Topology Vol. 27, No. 4, pp. 387–392, 1988
Printed in Great Britain.

0040-9383/88 $3.00+.00

A CONJECTURE OF BARRATT–JONES–MAHOWALD CONCERNING FRAMED MANIFOLDS HAVING KERVAIRE INVARIANT ONE

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(*Received in revised form* 21 *July* 1987)

§1. INTRODUCTION

TO SETTLE the question of the existence or non-existence of a framed manifold having a non-trivial Kervaire invariant (or Arf invariant) is one of the main long-standing problems in algebraic topology. The Kervaire invariant is a $\mathbb{Z}/2$-valued invariant which may be formulated in many contexts. Originally it occurred as an invariant in framed surgery theory and for this approach the reader may consult [12] for example. It is reformulated in [7] in terms of the Adams spectral sequence for the stable homotopy of spheres. In particular the only open cases were reduced to determining whether $h_k^2 \in \mathrm{Ext}_{\mathscr{A}}^{2,2^{k+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$ is an infinite cycle, producing a non-trivial element θ_k in the $2^{k+1}-2$ stem of the stable homotopy of spheres. More recently the Kahn–Priddy theorem [8] and the algebraic Kahn–Priddy theorem [10] have been used to convert it to a problem in the stable homotopy of infinite dimensional real projective space $\mathbb{R}P^\infty$. The Kahn–Priddy theorem gives a stable map τ: $\mathbb{R}P^\infty \to S^0$ which is a split surjection of stable homotopy groups (localized at the prime 2)

$$(1.1) \qquad \pi_*^S(\mathbb{R}P^\infty)_{(2)} \xrightarrow{\pi_*\tau} \pi_*^S(S^0)_{(2)}.$$

The algebraic Kahn–Priddy theorem states that the map of Ext groups, induced by τ, is an epimorphism

$$(1.2) \qquad \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(\mathbb{R}P^\infty), \mathbb{Z}/2) \twoheadrightarrow \mathrm{Ext}_{\mathscr{A}}^{s+1,t+1}(\mathbb{Z}/2, \mathbb{Z}/2).$$

It is well known that, (see [11] for example), together they reduce the question of the existence of θ_{k+1} to the existence of a stable map g: $S^{2^{k+1}-2} \to \mathbb{R}P^{2^{k+1}-2}$ which is detected by Sq^{2^k}, i.e. Sq^{2^k} is nonzero on $H_{2^{k+1}-1}$ (Cone g). Notice that this reduces it from a problem about secondary operations to one about primary operations. What this paper does is show that this question about primary operations in cohomology is equivalent to a question about e-invariants in K-theory. We do this by proving the following theorem.

(1.3) THEOREM. *Let* g: $\Sigma^N S^n \to \Sigma^N \mathbb{R}P^n$, $(n=2t-2,\ t=2^k,\ k\geq 2)$ *be a map representing* $[f] \in \pi_n^S(\mathbb{R}P^\infty)$. *Then the following are equivalent*:

(a) $[f] \in \pi_n^S(\mathbb{R}P^\infty)$ has non-trivial Kervaire invariant in the sense mentioned above.
(b) $g_* = f_*$: $jo_n(S^n) \to jo_n(\mathbb{R}P^\infty)$ *is non-trivial.*
(c) g has $KU_* - e$-invariant equal to $((3^t-1)/4)(2s+1)$ in (2.2).
(d) g (or f) has $bo_* - e$-invariant of the form $2^{k+1}(2v+1)$ in $bo_{n+1}(\mathbb{R}P^\infty) \cong \mathbb{Z}/2^t$.

In [5] it was shown that (b) implies (a), in §1.3, and it was conjectured there that the converse holds. The formulation of the Kervaire invariant problem as in §1.3 (c)/(d) is much easier to work with than that of (a) or (b). As an (easy) exercise the reader is invited to derive all the Hopf invariant results of [5] from this formulation.

§2 will show the equivalence of (b), (c) and (d). We conclude this section by giving a proof, modulo some computational lemmas proved in §3, of the equivalence of (a) and (d) in (1.3).

