Khrohn (1977), Beresin (1980), Cartier and DeWitt-Morette (1995), DeWitt-Moretee *et al*. (1979), Elworthy (1984), Smolyanov and Shavgulidze (1990), and references therein. In particular, one can find applications to semiclassical asymptotics in Albeverio *et al*. (1979), Albeverio and Hoegh-Krohn (1977), Maslov (1976), to differential equations on manifolds in Elworthy (1982), Cartier and DeWitt-Morette (1995), to stochastic and infinite dimensional generalizations of Schrödinger's equation in Albeverio *et al*. (1996), Albeverio *et al*. (1997), Kolokoltsov (1999), Truman and Zhao (1996), Tyukov (1999), to rigorous calculations of important quantities of quantum mechanics and quantum field theory in Beresin (1980), DeWitt-Morette *et al*. (1979), and the definition of the Feynman integral over the phase space in Dewitt-Morette *et al*. (1979), Smolyanov and Shavgulidze (1990), Slavnov and Faddeev (1988).

4. Concluding Remarks

The approach to the definition of the Feynman integral developed in this paper seems to be convenient for practical calculations of the path integrals. We are going to discuss this topic in detail in a future paper. Let us indicate here only main ideas. First of all, notice that the assumption on the potential to belong to the space $\mathscr{F}(\mathscr{R}^d)$, i.e. to be the Fourier transform of a finite complex measure (which was necessary for our construction of the path integral), is surely rather restrictive from the theoretical point of view. However, from the point of view of calculations, this assumption is not as bad, because any function from $C_0(\mathscr{R}^d)$ (in particular, any scattering potential) can be approximated uniformly by the elements of the space $\mathscr{F}(\mathscr{R}^d)$. Suppose for simplicity that $d = 1$. Then any function from $C_0(\mathscr{R})$ can be uniformly approximated by piecewise constant functions (sometimes called linear splines) with a compact support, i.e. by the functions of the form $V^h_{a,\delta}$, which vanishes outside the interval $[a, a + (k + 1)\delta]$ and which equals

$$\delta^{-1}[(h_{j+1} - h_j)x - h_{j+1}(a + j\delta) + h_j(a + j\delta + \delta)],$$

for $x \in [a + j\delta, a + (j + 1)\delta], j = 0, \ldots, k$, where a, δ are real constants, $\delta > 0$, and $h = (h_1, \ldots, h_k)$ is a vector, and where it is assumed that $h_0 = h_{k+1} = 0$. In other words $V^h_{a,\delta}$ is a piecewise linear function that equals h_j at the points $a + \delta j, j = 0, \ldots, k + 1$. One finds easily that the Fourier transform $\tilde{V}^h_{a,\delta}(p) = (2\pi)^{-1} \int e^{-ipx} V^h_{a,\delta}(x)$ of $V^h_{a,\delta}$ equals

$$V^h_{a,\delta}(p) = \frac{1 - \cos(\delta p)}{\pi \delta p^2} e^{-ipa} \sum_{j=1}^{k} h_j e^{-ip\delta j}.$$

Therefore $V^h_{a,\delta}$ belongs to $\mathscr{F}(\mathscr{R})$: it has the form V_μ with the finite complex measure $\mu(dp) = \tilde{V}^h_{a,\delta}(p)dp$, which has the analytic density $\tilde{V}^h_{a,\delta}(p)$ with respect to the Lebesgue measure of the real line.

Turning to the discussion of the computational aspects of formalas (12), (13) notice that these formulae are not quite convenient for calculations, because of the complex measure under the integral. However, as shown in our proof of Proposition 5, this measure is actually absolutely continuous with respect to some real measure of a certain (real) compound Poisson process. Writing the corresponding density explicitly will lead to an equivalent formula given already in terms of the real compound Poisson process. The generation of compound Poisson processes are well studied, see Devroye (1985), and can be used for the calculations of the corresponding path integral by means of the Monte-Carlo method.

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