UNIVERSAL ENVELOPING ALGEBRAS, VERMA MODULES, AND THE DEGREES OF A LIE GROUP NOTES FOR MATH 261, SPRING 2002

ALLEN KNUTSON

1. The definition, and the universal property

Given a vector space V, define the **tensor algebra** TV by

$$TV := \bigoplus_{n \in \mathbb{N}} V^{\otimes n}.$$

This has an obvious structure of an associative graded algebra.

It also has a simple universal property: if A is an associative algebra, and $\phi : V \to A$ is a linear map, then there exists a unique extension to an algebra map $\tilde{\phi} : TV \to A$. Put another way, the functor T from **Vec** to **Alg** is left adjoint to the forgetful functor. The example we saw of such a construction last term was the "group algebra" of a group, which was left adjoint to the "group of units" functor from **Alg** to **Grp**.

There's also a forgetful functor from **Alg** to **Lie**, taking an associative algebra to the Lie algebra whose bracket is defined by $[X, Y] := XY - YX$.

We now determine its left adjoint. Let $\mathfrak{g}$ be any old Lie algebra (over any old field), and A an associative algebra. Given a merely *linear* map from $\mathfrak{g}$ to the vector space underlying A, we get an associative algebra map $T\mathfrak{g} \to A$. If we can do better and give a *Lie* map from $\mathfrak{g}$ to the Lie algebra associated to A, then this map $T\mathfrak{g} \to A$ must include all

$$X \otimes Y - Y \otimes X - [X, Y], \quad X \in \mathfrak{g}$$

in its kernel.

So define the **universal enveloping algebra** $U\mathfrak{g}$ of $\mathfrak{g}$ as the quotient of the tensor algebra $T\mathfrak{g}$ by the ideal generated by these relations. This has the desired property: any Lie map $\mathfrak{g} \to A$ extends uniquely to an associative map $U\mathfrak{g} \to A$.

In particular, every representation of $\mathfrak{g}$ (such as the differenital of a representation of $\mathfrak{G}$) gives a module over the algebra $U\mathfrak{g}$.

2. The Poincaré-Birkhoff-Witt theorem

Since the relations $XY - YX - [X, Y]$ mix degree 2 terms in $T\mathfrak{g}$ with degree 1 terms, the quotient algebra $U\mathfrak{g}$ isn't graded – it's only filtered: if $(U\mathfrak{g})_n$ is defined as the image of $\oplus_{i \leq n} \mathfrak{g}^{\otimes i}$, then $(U\mathfrak{g})_m (U\mathfrak{g})_n \leq (U\mathfrak{g})_{m+n}$.

Therefore the **associated graded** space $\oplus_n (U\mathfrak{g})_n / (U\mathfrak{g})_{n-1}$ possesses a well-defined graded algebra structure. There is a naïve guess as to what this graded algebra is: just replace each of the filtered relations $XY - YX - [X, Y]$ by its top degree part, $XY - YX$. That is the **symmetric algebra** $Sym(\mathfrak{g}) = T\mathfrak{g} / (\{XY - YX\}_{X,Y \in \mathfrak{g}})$.

One can push this far enough to show that there is a well-defined map from $\mathrm{Sym}(\mathfrak{g})$ onto the associated graded of $U\mathfrak{g}$.

Theorem (Poincaré-Birkhoff-Witt). *This map* $\mathrm{Sym}(\mathfrak{g}) \twoheadrightarrow U\mathfrak{g}$ *is an isomorphism.*

We only sketch the proof. If the theorem were true, we'd know how to think of the vector space $\mathrm{Sym}(\mathfrak{g})$ as a module over $U\mathfrak{g}$. Then we could restrict it to a Lie representation of $\mathfrak{g}$. It's actually easy to write down this action, and check that it's well-defined for $\mathfrak{g}$. Then by the universal property, it extends to an action of $U\mathfrak{g}$. This turns out to give the map backwards from $U\mathfrak{g}$ to $\mathrm{Sym}(\mathfrak{g})$. QED.

