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On the zeros of certain self-reciprocal polynomials

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Abstract

We investigate the distribution of zeros around the unit circle of real self-reciprocal polynomials of even degrees with five terms whose absolute values of middle coefficients equal the sum of all other coefficients. Furthermore, it also give a new inequality and other Eneström–Kakeya types of results as by-products of this investigation.

Keywords: Self-reciprocal polynomials; Zeros; Unit circle

1. Introduction and statements of results

Throughout this paper, U denotes the unit circle and n is a positive integer. A polynomial $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is said to be a self-inversive polynomial of degree n if it satisfies $a_n \neq 0$ and $P(z) = \mu P^*(z)$, where $|\mu| = 1$ and

$$P^*(z) = z^n \overline{P(1/\bar{z})} = \overline{a_0} z^n + \overline{a_1} z^{n-1} + \cdots + \overline{a_n}.$$

In particular, if $P(z) = z^n P(1/z)$, $P(z)$ is called be self-reciprocal. Thus the zeros of a self-reciprocal polynomial either lie on U or occur in pairs conjugate to U. Since the class of self-inversive polynomials of degree n includes polynomials of degree n which have all their zeros on U, it is interesting to mention the condition for a self-inversive polynomial having all its zeros on U. Besides problems for the zeros, a self-reciprocal polynomial $P(z)$ of degree n has the remarkable relation (see [10, p. 153])

$$\max_{z\in U}\left|P'(z)\right| = \frac{n}{2}\max_{z\in U}\left|P(z)\right|.$$

Furthermore, every point of maximum modulus of $P(z)$ on U is also a point of maximum modulus of $P'(z)$. In particular, this relation holds for all polynomials of degree n whose zeros lie on U.

A useful tool for showing a self-inversive polynomial having all its zeros on U is due to Cohn [4].

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Theorem 1 *(Cohn). Let $P(z)$ be a self-inversive polynomial of degree n. Suppose that $P(z)$ has exactly τ zeros on U (counted according to multiplicity) and exactly ν critical points in the closed unit disc (counted according to multiplicity). Then*

$$\tau = 2(\nu + 1) - n.$$

Thus a necessary and sufficient condition for all zeros of a self-inversive polynomial $P(z)$ to lie on U is that all zeros of $P'(z)$ lie inside or on U. Another tool to show a self-inversive polynomial having all its zeros on U can be found in Chebyshev transformation. Studying the spectral properties of the Coxeter transformation, Lakatos [7] obtained sufficient conditions for self-reciprocal polynomials having all their zeros on U by using Chebyshev transformation. Schinzel [9] generalized Lakatos's results [7] to self-inversive polynomials. Lakatos and Losonczi [8] again used Chebyshev transformation and improved Lakatos's previous results [7] and Schinzel's results [9] for polynomials of odd degrees. However their method didn't work for even degree polynomials.

Our first goal in this paper is to investigate the distribution of zeros around U of some rather specialized real self-reciprocal polynomials of even degrees. More precisely, we will study the distribution of zeros of real self-reciprocal polynomials of even degrees with five terms whose absolute values of middle coefficients equal the sum of all other coefficients. Furthermore, it will give a new inequality and other Eneström–Kakeya [6] types of results as by-products of this investigation.

All real self-reciprocal polynomials of even degrees with three terms whose absolute values of middle coefficients equal the sum of all other coefficients are of the form

$$Az^{2m} \pm 2Az^m + A = A\left(z^m \pm 1\right)^2,$$

and have all their zeros on U. This induces our attention naturally how zeros of the same kinds of self-reciprocal polynomials with five terms are located around U. In fact, there are exactly four types of such polynomials as follows. For integers m, n with $m > n > 0$ and positive real numbers a, b, we let

$$\begin{aligned}
P^+(z) &= az^{2m} - bz^{m+n} + 2(a-b)z^m - bz^{m-n} + a,\\
P^-(z) &= az^{2m} - bz^{m+n} - 2(a-b)z^m - bz^{m-n} + a,\\
Q^+(z) &= az^{2m} + bz^{m+n} + 2(a+b)z^m + bz^{m-n} + a,\\
Q^-(z) &= az^{2m} + bz^{m+n} - 2(a+b)z^m + bz^{m-n} + a.
\end{aligned}$$

In Section 2, we will prove the results below. Throughout these theorems, we assume that m, n are integers with $m > n > 0$ and a, b are positive real numbers as above.