Proof. Remember we are trying to determine whether f is null homotopic, i.e. whether, in the cofibration

(1.4) $$S^n \xrightarrow{f} \mathbb{R}P^\infty \xrightarrow{h} \mathrm{Cone}\, f \xrightarrow{c} S^{n+1},$$

it happens that $\mathrm{Cone}\, f \simeq \mathbb{R}P^\infty \vee S^{n+1}$. Ignoring the action of the Steenrod algebra, we find that

(1.5) $$H\mathbb{Z}/2_*(\mathrm{Cone}\, f) \cong H\mathbb{Z}/2_*(\mathbb{R}P^\infty) \oplus H\mathbb{Z}/2_*(S^{n+1}).$$

However, one can also consider the action of the Steenrod algebra on $H\mathbb{Z}/2_*(\mathrm{Cone} f)$. In particular if Sq^{2^k} is nonzero on $H_{2^{k+1}-1}(\mathrm{Cone} f) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, we say f is detected by Sq^{2^k}. (One should note that one of the $\mathbb{Z}/2$'s comes from a cell in $\mathbb{R}P^\infty$ and it is known that Sq^{2^k} is zero on it.) It is important for our proof to notice that, in this case, detection by Sq^{2^k} is equivalent to detection by $Sq^{2^k}+b$ where b is a decomposable element of degree 2^k in the mod 2 Steenrod algebra, because b must be zero on $H_{2^{k+1}-1}$ (Cone f). There are two ways to see this. If f is detected by Sq^{2^a} for $a<k$, then our proof of the main theorem would imply that $[f]$ has a jo_* Hurewicz image of order greater than two which contradicts §2.8. Alternatively one can look at (4.6) of [5] to learn that $\mathrm{Ext}^{1,2^{k+1}-1}(H^*\mathbb{R}P^\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2$ generated by an element whose survival to E_∞ would indicate a map detected by Sq^{2^k}.

Working with bo homology we get

(1.6)
$$\begin{array}{ccccc} 0\to bo_{2^{k+1}-1}(\mathbb{R}P^\infty) & \xrightarrow{bo_*h} & bo_{2^{k+1}-1}(\mathrm{Cone}\, f) & \xrightarrow{bo_*c} & bo_{2^{k+1}-1}(S^{2^{k+1}-1})\to 0 \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ \mathbb{Z}/2^{2^k}\langle\beta_{2^{k-1}}\rangle & & \mathbb{Z}/2^{2^k}\langle\beta_{2^{k-1}}\rangle \oplus \mathbb{Z}_{(2)}\langle F\rangle & & \mathbb{Z}_{(2)}\langle T\rangle \end{array}$$

(see [14] for the calculation of $bo_*\mathbb{R}P^\infty$). This sequence splits as bo_* modules and in a further attempt to detect f we consider the action of ψ^3-1 on $bo_*\mathrm{Cone} f$. F has been chosen to be a preimage of T. Since $\psi^3 T = T$, the naturality of ψ^3 tells us that $(\psi^3-1)(F) \in \mathrm{Ker}\, bo_*c = \mathrm{Im}\, bo_*h$. So $(\psi^3-1)F = \lambda\beta_{2^{k-1}}$ for some $\lambda \in \mathbb{Z}/2^{2^k}$. Since (see [14])

$$(\psi^3-1)(\beta_{2^{k-1}}) = (9^{2^{k-1}}-1)\beta_{2^{k-1}} = (2s+1)2^{k+2}\beta_{2^{k+1}},$$

if $\lambda | 2^{k+2}$ ($\lambda \neq 2^{k+2}$) then Cone f is not homotopic to $\mathbb{R}P^\infty \vee S^{n+1}$ and we say that f has a nonzero e-invariant.

To relate these two methods of detection we use the following commutative diagram

(1.7)
$$\begin{array}{ccccc} H\mathbb{Z}/2_{2^{k+1}-1}(\mathrm{Cone}\, f) & \leftarrow & bo_{2^{k+1}-1}(\mathrm{Cone}\, f) & \xrightarrow{\psi^3-1} & bo_{2^{k+1}-1}(\mathrm{Cone}\, f) \\ \downarrow Sq^{2^k}+a & & \downarrow & & \downarrow (\psi^3-9)\ldots(\psi^3-9^{2^{k-2}-1}) \\ H\mathbb{Z}/2_{2^k-1}(\mathrm{Cone}\, f) & \overset{\cong}{\leftarrow} & (\Sigma^{2^k}bo)^{(2^{k-1}-1)}_{2^{k+1}-1}(\mathrm{Cone}\, f) & \xrightarrow{j^{2^{k-3}}\circ\lambda} & bo_{2^{k+1}-1}(\mathrm{Cone}\, f) \\ & & \cong\downarrow & & \\ & & \mathbb{Z}/2 & & \end{array}$$

which can be recovered from Theorem B and Theorem 4.2 in [15] (for the definition of θ, λ, j and $bo^{(n)}$ see Theorem B there).

