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Lie groups: Beyond an introduction, by Anthony W. Knapp, Birkhäuser, Boston, MA, 1996, xv+604 pp., \$49.50, ISBN 0-8176-3926-8, ISBN 3-7643-3926-8

From its beginnings with Sophus Lie, the theory of Lie groups was concerned with the explicit description of the group law in coordinates. Such questions can have nice answers only in coordinate neighborhoods of the identity. Accordingly many results were expressed in terms of a notion of "local Lie group". This is a small neighborhood U of the identity endowed with as much of the group law as fits into U. (That every local Lie group arises from a Lie group is a rather difficult theorem.) Two accounts of Lie theory from this perspective may be found in [3] (available in Cartan's collected works) and [7].

The modern approach to Lie groups was first enunciated in Chevalley's text [4]. Chevalley worked everywhere with a global Lie group, that is, with an analytic manifold G endowed with a group structure making multiplication and inversion analytic. The *Lie algebra* of G is the collection $\mathfrak{g}$ of vector fields on G invariant under left translation; the Lie bracket is commutator of vector fields. Chevalley formulated and proved a family of results making a clear and powerful "dictionary" between Lie algebras and Lie groups. Most of the results were older; work of Schreier in the 1920s on covering groups had completed the global theory. Nevertheless it was Chevalley's text that framed questions we now take for granted: given a Lie algebra $\mathfrak{g}$, how can we describe (the various possibilities for) the corresponding Lie group G? what are the subalgebras $\mathfrak{h} \subset \mathfrak{g}$ and the corresponding subgroups $H \subset G$?

Knapp's book is an introduction to the answers to many of these questions. I'll begin by wandering through some of the material; a more systematic list of what is in the book will come later.

One way to understand many results about Lie groups is as generalizations of ideas from linear algebra. From this point of view the most fundamental example of a Lie group is $GL(n, \mathbb{R})$, the group of $n \times n$ invertible matrices. The Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ consists of all $n \times n$ matrices, and the Lie bracket is commutator of matrices: $[X, Y] = XY - YX$.

Here is an example of a linear algebra result we would like to generalize.

Proposition 1 (Polar decomposition)**.**
1. *Any $n \times n$ invertible matrix g has a unique factorization $g = kp$, with k an orthogonal matrix and p a positive definite symmetric matrix.*
2. *Any positive definite symmetric matrix p has a unique representation $p = \exp(X)$, with X a symmetric matrix.*
3. *The collection $O(n)$ of all orthogonal matrices is a compact Lie group.*
4. *The collection $\mathfrak{s}$ of all symmetric matrices is a vector space, diffeomorphic by the exponential map to the collection S of all positive definite symmetric matrices.*

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5. *The group* $GL(n,\mathbb{R})$ *is diffeomorphic to the product* $O(n)\times\mathfrak{s}$ *of a compact group and a vector space, by the map*

$$(k,X)\mapsto k\exp(X).$$

Here (rephrased a bit from Proposition 1.122 in Knapp's book) is a first step towards a generalization.

Proposition 2 (Cartan decomposition)**.** *Suppose* $G\subset GL(n,\mathbb{R})$ *is any subgroup defined by polynomial equations in the matrix entries and closed under transpose of matrices. Define* $K=G\cap O(n)$ *to be the group of orthogonal matrices in* G*, and* $\mathfrak{p}=\mathfrak{g}\cap\mathfrak{s}$ *to be the space of symmetric matrices in the Lie algebra of* G*. Finally put* $P=\exp(\mathfrak{p})\subset G$*.*

1. *The polar decomposition* $g=kp$ *of any element of* G *has* $k\in K$ *and* $p\in P$*.*
2. *The group* G *is diffeomorphic to the product* $K\times\mathfrak{p}$ *of a compact group and a vector space, by the map*

$$(k,X)\mapsto k\exp(X).$$

Easy examples of groups G satisfying the hypotheses of Proposition 2 include all the classical matrix groups, such as the group $O(p,q)$ of matrices preserving the quadratic form

$$Q(x_1,\dots,x_{p+q})=x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2.$$

The structure theory for semisimple Lie algebras implies that any semisimple adjoint group can be realized as a group of matrices satisfying the hypotheses of Proposition 2; this is more or less the content of Theorem 6.31 in Knapp's book.