Theorem 2. *For $a \geqslant b > 0$, all zeros of $P^+(z)$ and $P^-(z)$ lie on U.*

Theorem 3. *Let d be the greatest common divisor of $2m$, $m+n$ and $m-n$. Then we have the following*:

(a) *If $d \mid m$, then $Q^+(z)$ has no zeros on U.*
(b) *If $d \nmid m$ and $m = dk + r$ for some integers k and r with $1 \leqslant r \leqslant d-1$, then $Q^+(z)$ has exactly $d/2$ zeros on U without counting multiplicities. Such zeros are the $d/2$th roots of -1.*

Theorem 4. *Let d be the greatest common divisor of $2m$, $m+n$ and $m-n$. Then we have the following*:

(a) *If $d \mid m$, then $Q^-(z)$ has exactly d zeros on U without counting multiplicities. Such zeros are the dth roots of -1.*
(b) *If $d \nmid m$ and $m = dk + r$ for some integers k and r with $1 \leqslant r \leqslant d-1$, then $Q^-(z)$ has exactly $d/2$ zeros on U without counting multiplicities. Such zeros are the $d/2$th roots of -1.*

The polynomials $P^+(z)$ and $P^-(z)$ are of interest. For example, a special case of $P^+(z)$ with $a = m^2$ and $b = n^2$, i.e.,

$$T(z) := m^2z^{2m} - n^2z^{m+n} + 2\left(m^2 - n^2\right)z^m - n^2z^{m-n} + m^2,$$

plays a role in the study of inequalities. In fact, it follows from

$$T(z) > 0 \quad \text{for } 0 < z < 1 \tag{1.1}$$

that we have a new inequality below. The inequality (1.1) will be shown in the proof of Proposition 5.

Proposition 5. *For $y > x > 0$ and $0 < \lambda < 1$, we have an inequality*

$$x^2 + 2(1-\lambda^2)xy + y^2 > \lambda^2(x^{1+\lambda}y^{1-\lambda} + x^{1-\lambda}y^{1+\lambda}). \tag{1.2}$$

We observe that, for $\lambda = 0$, (1.2) becomes $(x+y)^2 > 0$, and for $\lambda = 1$, (1.2) becomes $x^2 + y^2 > x^2 + y^2$ which is not true. The proof of Proposition 5 will be given in Section 3. Also the zeros of a special case of $P^-(z)$ with $a = n^2$ and $b = m^2$ are those of the equation

$$z^n\left(\frac{1}{m}\frac{z^m - 1}{z - 1}\right)^2 = z^m\left(\frac{1}{n}\frac{z^n - 1}{z - 1}\right)^2.$$

This is interesting because $\frac{z^k-1}{z-1}$ is an analogue of k. Furthermore, some factors of $P^-(z)$ seem to be related to Eneström–Kakeya [6] types of problems.

Theorem 6 *(Eneström–Kakeya). Let $a_0, a_1, \ldots, a_n$ be real numbers satisfying*

$$a_0 \geqslant a_1 \geqslant \cdots \geqslant a_n > 0.$$

Then the polynomial $P(z) = \sum_{k=0}^n a_k z^k$ has no zeros inside U.

For the proof of Eneström–Kakeya Theorem above, see p. 12 of [2]. If $m - n = 1$, then the self-reciprocal polynomials

$$U(z) = \frac{P^-(z)}{(z-1)^2} = \sum_{k=0}^{2m-2} a_k z^k,$$

where $a_k = b + k(b - a)$, seem to have their coefficients in increasing order. We will prove in Section 3 the theorem below about the distribution of zeros of some generalized polynomials of $U(z)$ above.

Theorem 7. *Let $P(z) = \sum_{k=0}^{2m+1} a_k z^k$ be a real self-reciprocal polynomial and $a_k = 1 + kr$, $k = 1, 2, \ldots, m$. Then we have the following*:

(a) *If $r < -\frac{2}{m}$, then $P(z)$ has $2m - 1$ zeros on U and a zero in $(0, 1)$.*
(b) *If $-\frac{2}{m} < r \leqslant 2$, then $P(z)$ has all its zeros on U.*
(c) *If $2 < r < 2 + \frac{2}{m}$ and m is even, then $P(z)$ has all its zeros on U.*
(d) *If $r = 2 + \frac{2}{m}$ and m is even, then $P(z)$ has all its zeros on U.*
(e) *If $r > 2 + \frac{2}{m}$, then $P(z)$ has $2m - 1$ zeros on U.*

In above theorem, the three cases "$2 < r < 2 + \frac{2}{m}$ and m is odd," "$r = -\frac{2}{m}$," "$r = 2 + \frac{2}{m}$ and m is odd" remain open problems.

