STRING SURFACES, STRING INDEXES AND GENERA OF LINKS

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Abstract. We show that every link can be presented as a boundary of a flat string surface. We define the string index and the flat string indexes of a link. Then we provide some upperbounds for these string indexes by using braid representatives and canonical Seifert surface. We study the relation between these string indexes and the genera of links. For applications, we calculate the string index of pretzel links.

1. Introduction

One of classical goals in knot theory is to find a suitable representative or an invariant for a given link that can lead us to the classification of links. The closure of a braid in classical Artin group is one of them [2, 27]. Although it has been a very powerful tool to overcome several hard questions, it never meets the original purpose because of the lack of control on the second Markov move. Representation theory of the quantum groups has extended the polynomial invariants of links to the next level, but it has to be explored further [26].

Milnor invariants brought a very different representative, "a string link" and there are successful classifications of links of small number of components up to link homotopy [15,16]. On the other hand, compact orientable surfaces, *Seifert surfaces* play a key role in the study of links and 3-manifolds. The existence of such a surface was first proven by Seifert using an algorithm on a diagram of L, named after him as *Seifert's algorithm* [22]. Some of Seifert surfaces feature extra structures. For example, Seifert surfaces obtained by plumbings annuli have been studied extensively for the fibreness of links and surfaces [7,8,11,18–20,23]. The existence of these plumbing surfaces was shown by several authors [5, 12]. However, these plumbing surfaces have natural geometric restrictions to avoid the ambiguities in the construction. For geometrically flexible Seifert surfaces, we consider *(flat) string surfaces* as closed (zero, respectively) framed string links as described in Section 2. Originally these framed string links were used for quantum link invariants [24] but here we will use a different connection of the top and the bottom of the strings.

The flexibility of string links allows us to find a string surface for a given link without a difficulty, however for a flat string surface, it is completely different. The aim of the present article is to show the existence of such a flat string surface. Also, we discuss the efficiency of the number of strings required to present a link by a closed (zero) framed n-string link. At last, we relate these string surfaces and genera of links.

The outline of this article is as follows. First, we will provide precise definitions of string surfaces and string indexes in Section 2. In Section 3, we show the existence of string surfaces and flat string surfaces. Then we find some upperbounds for string indexes. In

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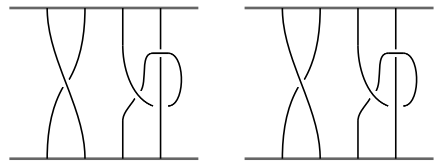

Figure 1. A 4-string link and a framed 4-string link.

Section 4, we study the relations between these string indexes and the genera of links which will provide us some lowerbounds for these string indices. Consequently, we find the exact string index for some pretzel links.

2. Preliminaries and definitions

Let D^2 be a disc in the xy-plane which intersects the x-axis. An *n-string link S* is an ordered collection of n disjoint arcs properly embedded in $D^2 \times [0, 1]$ in such a way, the i-th arc ends in the points $p_{s_i} \times 0$ and $p_{t_i} \times 1$, where $p_{s_i} = (x_i, 0)$, $p_{t_i} \in \{(x_i, 0)|i = 1, 2, \ldots, n\}$ taken from some prescribed points on the x-axis, enumerated in the natural order $x_1 < x_2 < \ldots < x_n$. In particular, if $p_{s_i} = p_{t_i}$ for all i, we say the n-string link is *pure*.