Proposition 2 says that many questions about the topology of semisimple Lie groups can be reduced to the case of compact groups. The topology of compact groups is a wonderful subject in its own right. What appear in Chapter IV of Knapp's book are just some of the foundations, selected for their importance in representation theory and in the structure of noncompact groups. Here is an example. Recall that a *torus* in a compact Lie group is a subgroup isomorphic to $\mathbb{R}^n/\mathbb{Z}^n$, and a *maximal torus* is a torus not properly contained in any other.

Theorem 3. *Suppose* G *is a compact connected Lie group and* $T\subset G$ *is a maximal torus.*

1. *Every conjugacy class in* G *meets* T*.*
2. *Any torus in* G *is conjugate to a subtorus of* T*.*
3. *Two elements* $t,t'\in T$ *are conjugate in* G *if and only if they are conjugate by the normalizer of* T *in* G*.*
4. *If* $T_0\subset T$ *is a subtorus, then the centralizer of* T_0 *in* G *is a compact connected subgroup of* G*.*

All of this is proved in Chapter IV of Knapp's book (Theorem 4.36, Corollary 4.51, and Proposition 4.53).

Proposition 2 provides information about the global structure of semisimple Lie groups. A defining characteristic of semisimple Lie groups is that they have very few normal subgroups. A more elementary problem is the analysis of global structure in the presence of normal subgroups, and this Knapp addresses near the beginning of his book. Here is a basic definition, taken from section 12 of Knapp's Chapter I.

Definition 4. Suppose G and H are Lie groups. An *action of G on H by automorphisms* is a smooth map $\tau\colon G \times H \to H$, with the following two properties. First, for each $g \in G$, the map $\tau(g, \cdot)$ from H to H is an automorphism of H. Second, the map from G to $\mathrm{Aut}(H)$ (sending g to $\tau(g, \cdot)$) is a group homomorphism. These two properties can be written

$$\tau(g, h_1 h_2) = \tau(g, h_1)\tau(g, h_2), \qquad \tau(1, h) = h, \qquad \tau(g_1 g_2, h) = \tau(g_1, \tau(g_2, h)).$$

Given an action of G on H, the *semidirect product of G by H* is the Lie group $G \times_\tau H$ whose underlying manifold is $G \times H$, with group law

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, \tau(g_2^{-1}, h_1)h_2).$$

This semidirect product contains H as a normal subgroup and G as a subgroup meeting H only in the identity.

There is a parallel notion of semidirect products of Lie algebras, given in Knapp's Proposition 1.22.

Definition 5. Suppose $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras. An *action of $\mathfrak{g}$ on $\mathfrak{h}$ by derivations* is a linear map $\pi\colon \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$, with the following two properties. First, for each $X \in \mathfrak{g}$, the map $\tau(X, \cdot)$ from $\mathfrak{h}$ to $\mathfrak{h}$ is a derivation of $\mathfrak{h}$. Second, the map from $\mathfrak{g}$ to $\mathrm{Der}(\mathfrak{h})$ is a Lie algebra homomorphism. These two properties can be written

$$\pi(X, [Y_1 Y_2]) = [\pi(X, Y_1), Y_2] + [Y_1, \pi(X, Y_2)],$$

$$\pi([X_1, X_2], Y) = \pi(X_1, \pi(X_2, Y)) - \pi(X_2, \pi(X_1, Y)).$$

Given an action of $\mathfrak{g}$ on $\mathfrak{h}$ by derivations, the *semidirect product of $\mathfrak{g}$ by $\mathfrak{h}$* is the Lie algebra $\mathfrak{g} \times_\pi \mathfrak{h}$ whose underlying vector space is $\mathfrak{g} \times \mathfrak{h}$, with Lie bracket

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2] + \pi(X_1, Y_2) - \pi(X_2, Y_1)).$$

This semidirect product contains $\mathfrak{h}$ as an ideal and $\mathfrak{g}$ as a subalgebra meeting $\mathfrak{h}$ only at 0.