2. Four types of self-reciprocal polynomials

We will often use Theorem 1 to prove our results Theorems 2, 3 and 4 in Section 1. We first prove Theorem 2.

Proof of Theorem 2. Let

$$p(z) := \frac{[P^+(z)]'}{z^{m-n-1}} = 2amz^{m+n} - b(m+n)z^{2n} + 2(a-b)mz^n - b(m-n).$$

For $\epsilon > 0$, we define the polynomial

$$p_\epsilon(z) := (2am + \epsilon)z^{m+n} - b(m+n)z^{2n} + 2(a-b)mz^n - b(m-n).$$

Then, for $|z| = 1$, we have

$$\begin{aligned}\left|(2am + \epsilon)z^{m+n}\right| &= (2am + \epsilon) > 2am \\ &= b(m+n) + 2(a-b)m + b(m-n) \\ &\geqslant \left|-b(m+n)z^{2n} + 2(a-b)mz^n - b(m-n)\right|.\end{aligned}$$

By Rouché's theorem, $p_\epsilon(z)$ has all its zeros strictly inside U. This implies that all zeros of $p(z)$ lie inside or on U, and then Theorem 1 completes the proof. The result for $P^-(z)$ can be proved in the same way. □

We can also prove all zeros of $P^-(z)$ lying on U by using the following theorem [3].

Theorem 8 *(Chen). A necessary and sufficient condition for all the zeros of $P_n(z) = \sum_{k=0}^{n} a_k z^k$, $a_n \neq 0$ with complex coefficients to lie on U is that there is a polynomial $q_{n-l}(z)$ with all its zeros inside or on U such that*

$$P_n(z) = z^l q_{n-l}(z) + e^{i\theta} q^*_{n-l}(z),$$

where $q^*_{n-l}(z) = z^{n-l}\overline{q_{n-l}(1/\bar{z})}$, for some nonnegative integer l and real θ.

The suitable $q_{n-l}(z)$ in Theorem 8 for $P^-(z)$ is

$$az^m - bz^n - (a-b).$$

In fact, we have

$$\begin{aligned}P^-(z) &= z^m\left[az^m - bz^n - (a-b)\right] + \left[-(a-b)z^m - bz^{m-n} + a\right] \\ &= z^m\left[az^m - bz^n - (a-b)\right] + \left[az^m - bz^n - (a-b)\right]^*\end{aligned}$$

and $az^m - bz^n - (a-b)$ has all its zeros inside or on U (see p. 227 of [1] for the zero distribution of $az^m - bz^n - (a-b)$). The problems about the number of zeros on U for $P^+(z)$ and $P^-(z)$ when $b > a > 0$ remain open.

We prove Theorems 3 and 4.

Proof of Theorem 3. For $\epsilon > 0$, we define the polynomial

$$Q^+_\epsilon(z) = az^{2m} + bz^{m+n} + \big(2(a+b) + \epsilon\big)z^m + bz^{m-n} + a.$$

For $|z| = 1$,

$$\left|Q^+_\epsilon(z)\right| \geqslant 2(a+b) + \epsilon - 2a - 2b = \epsilon > 0,$$

which implies that $Q^+_\epsilon(z)$ does not have a zero on U. Also it follows from Rouché's theorem that $Q^+_\epsilon(z)$ has exactly m zeros strictly inside U, say $\alpha_1, \ldots, \alpha_m$. Suppose that, as $\epsilon \to 0$, some of these tend to U, say

$$\alpha_j \to e^{i\theta_j}, \quad \theta_j \in \mathbb{R}.$$

Since $Q^+(z_j) = 0$, where $z_j = e^{i\theta_j}$, we have

$$\left|az_j^{2m} + bz_j^{m+n} + bz_j^{m-n} + a\right| = 2(a+b).$$

This equality holds only if the four points z_j^{2m}, z_j^{m+n}, z_j^{m-n}, 1 have the same argument, so

$$(2m)\theta_j \equiv (m+n)\theta_j \equiv (m-n)\theta_j \equiv 0 \pmod{2\pi}.$$

Hence $e^{i\theta_j}$ is a dth root of unity, where

$$d = \gcd(2m, m+n, m-n). \tag{2.1}$$

If $d \mid m$, then

$$\begin{aligned} Q^+(w) &= aw^{2m} + bw^{m+n} + 2(a+b)w^m + bw^{m-n} + a \\ &= a + b + 2(a+b)w^m + b + a \\ &= 2(a+b)\big(1 + w^m\big) = 4(a+b) \neq 0. \end{aligned}$$