Now, we define a *framed string link* (S, μ) where S is an n-string link and $\mu = (m_1, m_2, \ldots, m_n)$ is a framing on S. A 4-string link and a framed 4-string link are illustrated in Fig. 1. To obtain a surface from a framed string link, we attach a disc consistent to the framing of the framed string link. The key idea is to put the framed string link into the upper half space, $\{(x, y, z)|y > 0\}$ and a disc $\mathcal{D}$ to the lower half space, $\{(x, y, z)|y < 0\}$ as shown in Fig. 2 such that the framing m_i is the linking number between the oriented arc α represented by i-th string of S and any path from p_{t_i} to p_{s_i} in $\mathcal{D}$ and its push up α^+ toward to the positive normal direction as indicated "+" in Fig. 2, because the surface we obtained from a closed framed string link is oriented. Indeed, a framing on a string represents n-full twists on the surface. It is fairly easy to see that the linking number we discussed does not depend on the choice of a path on $\mathcal{D}$. We name these surfaces *string surfaces*. In particular, if all framing of a framed string link in a string surface are zero, we call it a *flat string surface*. Although, it is easy to see the existence of a string surface for links, it is not clear for the existence of a flat string surface of a given link. We will provide the existence of flat string surfaces in Theorem 3.5 and 3.7. Then we can define the string index and flat string index as follow. The *string index* of a link L, denoted by $S(L)$ is n if there exists a string surface $\mathcal{F}$ which is obtained from an n-string link S such that the boundary of the surface $\mathcal{F}$ is L but there does not exist any string surface which is obtained from a k-string link T for any positive integer k less than n. Similarly we can define that the *flat string index* of a link L, denoted by $FS(L)$ to be the minimal number of strings of a flat string surface whose boundary is L.

3. Existence of flat string surfaces

Throughout the section, we will assume all links are not splitable. Otherwise, we can handle each component separately. For standard definitions and notations in knot theory, we refer to [1].

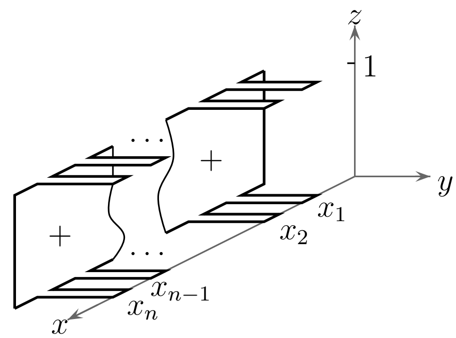

FIGURE 2. The standard disc $\mathcal{D}$ in the lower half plane $\{(x, y, z)|y < 0\}$ to obtain the string surface $\mathcal{F}$ from a framed n-string link.

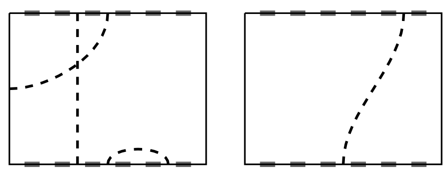

FIGURE 3. Lines in $\overline{\mathcal{D}}$, in the left figure, which represent a half twist which is allowed and a line in $\overline{\mathcal{D}}$ which represents a half twist which is prohibited in the right figure.

In the following subsections, we will isotop a disc $\overline{\mathcal{D}}$ to the standard one $\mathcal{D}$ in Fig. 2 to obtain (flat) string surfaces. First, we can isotope the boundary $2n$ line segments to the desired position. The group of homeomorphisms on the disc which fix the $2n$ points setwisely are generated by homeomorphisms which interchange two adjacent points by a 180 degree twist [2]. However, the disc $\overline{\mathcal{D}}$ in theorems which address the existence of string surfaces and flat string surfaces is a disc which is the union of n-discs joined by $(n-1)$-half twisted bands. Thus, instead of using the full group of homeomorphisms, we can untwist these half twisted bands to obtain a flat disc. But when we untwist these twisted bands, $\overline{\mathcal{D}}$ does require to keep the framed string link structure so we can obtain the desired disc $\mathcal{D}$. This condition will be examined in the following lemma.