Gröbner basis fanatics will note that the original set of relations is a noncommutative Gröbner basis (which also proves PBW).

3. The quasi-classical limit

Instead of imposing $XY - YX = [X, Y]$, introduce a new variable $\hbar$, and impose $XY - YX = \hbar[X, Y]$. So the quotient, as a vector space, is $U\mathfrak{g}[\hbar]$. The PBW theorem says we get a family of algebra structures, on the same vector space, as we take $\hbar$ actually equal to a parameter and let it vary.

Exercise. Show that setting $\hbar$ to a number gives us $U\mathfrak{g}$, so long as the number isn't zero.

If we let $\hbar^2 = 0$ (but not $\hbar = 0$), then any product of elements in $U\mathfrak{g}$ has two terms:

$$pq = p \cdot q + \hbar\{p, q\}$$

where the $p \cdot q$ is the commutative product from $\mathrm{Sym}(\mathfrak{g})$.

Exercise. Show that $\{p, q\}$

- is antisymmetric;
- satisfies the Jacobi identity (so it is a Lie bracket);
- satisfies the Leibniz rule $\{p, qr\} = \{p, q\}r + q\{p, r\}$.

This is called a **Poisson bracket** on $\mathrm{Sym}(\mathfrak{g})$. Geometrically, it corresponds to a 2-tensor on $\mathfrak{g}^* = \mathrm{Spec}\mathrm{Sym}(\mathfrak{g})$.

Since it is a 2-tensor, we can contract it with cotangent vectors and get vectors. One can show that the image in the tangent space to $\lambda \in \mathfrak{g}^*$ is the tangent space to the coadjoint orbit through λ, and indeed the tensor is the "inverse" to the symplectic form we defined on coadjoint orbits!

4. Verma modules

We bring up another couple of adjoint functors. Given a subring R of S (or more generally, a ring homomorphism, but let's stick with subring), we can restrict the action of S on an S-module to an R-action, giving a forgetful functor from S-**Mod** to R-**Mod**.

This functor has a left adjoint called **extension of scalars**, taking an R-module A to the S-module $S \otimes_R A$. (Note: this doesn't require commutativity, since S is an R-bimodule.)

If we'd been more into rep theory of finite groups, we would have used this to extend a rep V of a finite group H to one of an overgroup G, making $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$. The dimension goes up by the factor $|G|/|H|$. This is mostly why we've avoided it for Lie groups, because we've avoided infinite-dimensional representations.

But now's the time for them to appear. Let B is a Borel subgroup of a complex Lie group G, so $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$, and $\lambda \in \mathfrak{t}^*$ a weight. Then let $\mathbb{C}_\lambda$ be the one-dimensional representation of $\mathfrak{b}$, and define

$$\mathrm{Verm}_\lambda := U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda.$$

This gives a module for $U\mathfrak{g}$, and therefore a Lie representation of $\mathfrak{g}$; it does *not* give a representation for G. We're used to the idea "if G's simply connected, then by exponentiation we can just extend an algebra map to a group map", but that requires that the exponential map converge, and in these infinite-dimensional cases it doesn't.

These Verma modules satisfy a universal property. Let Rep_λ denote the category of $\mathfrak{g}$-modules equipped with a chosen high weight vector of weight λ. Then Verm_λ is the initial object; it has a unique map to any module in this category, taking its high weight vector to theirs.

5. The center of the universal enveloping algebra

In this one we assume G is the complexification of a compact group, and in particular that $\mathfrak{g} \cong \mathfrak{g}^*$ as G-reps.

If we note that the degeneration of $U\mathfrak{g}$ to $\mathrm{Sym}(\mathfrak{g})$ is G-equivariant, we can follow the G-invariant subspace:

$$Z(U\mathfrak{g}) = (U\mathfrak{g})^G \; \mathrm{Sym}(\mathfrak{g})^G$$

The latter is G-invariant polynomials on $\mathfrak{g}^*$ – or let us say $\mathfrak{g}$ – which are determined by their restriction to $\mathfrak{t}$, giving a map $\mathrm{Sym}(\mathfrak{g})^G \hookrightarrow \mathrm{Fun}(\mathfrak{t})^W$.