Thus $Q^+(z)$ has no zeros on U. We now suppose that $d \nmid m$ and $m = dk + r$ for some integers k, r with $1 \leqslant r \leqslant d - 1$. Then d must be even since, for d odd, $d \mid 2m$ and so $d \mid m$. Also $d \mid 2m$ implies that $d \mid 2r$. Letting $2r = du$ for some positive integer u, we have $du/2 < d$ and so $u = 1$, i.e., $d = 2r$. Now

$$\begin{aligned} Q^+(w) &= aw^{2m} + bw^{m+n} + 2(a+b)w^m + bw^{m-n} + a \\ &= a + b + 2(a+b)w^m + b + a \\ &= 2(a+b)\big(1 + w^m\big) = 2(a+b)\big(1 + w^{d/2}\big). \end{aligned}$$

Since the $d/2$th roots of -1 are contained in the $d/2$th roots of unity, $Q^+(z)$ has exactly $d/2$ zeros on U without counting multiplicities. Such zeros are the $d/2$th roots of -1. □

Next we give a proof of Theorem 4 which is very similar to that of Theorem 3.

Proof of Theorem 4. For $\epsilon > 0$, we define the polynomial

$$Q_\epsilon^-(z) = az^{2m} + bz^{m+n} - \big(2(a+b) + \epsilon\big)z^m + bz^{m-n} + a.$$

With $Q_\epsilon^-(z)$, we follow exactly same procedure of the proof of Theorem 3 until (2.1). Then we can see that if there exist zeros of $Q_\epsilon^-(z)$ on U, they must be dth roots of unity. If $d \mid m$, then

$$\begin{aligned} Q^-(w) &= aw^{2m} + bw^{m+n} - 2(a+b)w^m + bw^{m-n} + a \\ &= a + b - 2(a+b)w^m + b + a \\ &= 2(a+b)\big(1 - w^m\big). \end{aligned}$$

Hence $Q^-(w)$ has exactly d zeros on U that are the dth roots of unity. We now suppose that $d \nmid m$ and $m = dk + r$ for some integers k, r with $1 \leqslant r \leqslant d - 1$. Then $d = 2r$ as in the proof of Theorem 3. Now

$$\begin{aligned} Q^-(w) &= aw^{2m} + bw^{m+n} - 2(a+b)w^m + bw^{m-n} + a \\ &= a + b - 2(a+b)w^m + b + a \\ &= 2(a+b)\big(1 - w^m\big) = 2(a+b)\big(1 - w^r\big). \end{aligned}$$

Since the $d/2$th roots of -1 are contained in the $d/2$th roots of unity, $Q^+(z)$ has exactly $d/2$ zeros on U without counting multiplicities. Such zeros are the $d/2$th roots of -1. □

3. An inequality and Eneström–Kakeya types of problems

In this section, we prove Proposition 5 and Theorem 7 in Section 1.

Proof of Proposition 5. Let, for $0 < z < 1$,

$$T(z) := m^2z^{2m} - n^2z^{m+n} + 2\big(m^2 - n^2\big)z^m - n^2z^{m-n} + m^2,$$

where $m > n > 0$. Then we have

$$T(z) > 0$$

for $0 < z < 1$. This is because

$$T(z) = z^{-n}\big(\big(m^2z^n\big(z^m + 1\big)^2\big) - n^2z^m\big(z^n + 1\big)^2\big), \quad z^n > z^m,$$

and

$$m^2(z^m+1)^2 > n^2(z^n+1)^2.$$

In fact

$$mz^m - nz^n + m - n > nz^m - mz^n + m - n > 0.$$

For the second inequality of the above, see [5, pp. 39–42]. We replace z by $(\frac{x}{y})^{\frac{1}{m}}$ (with $y > x$) so that we obtain

$$m^2\left(\frac{x}{y}\right)^2 - n^2\left(\frac{x}{y}\right)^{1+\frac{n}{m}} + 2(m^2-n^2)\left(\frac{x}{y}\right) - n^2\left(\frac{x}{y}\right)^{1-\frac{n}{m}} + m^2 > 0.$$