Lemma 3.1. *Let $\overline{\mathcal{D}}$ be a disc in the upper half plane $\{(x, y, z)|y > 0\}$ such that its boundary touches xz-plane exactly $2n$ line segments, the first half on the line $z = 0$ and the other half on the line $z = 1$ as shown in Fig. 2 but $\overline{\mathcal{D}}$ may not be necessary standardly embedded as in Fig. 2 but is the union of n-discs joined by $(n-1)$ half twisted bands. Suppose the upper n line segments are connected to the lower n line segments through a framed n-string link (S, μ). Then $\overline{\mathcal{D}}$ can be deformed to the standard disc $\mathcal{D}$ as illustrated in Fig. 2 only by $(n-1)$ half twists such that the upper n line segments are connected to the lower n line segments through a framed n-string link (T, μ).*

Proof. Each twist band which was attached to join n-disc can be drawn as a line β which connects two boundary points in the intersecting part of the plane $y = 0$ and on the standard disc $\mathcal{D}$ in Fig 2. Depend on the positions of two points, we can divide three cases

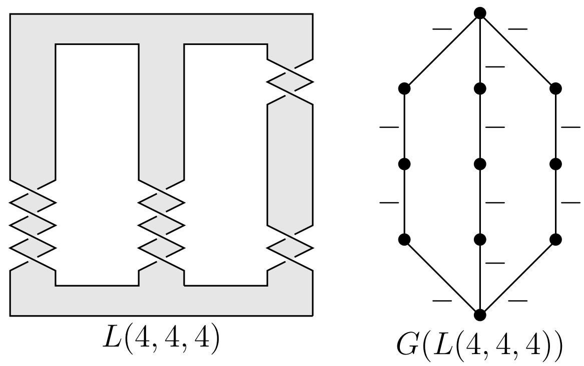

FIGURE 4. A canonical Seifert surface of a pretzel link $L(4,4,4)$ and its corresponding induced graph $G(L(4,4,4))$.

as illustrated in Fig. 3. If both ends of the line are connected to the boundary whose z value is 0 or 1 or one is connected to a side, then the untwist move does not change the string link structure nor the framing of a framed string link. If the line connects a point on the bottom part of the boundary whose z value is 0 and a point on the top part of the boundary whose z value is 1, we can count the numbers of points which is illustrated as thick gray short lines in Fig 2 in a sub-disc divided by the line β. Although there are two possible sub-discs, our first claim is that the numbers of points on top and line have to be the same. Otherwise, there exists a string transversal to the line β as depicted in the right hand side of Fig 3. Since it is framed, the neighborhood of the line presented by the string and the path joining both ends of the string in $\mathcal{D}$ is Möbius bands thus, the surface is no longer orientable. Then the untwist move flips some of strings but it still does not change string link structure nor the framing of a framed string link. Therefore, we obtain the standard disc $\mathcal{D}$ such that the upper n line segments are connected to the lower n line segments through a framed n-string link (T, μ). □

We remark that the framing of a framed n-string link (S, μ) was not changed through the isotopy described in Lemma 3.1 to (T, μ).

3.1. **String surfaces and string indexes.** From a given Seifert surface of L, by sling the band which presenting the generators of the homology of the surface, one can obtain a framed string surface without any trouble. Thus, we find an upperbound of string index of a link L. A Seifert surface $\mathcal{F}$ of a link L by applying Seifert's algorithm to a link diagram $D(L)$ as shown in Fig. 4 is called a *canonical Seifert surface*. From such a canonical Seifert surface, we construct a (signed) graph $G(L)$ by collapsing discs to vertices and half twist bands to signed edges as illustrated in the right side of Fig 4. Let us call it an *induced graph* of a link L. It is fairly easy to see that the number of Seifert circles (half twisted bands), denoted by $s(S)(c(S))$, is the cardinality of the vertex set (edge set, respectively). It is clear that if Seifert surface $\mathcal{F}$ is connected, its induced graph $G(L)$ is also connected. For terms in graph theory, we refer to [10]. It is fairly well known that the number of edges of spanning tree of a connected graph with n vertices is $n-1$. Now we are set to state a theorem for an upperbound of the string index from the Seifert's algorithm.