An Adams spectral sequence calculation (see Lemma 3.1) will show that $bo_{2^k-1}^{(2^{k-1}-1)}$ (Cone f) $\cong \mathbb{Z}/2$ and that the map to $H\mathbb{Z}/2_{2^k-1}$ (Cone f) is an isomorphism. Similarly, studying the cofibration associated to Cone f indicates that in

$$\begin{array}{ccc} bo_{2^{k+1}-1}(\text{Cone } f) & \longrightarrow & H\mathbb{Z}/2_{2^{k+1}-1}(\text{Cone } f) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{Z}/2^{2^k}\langle\beta_{2^k-1}\rangle \oplus \mathbb{Z}_{(2)}\langle F\rangle & & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \end{array}$$

F is sent to the generator of the $\mathbb{Z}/2$ coming from the cell attached by f. Hence f is detected by Sq^{2^k} if and only if $\theta_{2^k-2}F$ is non-zero. Finally Lemmas (3.2) and (3.3) will show that

$$\begin{aligned}(\psi^3-1)(\psi^3-9)\ldots(\psi^3-9^{2^{k-2}-1})(F) &= (2s+1)\cdot\lambda\cdot 2^{2^k-k-2}\beta_{2^k-2} \\ &= 2^{2^k-1}\cdot\theta_{2^k-2}(F)\end{aligned}$$

Thus $\theta_{2^k-2}F$ is nonzero if and only if $v_2(\lambda)=k+1$ i.e. f has a nonzero e invariant (one should notice that this immediately gives the fact that f is not divisible by 2). Q.E.D.

§2. k-THEORY e-INVARIANTS

Suppose that $t=2^k$, $n=2t-2$ and that $g\colon \Sigma^N S^n \to \Sigma^N \mathbb{R}P^n$ is a map which represents $[f]$ in §1.3. Let KU_* denote reduced (periodic) unitary K-homology [1, p.134] and let bu_*, bo_* denote connective unitary and orthogonal K-theory respectively [2, p. 146]. Let $C(g)$ denote the cofibre of g and consider the resulting K-theory sequences.

(2.1)

$$\begin{array}{ccccccc} \mathbb{Z}/2^{t-1}\cong KU_{n+1}(\mathbb{R}P^n) & \rightarrowtail & KU_{n+1}(C(g)) & \overset{\delta}{\twoheadrightarrow} & KU_n(S^n)\cong\mathbb{Z} \\ \uparrow\cong & & \uparrow & & \uparrow\cong \\ bu_{n+1}(\mathbb{R}P^n) & \rightarrowtail & bu_{n+1}(C(g)) & \overset{\delta}{\twoheadrightarrow} & bu_n(S^n) \\ \uparrow\cong & & \uparrow & & \uparrow\cong \\ \mathbb{Z}/2^{t-1}\cong bo_{n+1}(\mathbb{R}P^n) & \rightarrow & bo_{n+1}(C(g)) & \overset{\delta}{\twoheadrightarrow} & bo_n(S^n) \end{array}$$

Let $F\in KU_{n+1}(C(g))$ be such that $\delta(F)$ is a generator and let ψ^3 denote the Adams operation, as usual. The KU_*-e-invariant of g is given by

(2.2) $$(\psi^3-1)(F)\in KU_{n+1}(\mathbb{R}P^n)\cong\mathbb{Z}/2^{t-1}.$$

Since $\psi^3(x)=3^t x$ for $x\in KU_{n+1}(\mathbb{R}P^n)$ we see that (2.2) is well-defined modulo

$$(3^t-1)\mathbb{Z}/2^{t-1}=2^{k+2}\mathbb{Z}/2^{t-1}, \text{ if } t=2^k \text{ and } k\geq 2.$$

From (2.1) one sees that one may equally well define the e-invariant of (2.2) by means of bu_* or bo_*. In particular we have the following results since the canonical map

$$bo_{n+1}(\mathbb{R}P^n)\cong\mathbb{Z}/2^{t-1}\to bo_{n+1}(\mathbb{R}P^\infty)\cong\mathbb{Z}/2^t$$

is injective when $n=2t-2$, $t=2k$.