Chevalley's dictionary between Lie groups and Lie algebras almost immediately proves the following result (Knapp's Theorem 1.102).

Theorem 6. *Suppose G and H are connected, simply connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and suppose π is an action of $\mathfrak{g}$ on $\mathfrak{h}$ by derivations. Then there is a unique action τ of G on H by automorphisms, with differential π. The semidirect product group $G \times_\tau H$ is a connected, simply connected Lie group with Lie algebra $\mathfrak{g} \times_\pi \mathfrak{h}$.*

Knapp shows how to construct all simply connected Lie groups by iterating this semidirect product construction, beginning with the real line and with semisimple groups. Because of such results, one can sensibly focus on the structure of semisimple groups, and this Knapp does in the last half of his book. Among many other things, he proves the Iwasawa decomposition (generalizing the Gram-Schmidt orthogonalization process to semisimple groups), the classification of real semisimple Lie algebras, and the Bruhat decomposition (generalizing the cell decomposition of projective space to a wide class of compact homogeneous spaces).

Having skimmed some of the cream from the book, let us discuss the contents in more detail. The reader is assumed to know about Chevalley's dictionary between Lie groups and Lie algebras, but even this is carefully and thoroughly summarized (with references to proofs) in the tenth section of Chapter I. Chapter I begins with a nice introduction to Lie algebras, more or less along the lines of the first two chapters of [6]. One difference is that Knapp pays more attention to Lie algebras over fields that are not algebraically closed (since the case of $\mathbb{R}$ is central to the rest of the book). The end of Chapter I deals with Lie groups and the first consequences of the Lie-algebraic results for their global structure. Theorem 6 above is typical.

Chapter II concerns the structure of complex semisimple Lie algebras, more or less as in [8] or in Chapters III–V of [6]. Chapter III is about the universal enveloping algebra (of a Lie algebra) and the Poincaré-Birkhoff-Witt theorem; this is an important technical tool for the representation theory in Chapter V.

Chapter IV concerns the structure and representation theory of compact Lie groups, including the Peter-Weyl theorem and Theorem 3 above. Chapter V describes the irreducible representations in detail, culminating in the Weyl character formula. This material is similar to Chapter VI of [6]; the main difference is that analytic interpretations of the results are not available in [6].

Chapters VI and VII concern the structure of semisimple groups. I have already mentioned some of the highlights. Chapter VI concludes with the classification of real semisimple Lie algebras. The classification is based on a maximally compact Cartan subgroup. This is a change from the traditional approach (found in [5], for example) based on a maximally split Cartan subgroup. The traditional approach has advantages (for example, it generalizes well to other local fields), but Knapp's approach is a much better introduction to current work on infinite-dimensional representation theory, where Zuckerman's "cohomological induction" functors play a central rôle. In any case, Knapp provides in section 11 of Chapter VI a complete dictionary from his version of the classification to the traditional one.

Chapter VIII is about integration on Lie groups in general and semisimple groups in particular. One can find here the basic general facts relating Haar measures on G, a subgroup H, and the homogeneous space G/H; integration formulas corresponding to the Iwasawa and Bruhat decompositions follow. Weyl's formula for relating integration on a compact group to a maximal torus is proved, and Harish-Chandra's generalization to semisimple groups is stated.