Theorem 3.2. *Let $\mathcal{F}$ be a canonical Seifert surface of a link diagram S of L with $s(S)$ Seifert circles and $c(S)$ half twisted bands. Let T be a spanning tree of its induced graph $G(L)$ which has exactly $s(S)-1$ edges. Then the string index of L, $S(L) \leq c(S)-s(S)+1$.*

Proof. Let D be the disc corresponding to the spanning tree T. For any twisted band $t(e)$ with a sign $\epsilon(e)$ which is not a part of the disc D, *i.e.*, an edge e in $E(G)-T$ in its induced graph $G(L)$ we described above, we can choose a unique path W in the spanning tree T which joins the ends of the edge e. Let k be the sign sum of edges in the path W. Then $k+\epsilon(e)$ has to be an even integer because $\mathcal{F}$ is oriented surface. Let $n = \dfrac{k+\epsilon(e)}{2}$. Since $\mathcal{F}$ is oriented, each vertices of $G(L)$ can be labeled by 0 or 1 such that every two adjacent vertices have different labels. Using this label, we can determine which end of twisted band will be isotop to $D^2\times 0$ or $D^2\times 1$. The resulting $\mathcal{D}$ satisfies the requirement of the hypothesis of Lemma 3.1 with $c(S)-s(S)+1$ strings. Therefore, we can isotop $\mathcal{D}$ to obtain a string surface $\mathcal{F}$. Thus, its string index is less than or equal to $c(S)-s(S)+1$. □

In the following example, we calculate a string index of a link.

Example 3.3. *Let L be the pretzel link $L(4,4,4)$ as depicted in Fig. 4, The string index of $L(4,4,4)$ is 2.*

Proof. We use the shaded region as a Seifert surface of the link $L(4,4,4)$. First, we find that $s(S)=11, c(S)=12$ from the figure. By applying Theorem 3.2, we find that the string index of L is less than or equal to $c(S)-s(S)+1 = 12-11+1 = 2$. But only links of string index 1 are the closure of 2-braid $(\sigma_1)^{2n}$. The HOMFLY polynomial of $L(4,4,4)$ is y^2x^{-2} $+2x^{-3}y$ $+x^{-6}y^2$ $+x^{-8}(1-3y^2)$ $+2x^{-9}y$ $+x^{-10}(3y^3-2-y^{-1}+y^{-2})$ $+x^{-12}(-3-y^{-1}+2y^{-2})$ $+2x^{-14}2y^{-2}$. By the celebrated theorem by Frank-Williams [6], the braid index of $L(4,4,4) \geq \dfrac{(-2)-(-14)}{2}+1 = 7$. Therefore, $L(4,4,4)$ is not a closed 2-braids, thus the string index of L has to be 2. □

Example 3.3 demonstrates that the inequality in Theorem 3.2 is sharp. Furthermore, we provide Theorem 4.2 which determines the string index for links with a property.

But as mentioned before, it is not clear whether a zero framing string surface exists or not. To show the existence of flat string surfaces of a given link L, we first present the link L as the closure of a braid in classical Artin group. Then by using the first Markov move, we may use a different braid representative $(\sigma_1\sigma_2\ldots\sigma_{n-2}\ \sigma_{n-1})\ \beta\ (\sigma_1\sigma_2\ldots\sigma_{n-2}\ \sigma_{n-1})^{-1}$ for L. Using this braid representative of L, we can find the following upperbound for the string index of L for which all the framing of strings are either 0 or 1.

Theorem 3.4. *Let L be a closed n-braid $\overline{\beta}$ where the braid β can be written as $\sigma_1\sigma_2\ldots\sigma_{n-2}$ $\sigma_{n-1}W$ and the length of the word W is m. Then the string index of L is less than or equal to m, i.e., $S(L) \leq m$.*

Proof. First we pick a disc $\mathcal{D}$ which is obtained from n disjoint disks by attaching $(n-1)$ twisted bands presented by $\sigma_1\sigma_2\ldots\sigma_{n-2}\ \sigma_{n-1}$. For each letter in the word W, we attach an half twisted band to $\mathcal{D}$ which will present a string in the desired string surface. First, we untwist n disjoint disks alternatively to $D^2\times 0$ or $D^2\times 1$ to obtain a framed string link,