(2.3) LEMMA. *Let n, t, k ($k\geqslant 2$) and g be as in §1.3. Then the KU_*-e-invariant of g equals $((3^t-1)/4)(2s+1)=2^k(2u+1)$ in $\mathbb{Z}/2^{t-1}$ if and only if the bo_*-e-invariant of f equals $2^{k+1}(2v+1)$ in $\mathbb{Z}/2^t\cong bo_{n+1}(\mathbb{R}P^\infty)$.*

(2.4) Now let $b\text{spin} = bo\langle 4\rangle$ be the 3-connected cover of bo [2, p. 146].

In a more refined manner one may consider the e-invariant defined by studying

$$\psi^3 - 1\colon bo_{n+1}(C(g)) \to b\text{spin}_{n+1}(C(g)). \tag{2.5}$$

It is straightforward to relate the "e-invariant" of (2.5) to the $bo_* - e$-invariant.

On the other hand, if we define a (2-localized) spectrum, jo, by the fibration

$$jo \to bo \xrightarrow{\psi^3 - 1} b\text{spin} \tag{2.6}$$

one may easily relate the induced map

$$g_* = f_*\colon jo_n(S^n) \to jo_n(\mathbb{R}P^n) \cong jo_n(\mathbb{R}P^\infty) \tag{2.7}$$

to (2.5) and thence to Lemma 2.3.

One finds the following relationship.

(2.8) Proposition. *Let f, g, n, t be as in* (1.3). *Then the e-invariant of* (2.2) *equals* $((3^t - 1)/4)(2s-1)$ *in* $\mathbb{Z}/2^{t-1}$ *if and only if* $g_* = f_*$ *is non-zero in* (2.7).

In fact, $2g_* = 0$ *in any case* [5].

§3. COMPUTATIONS

All that remains to be done to complete the proof of the main theorem (1.3) is to prove the following three lemmas.

(3.1) Lemma. $bo^{(2^{k-1}-1)}_{2^k-1}(\text{Cone } f) \cong \mathbb{Z}/2$ *and* $bo^{(2^{k-1}-1)}_{2^k-1}(\text{Cone } f) \to H\mathbb{Z}/2_{2^k-1}(\text{Cone } f)$ *is an isomorphism.*

(3.2) Lemma. $j^{2^k-3} \circ \lambda\colon bo^{(2^{k-1}-1)}_{2^k-1}(\text{Cone } f) \to bo_{2^{k+1}-1}(\text{Cone } f)$ *is an injection.*

$$\begin{array}{ccc} bo^{(2^{k-1}-1)}_{2^k-1}(\text{Cone } f) & \to & bo_{2^{k+1}-1}(\text{Cone } f) \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{Z}/2 & & \mathbb{Z}/2^{2^k} \end{array}$$

(3.3) Lemma. $(\psi^3 - 9)(\psi^3 - 9^2)\ldots(\psi^3 - 9^{2^{k-2}-1})(\beta_{2^{k-2}}) = 2^{2^k-k-2}(\beta_{2^{k-1}})$

Proof of 3.1. As promised we use Adams spectral sequences since the definition of $bo^{(n)}$ indicates that

$$\left\{\begin{array}{l} D_2^{s,t}(n) = \text{Ext}_{\mathscr{A}}^{s,t}(H^*(bo^{(n)}) \otimes H^*(X), \mathbb{Z}/2) \Rightarrow bo^{(n)}_{t-s}(X) \\ \text{and } \lambda\colon bo^{(n)} \to bo \text{ of (1.7) induces an isomorphism} \\ D^{s,t}(n) \cong \text{Ext}_{\mathscr{A}}^{s+n,t+n}(H^*(bo) \otimes H^*(X), \mathbb{Z}/2) \cong E_2^{s+n,t+n} \\ \text{where } E_2^{s,t} \Rightarrow bo_{t-s}(X) \text{ is an Adams spectral sequence.} \end{array}\right. \tag{3.5}$$