Letting $n/m = \lambda$ and dividing m^2y^{-2} of each side gives

$$x^2 - \lambda^2 x^{1+\lambda}y^{1-\lambda} + 2(1-\lambda^2)xy - \lambda^2 x^{1-\lambda}y^{1+\lambda} + y^2 > 0,$$

which completes the proof. □

Finally, we prove Theorem 7.

Proof of Theorem 7. Observe that

$$P(z) = \frac{z^{m+1}-1}{z-1} \cdot u(z),$$

where

$$u(z) = 1 + rz + rz^2 + \cdots + rz^m + z^{m+1}.$$

We first prove (a). Suppose that $r < -\frac{2}{m}$. Since $P(0) = 1$ and $P(1) = (m+1)(2+mr) < 0$, $P(z)$ has at least one zero in $(0, 1)$. Now it follows from the polynomial $\frac{z^{m+1}-1}{z-1}$ that $P(z)$ has at least m zeros on U that are $(m+1)$th roots of unity except 1. Also we may compute that

$$(z-1)u(z) = z^{m+2} + (r-1)z^{m+1} - (r-1)z - 1.$$

By Theorem 1, it suffices to show that $(z-1)u(z)$ has exactly m critical points inside or on U. Differentiating $(z-1)u(z)$ with respect to z gives

$$f(z) := (m+2)z^{m+1} + (r-1)(m+1)z^m - (r-1).$$

Then, for $|z| = 1$, we have

$$\left|(r-1)(m+1)z^m\right| = (m+1)(1-r) > (m+2) + (1-r) \geqslant \left|(m+2)z^{m+1} - (r-1)\right|,$$

and so $f(z)$ has m zeros inside U by Rouché's theorem. But it follows from $f(1) = 2 + rm < 0$ and $f(z) \to \infty$ as $z \to \infty$ that $f(z)$ has at least one zero in $(1, \infty)$. This implies that $f(z)$ has exactly m zeros inside or on U. This completes the proof of (a). Next we prove (b). The cases for $r = 0, 1$ are trivial. For $-\frac{2}{m} < r < 0$, it suffices to show that

$$u'(z) = r + 2rz + 3rz^2 + \cdots + mrz^{m-1} + (m+1)z^m$$

has all its zeros inside or on U by Theorem 1. This follows from Rouché's theorem since, for $|z| = 1$, we have

$$\left|r + 2rz + 3rz^2 + \cdots + mrz^{m-1}\right| \leqslant -r - 2r - 3r - \cdots - mr = -r\frac{m(m+1)}{m} < m + 1 = \left|(m+1)z^m\right|.$$

In case $0 < r < 1$, we consider

$$(z-1)^2 P(z) = q(z)(z^{m+1}-1), \tag{3.1}$$

where

$$q(z) = z^{m+2} + (r-1)z^{m+1} - (r-1)z - 1.$$

Observe that $q(z)$ is self-inversive and all the zeros of

$$q'(z) = (m+2)z^{m+1} + (r-1)(m+1)z^m - (r-1) \tag{3.2}$$

lie inside or on U by Rouché's theorem since for $|z| = 1$,

$$\left|(r-1)(m+1)z^m - (r-1)\right| < (1-r)(m+1) + (1-r) = (1-r)(m+2) < (m+2) = \left|(m+2)z^{m+1}\right|.$$

It follows from Theorem 1 that the polynomial $q(z)$ has all its zeros on U. This proves the case $0 < r < 1$ of (b). It remains the case $1 < r \leqslant 2$ to complete the proof of (b). A proof for the case $1 < r < 2$ is very similar to that of the case $0 < r < 1$ above. From (3.1), it is enough to show that all zeros of

$$q'(z) = (m+2)z^{m+1} + (r-1)(m+1)z^m - (r-1)$$

lie inside or on U by Theorem 1. For $|z| = 1$,

$$\left|(r-1)(m+1)z^m - (r-1)\right| < (r-1)(m+1) + (r-1) = (r-1)(m+2) < (m+2) = \left|(m+2)z^{m+1}\right|.$$