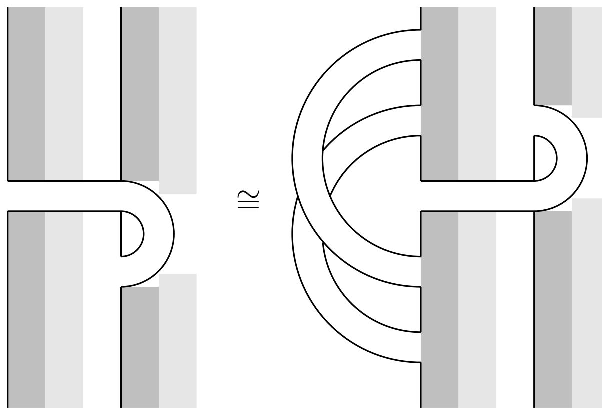

FIGURE 5. Changing the sign of a twisted band by adding two flat annuli.

thus $\mathcal{D}$ meets the hypothesis of Lemma 3.1. We can isotop $\mathcal{D}$ to the standard disc in Fig. 2 to obtain a string surface for L by Lemma 3.1. Furthermore, if we count the linking number carefully, one can see that each half twisted band corresponding to a negative letter, σ_i^{-1} has framing 0 and each half twisted band corresponding to a positive letter, σ_i has a framing 1. □

3.2. **Flat string surfaces and flat string indexes.** Key ingredients of constructing a flat string surface of a given link are Theorem 3.4 and a method to change the sign of twisted band by adding two flat annuli illustrated in Fig 5 first found by R. Furihata, M. Hirasawa and T. Kobayashi [5].

First, we find a flat string surface from a closed braid representative of a link L.

Theorem 3.5. *Let L be a closed n-braid with a braid word $\sigma_1\sigma_2\ldots\sigma_{n-2}\ \sigma_{n-1}W$ where the length of W is m and W has s positive letters, then there exists a flat string surface $\mathcal{F}$ with $m+2s$ strings such that $\partial\mathcal{F}$ is isotopic to L, i.e., $FS(L) \leq m+2s$.*

Proof. For a link presented by a closed braid, we have found a string surface obtained from an m-string link with each framing is either 0 or 1 and each string corresponding to a positive letter, σ_i has a framing 1 in Theorem 3.4. To replace this string of the framing 1 by strings of framing 0, we add two extra annuli as illustrated in the second figure in Fig. 7. Now all framing on the string and annuli are 0, however, it is no longer string surface. If we slide two annuli through the original string in the middle which has framing 0, it will change the framing of annuli as depicted in the third figure in Fig. 7. To get a flat string surface, we do start this process from the rightmost (from the topside) string of framing 1. Instead of sliding annuli we have put on top, we slide strings which are the righthand side of this string of the framing 1 through the first adjacent annulus as shown in Fig 7. This sliding does not change the flamings of strings which have been sliding. Since there does not exists any strings on righthand side of the annuli, we can consider the bottom part of new $\mathcal{D}$ start from the dashed line on the top of the second figure in Fig 7. Inductively we can remove all strings of framing 1 to get a flat string surface. The resulting surface has been changed by adding extra strings but the boundary of the surface $\mathcal{D}$ is still isotopic

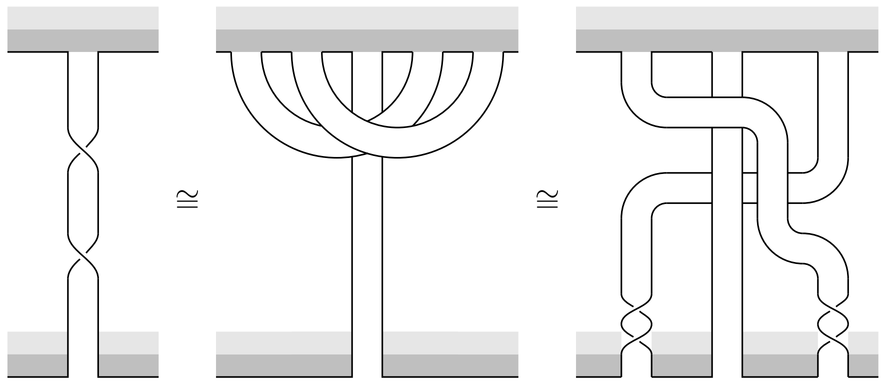

FIGURE 6. Replacing a string of a framing 1 by three flat annuli.