Now one simply turns to [14] to see that

(3.6)

$$\text{Ext}_{\mathscr{A}}^{m,m+2^k-1}(H^*(bo) \otimes H^*(\mathbb{R}P^\infty), \mathbb{Z}/2) \cong \begin{cases} 0 & \text{if } m > 2^{k-1} - 1 \\ \mathbb{Z}/2 & \text{if } 0 \le m \le 2^{k-1} - 1 \end{cases}$$

This completes the proof of the lemma but the authors feel that they are doing the readers a disservice if they do not point out the methods used in [14] for doing these calculations, since once the method is understood, anyone can duplicate the calculations on their own faster than they can look them up in [14] and these methods are at the heart of all the calculations in this paper. The fundamental fact is that if R is defined as the stable fibre of the Kahn–Priddy map

(3.7) $$S^{-1} \xrightarrow{h} R \to \mathbb{R}P^\infty \xrightarrow{\tau} S^0$$

then it turns out that

(3.8) $$R \wedge bo \simeq \vee_{n\geq 0}\Sigma^{4n-1}H\mathbb{Z}_{(2)}.$$

Thus to make calculations in $bo_*\mathbb{R}P^\infty$ one just studies the kernel and cokernel of $bo_*(h)$, the calculations being done using the Adams spectral sequence which collapses in both cases and the map bo_*h is determined by the fact that it always preserves Adams filtration.

Proof of (3.2). The proof of (4.1) showed that

(3.9) $$\begin{array}{ccc} \lambda\cdot bo^{(2^{k-1})}_{2^k-1}(\text{Cone } f) & \to & bo_{2^k-1}(\text{Cone } f) \\ \downarrow\cong & & \downarrow\cong \\ \mathbb{Z}/2 & & \mathbb{Z}/2^{2^{k-1}} \end{array}$$

is an inclusion. So it remains to show that

(3.10) $$\begin{array}{ccc} j^{2^{k-3}}\colon\ bo_{2^k-1}(\text{Cone } f) & \to & bo_{2^{k+1}-1}(\text{Cone } f) \\ \downarrow\cong & & \downarrow\cong \\ \mathbb{Z}/2^{2^{k-1}} & & \mathbb{Z}/2^{2^k} \end{array}$$

in an inclusion. But this is well known and can be proved by means of the Atiyah–Hirzebruch spectral sequence. Q.E.D.

Proof of (3.3). From [14] we know that $\psi^3(\beta_{2^k-1}) = 9^{2^{k-1}}\beta_{2^k-1}$. Notice that

(3.11) $$v_2(9^m-1) = v_2(m)+3.$$

For if m is odd this is obvious from the binomial expansion of $(8+1)^m = 9^m$ and if not then it follows by induction from the equation

$$(9^{2m}-1) = (9^m-1)^2 + 2(9^m-1).$$

Thus

$$\begin{aligned} &(\psi^3-9)\ldots(\psi^3-9^{2^{k-2}-1})(\beta_{2^k-1}) \\ &= (9^{2^{k-1}}-9)(9^{2^{k-1}}-9^2)\ldots.(9^{2^{k-1}}-9^{2^{k-2}-1})\beta_{2^k-1} \\ &= 9^a(9^{2^{k-1}-1})(9^{2^{k-1}-2}-1)\ldots(9^{2^{k-2}+1}-1)\beta_{2^k-1} \\ &= (2s+1)2^{3(2^{k-2}-1)}2^x\lambda\beta_{2^k-1} \end{aligned}$$

by (3.11), where

$$\begin{aligned} x &= v_2(2^{k-2}+1)+v_2(2^{k-2}+2)+\ \ldots\ +v_2(2^{k-1}-1) \\ &= v_2(1)+v_2(2)+\ \ldots\ +v_2(2^{k-2}-1) \\ &= v_2((2^{k-2}-1)!) \\ &= 2^{k-2}-1-(\alpha(2^{k-2}-1)) \end{aligned}$$

since $v_2(m!) = m-\alpha(m)$. Since $\alpha(2^{k-2}-1) = k-2$, we obtain the desired result.

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