By Rouché's theorem, $q'(z)$ has all its zeros inside U. The final case $r = 2$ can be easily checked from

$$(z-1)^2 P(z) = \left(z^{m+1} - 1\right)\left(z^{m+2} + z^{m+1} - z - 1\right) = \left(z^{m+1} - 1\right)(z+1)\left(z^{m+1} - 1\right).$$

Now we prove (c). For $2 < r < 2 + \frac{2}{m}$ and m is even, it suffices to show that $q(z)$ has all its zeros on U in (3.1). We consider the zeros of $q(-z)$ instead of those of $q(z)$. Then we have

$$q(-z) = (z-1)r(z),$$

where $r(z) = z^{m+1} + (2-r)z^m + (2-r)z^{m-1} + \cdots + (2-r)z + 1$. Observe that $r(z)$ is self-inversive and all zeros of

$$r'(z) = (m+1)z^m + m(2-r)z^{m-1} + \cdots + 2(2-r)z + (2-r)$$

lie inside or on U by Rouché's theorem since for $|z| = 1$,

$$\begin{aligned}
\left|(m+1)z^m\right| &= m+1 > (r-2)m(m+1)/2 \\
&= m(r-2) + (m-1)(r-2) + \cdots + 2(r-2) + (r-2) \\
&= \left|m(2-r)z^{m-1}\right| + \left|(m-1)(2-r)z^{m-2}\right| + \cdots + \left|2(2-r)z\right| + \left|(2-r)\right| \\
&\geqslant \left|m(2-r)z^{m-1} + (m-1)(2-r)z^{m-2} + \cdots + 2(2-r)z + (2-r)\right|.
\end{aligned}$$

It follows from Theorem 1 that the polynomial $r(z)$ has all its zeros on U. This completes the proof of (c). Now we prove (d). For $r = 2 + \frac{2}{m}$ and m is even, we follow exactly same procedure of the proof (b) until (3.2). In (3.2), we observe that

$$\begin{aligned}
q'(-z) &= (1 + 1/n)\left[-2nz^{2n+1} + (2n+1)z^{2n} - 1\right] \\
&= -(1 + 1/n)(z-1)^2\left(2nz^{2n-1} + (2n-1)z^{2n-2} + \cdots + 2z + 1\right),
\end{aligned}$$

where $m = 2n$, $n = 1, 2, \ldots$. By Theorem 6,

$$2nz^{2n-1} + (2n-1)z^{2n-2} + \cdots + 2z + 1$$

has all its zeros inside U. So all zeros of $q'(z)$ lie inside or on U. It remains to prove (e). In (3.1), it is enough to show that

$$q'(z) = (m+2)z^{m+1} + (r-1)(m+1)z^m - (r-1)$$

has exactly m zeros inside or on U by Theorem 1. For $|z| = 1$,

$$\left|(m+2)z^{m+1} - (r-1)\right| \leqslant m + 2 + r - 1 < (m+1)(r-1) = \left|(m+1)(r-1)z^m\right|.$$

By Rouché's theorem, $q'(z)$ has m zeros inside U. If m is odd, then $q'(-1) = 2(m+2) - r(m+2) < 0$ and $q'(z) \to \infty$ as $z \to -\infty$. If m is even, then $q'(-1) = -2m - 2 + rm > 0$ and $q'(z) \to -\infty$ as $z \to -\infty$. Hence $q'(z)$ has at least a zero $(-\infty, -1)$. Finally $q'(z)$ has exactly m zeros inside or on U. $\square$

It can be proved by Theorem 8 that the zeros of $P(z)$ has all its zeros on U if $0 \leqslant r \leqslant 2$. From (3.1) it suffices to show that

$$z^{m+2} + (r-1)z^{m+1} - (r-1)z - 1$$

has all its zeros on U. The suitable $q_{n-l}(z)$ in Theorem 8 for (3.3) is $z + r - 1$ since $0 \leqslant r \leqslant 2$. In fact, we have

$$z^{m+2} + (r-1)z^{m+1} - (r-1)z - 1 = z^{m+1}(z + r - 1) - \big[(r-1)z + 1\big] = z^{m+1}(z + r - 1) - [z + r - 1]^*$$

and $z + r - 1$ has all its zeros on U since $0 \leqslant r \leqslant 2$.

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