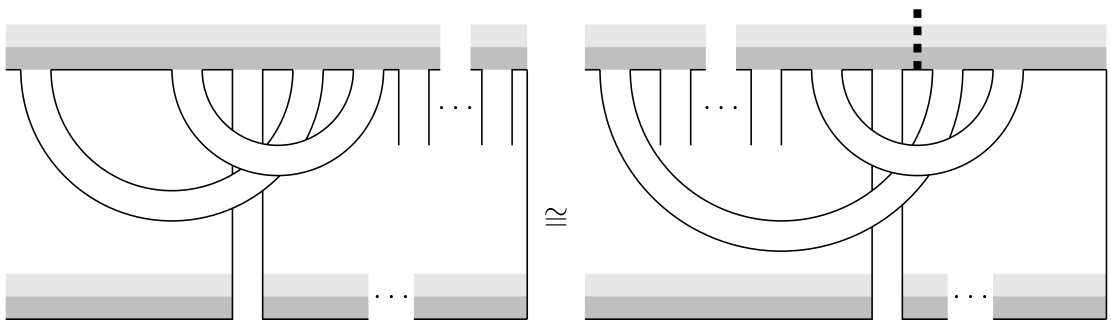

FIGURE 7. Sliding of strings to obtain a flat string surface.

to the original link L. Since we added $2s$ extra strings for each positive letters, the total number of strings in the flat string surface is $m + 2s$. It completes the proof. □

Now we want to find an upperbound from canonical Seifert surfaces. To obtain flat string surfaces, we have to choose a disk $\mathcal{D}$ carefully. The spanning tree of the induced graph of a closed braid is a path. Thus, there is no ambiguity about the choice of a spanning tree for a closed braid. For a general canonical Seifert surface, it could be drastically changed. Surprisingly, even if we choose any spanning tree, an alternative label on the tree with respect to the depth of the tree from a root whose valency is 1 satisfies the desired property as shown in the following lemma.

Lemma 3.6. *Let G be the induced graph of a Seifert surface. Let T be a spanning tree with an alternating signing with respect to the depth of the tree as depicted in Fig. 8. Let e be an edge in $E(G) - T$, the sign sum of the simple path in T which joins the end points of e is either* 1 *or* -1.

Proof. First, it is well known that for any two points in a tree, there exists a unique simple path from one to the other. Let e be an edge in $E(G)-T$. Let P be the unique simple path in T which joins the endpoints of e. Because of the orientability of the Seifert surface, the length of this simple path must be odd and the sum of signs on the simple path must be

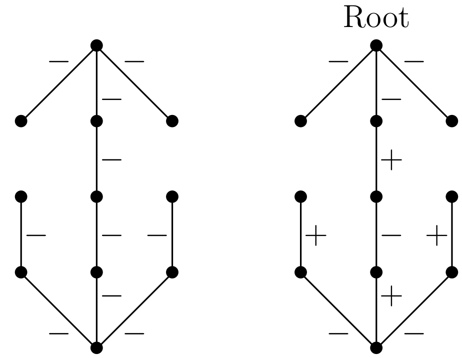

FIGURE 8. A spanning tree and its an alternating sign on the spanning tree with a root for the induced graph $G(L(4,4,4))$ in Fig. 4.