Exercise. Show the first equality above.

Example. If $G = GL_n(\mathbb{C})$, then we're looking at conjugation-invariant functions of a complex matrix. Since the diagonalizable matrices are dense, it suffices to look at S_n-invariant functions of the diagonal entries. We know these: they're "symmetric polynomials", or equivalently, they're polynomials in the elementary symmetric polynomials in the eigenvalues.

In particular, in this case $\mathrm{Fun}(\mathfrak{t})^W \cong \mathbb{C}[e_1, e_2, \ldots, e_n]$ where e_i is of degree i. This commutative ring is in fact a polynomial ring!

In fact the ring $Z(U\mathfrak{g})$ is always isomorphic to $\mathrm{Fun}(\mathfrak{t})^W$; there is an explicit isomorphism called the "Harish-Chandra homomorphism", whose details will not concern us.

6. The degrees and exponents of a Weyl group, and Coxeter elements

Theorem (Chevalley)**.** *Let* V *be a vector space with a nondegenerate symmetric form, and* W *a finite subgroup of* $O(V)$ *generated by reflections. Then* $\mathrm{Fun}(V)^W$ *is a polynomial ring. In particular, the degrees of its generators (as a graded ring) are well-defined.*

These degrees $\{d_i\}$ are called the **exponents of the group** W, and have a million interesting properties (our reference is Humphreys' *Reflection groups and Coxeter groups*).

Proposition.
- *The product of the degrees is* $|W|$.
- *The sum of the degrees is the rank of* G *plus the number of reflections in* W.
- *The Poincaré polynomial of* G/B *is* $\prod_i (1 - q^{2d_i})/(1 - q^2)$.
- *The cohomology ring* $H^*(BG)$ *is a polynomial ring, with generators in degrees* $\{2d_i\}$.

- *The cohomology ring* $H^*(G)$ *is an exterior algebra, with generators in degrees* $\{2d_i - 1\}$.

Exercise. Check these for $U(2)$, and the first three for $U(n)$.

Subtracting one from each d_i we get the **exponents** $\{e_i = d_i - 1\}$ of the Weyl group. (Don't be fooled: these *do* appear often enough to deserve their own name.) To really appreciate them, we first need to introduce Coxeter elements, as follows.

Start by multiplying all the simple reflections in W together exactly once. In S_n one gets n-cycles this way.

Exercise. Use the fact that Dynkin diagrams are trees to 2-color them "black" and "white". Then one particularly nice way to get a Coxeter element is to multiply all the black reflections together first, and then all the white ones. Show that the answer is well-defined, except for the Z_2 choice of which group is black and which is white. Can one use the affine diagram to break the black/white symmetry?

Exercise. Use the fact that Dynkin diagrams are trees to show that the conjugacy class of a Coxeter element is always well-defined. Hint: use induction, and the fact that moving an element from one end of a word to the other is a conjugation.

In particular, one can ask for the eigenvalues of the Coxeter element acting on $\mathfrak{t}$. Since it's a finite-order element (with order **the Coxeter number** h), the eigenvalues are hth roots of unity.

Proposition.

- *The eigenvalues of the Coxeter are* $\exp(2\pi i\, e/h)$*, as* e *varies over the exponents. (Hence the name.)*
- *If* $1 \leq e < h$ *and* $(e, h) = 1$*, then* e *is an exponent.*
- *The Coxeter number* h *is* $|\Delta|$ *divided by* $\dim \mathfrak{t}_{ss}$ *(from the commutator subgroup of* G*).*
- *If we make a partition whose rows have length the exponents, then the height of the* i*th column is the number of positive roots of height* i.

Example. Let $G = E_8$. Then by playing the find-the-highest-root game, we can determine $|\Delta| = 240$. So $h = 30$. Then we get seven es $1, 7, 11, 13, 17, 19, 23, 29$, and we're done!

Exercise. Express in these terms the height of the highest root.

Exercise. Check all these statements for $SU(n)$.