There are three appendices. The first is a course on tensor algebra: more or less elementary mathematics that is critical to the theory of Lie algebras, but which is often not found in an undergraduate curriculum. The second is a short and lucid account of Cartan's construction of a Lie group attached to any Lie algebra ("Lie's third theorem"); few if any textbooks include this result. The third appendix is a series of tables of information about root systems, Weyl groups, and the structure of real simple Lie algebras. On the last subject in particular, the tables in Knapp stand up very well in comparison with those on pages 514–520 of [5] and are a substantial improvement on pages 30–32 of [11]. (I found these other references easily, because my copies of these books fall open to these pages.)

This is a wonderful choice of material. Any graduate student interested in Lie groups, differential geometry, or representation theory will find useful ideas on almost every page. Each chapter is followed by a long collection of problems (roughly one for every two pages of text). The problems are interesting and enlightening; if enlightenment seems too distant, there are extensive hints at the back of the book.

The exposition, as in all of Knapp's books, is very careful and complete. A reader searching for an isolated result will appreciate the cross-references to notation and related material; it isn't necessary to read the entire book to make sense of one paragraph. The index is good, if not quite perfect: there is no entry for "maximal torus", for example. (The table of contents provides very clear hints about where to find this subject, however.) There is a very complete index of notation (where one can look up "T" to find maximal tori). The text has been proofread with extraordinary care. I found only half a dozen misprints in reading perhaps half of it. Most are as harmless as "Theoreem" on page xiv. The only substantive one I found is in the table on page 362: in the entry for $\mathfrak{so}(p,q)$ with $p+q$ odd, the restriction should be $p+q \geq 5$ rather than $p+q \geq 3$. (Without this restriction, there are more isomorphisms among the algebras listed in the table.) Most of Knapp's notational conventions are standard and reasonable. I was occasionally confused only by two: a semisimple group for him is by definition connected, and the element H_α in a Cartan subalgebra is the one corresponding to α under an invariant bilinear form, rather than the coroot.

A final question is what other sources there are for this material. Foremost is Helgason's text [5]; the first edition appeared in 1962. This contains almost all the material on semisimple groups in Knapp's book, together with much more on differential geometry and symmetric spaces. Helgason's book has served generations of students very well, and a student interested in differential geometry will still prefer it. Knapp's emphasis is on Lie theory; that is why he begins with Lie algebras while Helgason begins with Riemannian geometry. For the algebraically inclined, Knapp's approach will often be more palatable.

There are several valuable books about compact Lie groups. Želobenko's book [12] is a wonderful source for details about the representations of classical groups, but some important foundational material (like the classification of compact Lie groups) is only sketched. The book [2] of Bröcker and tom Dieck is another beautiful treatment of representation theory, but Lie algebras are systematically avoided; so for example the classification of compact Lie groups is omitted entirely. Adams' classic text [1] again treats the structure and representation theory of compact groups very well, but avoids the classification. Unfortunately I am not familiar with Varadarajan's text [9], and could not find a copy while writing this review. (The copy belonging to the MIT library was in circulation, and the one belonging to the department reading room had been stolen. Both of these facts are endorsements, of course.) By reputation Varadarajan's book is a good one, accessible to beginners and including much more than Bröcker and tom Dieck about the Lie algebra theory.

For a treatment of noncompact groups in book form, there are fewer choices. I have already mentioned [5]. Most of the structure theory for semisimple groups developed by Knapp may also be found for example in [10] and [11], but these books are aimed at much deeper problems in harmonic analysis and are not nearly as accessible.

Altogether this book is delightful and should serve many different audiences well. It would make a fine text for a second graduate course in Lie theory (whether aimed at Lie groups, finite-dimensional representation theory, or even just at the structure of Lie algebras). The presence of references and the absence of errors make it well-suited for self-study. A student learning about Lie algebras from [6], for example, could consult Knapp to find out quickly and clearly what that mathematics has to do with compact groups. Experts will be able to use it as a reference, both for formulations and proofs of basic results and for details about the examples from which the theory of semisimple groups is built. I am delighted to have this book in my (ever-widening) collection of Knapp's work; my only complaint is that the cover adds nothing to the collection's already ill-coordinated color scheme.

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