an odd number. Furthermore, if the path P does not passes the root, then the sum of odd number of alternating signs must be either 1 or -1 depending on the parity of the length of P. If P passes the root, it is a union of two paths, for which one is starting from the root and the other ending at the root, of alternating signs by the definitions of the tree and the alternating sign. However, one has odd length and the other has even length possibly zero. Therefore, any path P joining the end points of an edge e in $E(G) - T$ has a sign sum either 1 or -1. □

Using Lemma 3.6, we can consider the following minimum. First if the sign of an edge e in T does not coincide with a sign of the edge in the alternating sign, then we have to isotop the link by a type II Reidemeister move as shown in the left side of Fig. 9. Since we can completely reverse the sign of all edges in the spanning tree T, we may assume the total number of type II Reidemeister moves in the process is less than or equal to $\left\lceil \frac{s(S)-1}{2} \right\rceil$. Let us write β to be the total number of type II Reidemeister moves in the process described above. Now we set D the disc corresponding to the spanning tree T as depicted in the right side of Fig. 9. For each edge e in $E(G)-T$, if the sign of the edge is different from the sign sum of the edges in the path P which joins the endpoints of e, then the framing of the string presented by the edge e is zero because the linking number of α^+ and α which is a union of the line presenting e and the curve in $\mathcal{D}$ corresponding to the path P in the spanning tree T. Otherwise, we need to add three flat strings to make the half twisted band presented by the edge e. Let us remark that although we introduced two new edges from the type II Reidemeister move, the sign of one edge is different from the sign sum of the edges in the path P which joins the endpoints of e and the other's coincides. Thus, there will be a total 4β strings coming from type II Reidemeister moves. Let γ be the total number of the edges in $E(G) - T$ whose signs and the sign sums of the edges in the paths which join the endpoints of the edges are the same which will contribute 2γ extra strings to obtain a flat string surface. Since there are $c(S) - s(S) + 1$ edges in $E(G) - T$, by summarizing the these facts, we obtain the following theorem.

Theorem 3.7. *Let $\mathcal{F}$ be a canonical Seifert surface of a link L with $s(S)$ Seifert circles and $c(S)$ half twisted bands. Let G be the induced graph of a Seifert surface $\mathcal{F}$. Let T*

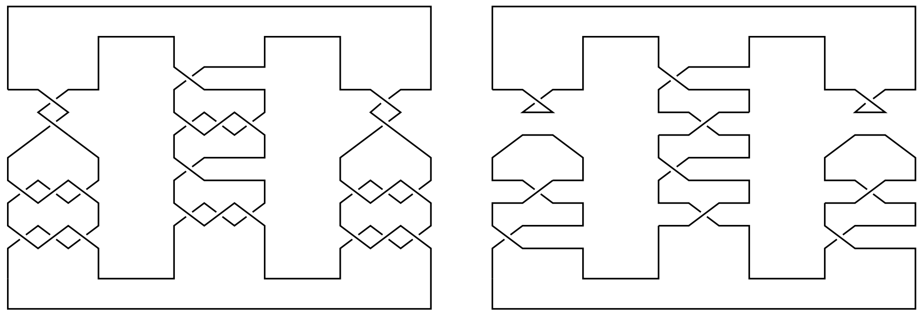

FIGURE 9. A modified link diagram from the alternating sign in Fig. 8 and a disc corresponding to the spanning tree with the alternating sign.

be a spanning tree with an alternating signing with respect to the depth of the tree. Let β *and* γ *be the numbers described above. Then the flat string index of* L *is bounded by* $c(S) - s(S) + 1 + 4\beta + 2\gamma$, *i.e.,*

$$FS(L) \leq c(S) - s(S) + 1 + 4\beta + 2\gamma.$$

For example, we need $12 - 11 + 1 + 4 \cdot 4 + 2 \cdot 2 = 22$ strings to the disc in Fig. 9 to obtain a flat string surface for a link L in Fig. 4.

4. RELATIONS BETWEEN STRING INDEXES AND GENERA OF LINKS

We will relate the string indexes with one of classical link invariants. Let us recall the definitions first. The *genus* of a link L is the minimal genus among all Seifert surfaces of L, denoted by $g(L)$. A Seifert surface $\mathcal{F}$ of L with the minimal genus $g(L)$ is called a *minimal genus Seifert surface* of L. A Seifert surface of L is said to be *canonical* if it is obtained from a diagram of L by applying Seifert's algorithm. Then the minimal genus among all canonical Seifert surfaces of L is called the *canonical genus* of L, denoted by $g_c(L)$. A Seifert surface $\mathcal{F}$ of L is said to be *free* if the fundamental group of the complement of $\mathcal{F}$, $\pi_1(\mathbb{S}^3 - \mathcal{F})$ is a free group. Then the minimal genus among all free Seifert surfaces of L is called the *free genus* for L, denoted by $g_f(L)$. Since any canonical Seifert surface is free, we have the following inequalities.

$$g(L) \leq g_f(L) \leq g_c(L).$$

There are many interesting results about the above inequalities [3, 4, 14, 17, 18, 21]. For the string index, first we find the following lemma.

Lemma 4.1. *Let* L *be a link, let* l *be the number of components of* L. *Then the string index of* L *is bounded as,*

$$2g(L) + l - 1 \leq S(L) \leq 2g_c(L) + l - 1.$$

Proof. Let V, E and F be the numbers of vertices, edges and faces, respectively in a minimal canonical embedding of $G(L)$. From Theorem 3.2, we find $S(L) \leq c(S) - s(S) + 1 =$

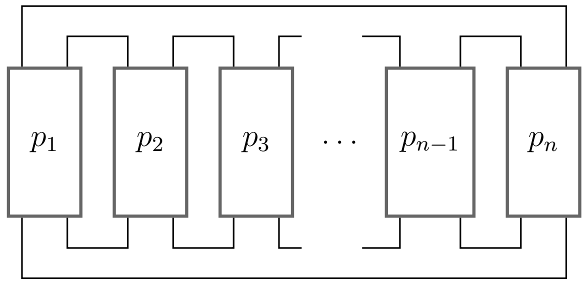

FIGURE 10. An n-pretzel link $L(p_1, p_2, \dots, p_n)$

$E - V + 1 = (E - V - F) + F + 1 = 2g_c(L) - 2 + F + 1 = 2g_c(L) + F - 1$. Since a string surface is a Seifert surface of L, the first inequality follows from the definition of the genus of L. □

Consequently, we find the following theorem.

Theorem 4.2. *If L is a link of l components and its minimal genus surface of genus $g(L)$ can be obtained by applying Seifert algorithm on a diagram of L, i.e., $g(L) = g_c(L)$, then $S(L) = 2g(L) + l - 1$.*

In fact, there are many links for which their genera and canonical genera are the same such as alternating links, closures of positive braids. For these links, we can find their string index by Corollary 4.2. One of well known but concrete examples is pretzel links as illustrated in Fig. 10. Genera of pretzel links are known [9] and it was shown that their genera and canonical genera are the same [13]. Therefore, we can find the following string index for pretzel links.

Corollary 4.3. *Let $K(p_1, o_2, o_3, \dots, o_n)$ be an n-pretzel knot with one even p_1. Let $\alpha = \sum_{i=2}^{n}$ sign(o_i) and β= sign(p_1). Suppose $|p_1|, |o_i| \geq 2$. Let*

$$\delta = \sum_{i=2}^{n} (|o_i| - 1).$$

Then the string index $S(K)$ of K,

$$S(K) = \begin{cases} \delta + 2 & \text{if } n \text{ is odd and } \alpha \neq 0, \\ \delta & \text{if } n \text{ is even and } \alpha = 0, \\ |p_1| + \delta & \text{if } n \text{ is even and } \alpha + \beta \neq 0, \\ |p_1| + \delta - 2 & \text{if } n \text{ is even and } \alpha + \beta = 0. \end{cases}$$

Many other cases of pretzel links are known but we will omit them since they are just straight consequences of Theorem 4.2.

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