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Some identities of symmetry for the generalized q-Euler polynomials

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ABSTRACT

By the symmetric properties of Drichlet's type multiple $q-l$-function, we establish various identities concerning the generalized higher-order q-Euler polynomials. Furthermore, we give some interesting relationship between the power sums and the generalized higher-order q-Euler polynomials.

1. Introduction

Let χ be a Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. As is well known, the generalized higher-order Euler polynomials are defined by the generating function to be

$$\left(2\sum_{a=0}^{d-1}\frac{(-1)^a\chi(a)e^{at}}{e^{dt}+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi}^{(r)}(x)\frac{t^n}{n!}. \tag{1.1}$$

When $x=0$, $E_{n,\chi}^{(r)} = E_{n,\chi}^{(r)}(0)$ are called the generalized Euler numbers attached to χ of order $r(\in \mathbb{N})$.

For $q \in \mathbb{C}$ with $|q|<1$, the q-number is defined by $[x]_q = \frac{1-q^x}{1-q}$.

Note that $\lim_{q\to 1}[x]_q = x$. In [7], Kim considered q-extension of generalized higher-order Euler polynomials attached to χ as follows:

$$F_{q,\chi}^{(r)}(t,x) = [2]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-q)^{m_1+\cdots+m_r}\left(\Pi_{i=1}^r \chi(m_i)\right) e^{[m_1+\cdots+m_r+x]_q t} = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x)\frac{t^n}{n!}. \tag{1.2}$$

Note that

$$\lim_{q\to 1} F_{q,\chi}^{(r)}(t,x) = \left(2\sum_{a=0}^{d-1}\frac{(-1)^a\chi(a)e^{at}}{e^{dt}+1}\right)^r e^{xt}.$$

For $s \in \mathbb{C}$ and $x \in \mathbb{R}$ with $x \neq 0,-1,-2,\ldots$, Kim defined Dirichlet-type multiple $q-l$-funtion which is given by

$$l_{q,r}(s,x|\chi) = [2]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(-q)^{m_1+\ldots+m_r}\left(\Pi_{i=1}^r\chi(m_i)\right)}{[m_1+\cdots+m_r+x]_q^s} = \frac{1}{\Gamma(s)}\int_0^{\infty} F_{q,\chi}^{(r)}(-t,x)t^{s-1}dt, \quad \text{(see [7])}. \tag{1.3}$$

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Applying the Laurent series and Cauchy residue theorem in (1.2) and (1.3), we get

$$l_{q,r}(-n,x|\chi)=E^{(r)}_{n,\chi,q}(x),\quad \text{where } n\in\mathbb{Z}_{\geqslant 0}. \tag{1.4}$$

When $x=0$, $E^{(r)}_{n,\chi,q}=E^{(r)}_{n,\chi,q}(0)$ are called the generalized q-Euler numbers attached to χ of order r. From (1.2), we note that

$$E^{(r)}_{n,\chi,q}(x)=\sum_{l=0}^{n}\binom{n}{l}q^{lx}E^{(r)}_{l,\chi,q}[x]_q^{n-l}=\left(q^xE^{(r)}_{\chi,q}+[x]_q\right)^n \tag{1.5}$$

with the usual convention about replacing $\left(E^{(r)}_{\chi,q}\right)^n$ by $E^{(r)}_{n,\chi,q}$, (see [1–13]).

In this paper, we investigate properties of symmetry in two variables related to multiple q—l-function which interpolates generalized higher-order q-Euler polynomials attached to χ at negative integers. From our investigation, we derive identities of symmetry in two variables related to generalized higher-order q-Euler polynomials attached to χ. Recently, several authors have studied q-extensions of Euler polynomials due to Kim (see [1–3,9–13]).

2. Symmetry of q-power sum and the generalized q-Euler polynomials

For $a,b\in\mathbb{N}$ with $a\equiv 1 \pmod 2$ and $b\equiv 1 \pmod 2$, we observe that

$$\frac{1}{[2]^r_{q^a}}l_{q^a,r}\left(s,bx+\frac{b}{a}(j_1+\cdots+j_r)|\chi\right)=[a]_q^s\sum_{n_1,\ldots,n_r=0}^{\infty}\sum_{i_1,\ldots,i_r=0}^{db-1}\frac{(-1)^{\sum_{l=1}^r(i_l+n_l)}q^{a\sum_{l=1}^r(i_l+bdn_l)}\left(\Pi_{l=1}^r\chi(i_l)\right)}{\left[ab\left(x+d\sum_{l=1}^rn_l\right)+b\sum_{l=1}^rj_l+a\sum_{l=1}^ri_l\right]_q^s}. \tag{2.1}$$

From (2.1), we have

$$\begin{aligned}&\frac{[b]_q^s}{[2]^r_{q^a}}\sum_{j_1,\ldots,j_r=0}^{da-1}(-1)^{\sum_{l=1}^rj_l}\left(\Pi_{l=1}^r\chi(j_l)\right)q^{b\sum_{l=1}^rj_l}l_{q^a,r}\left(s,bx+\frac{b}{a}\sum_{l=1}^rj_l|\chi\right)\\&=[a]_q^s[b]_q^s\sum_{i_1,\ldots,i_r=0}^{db-1}\sum_{j_1,\ldots,j_r=0}^{da-1}\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{(-1)^{\sum_{l=1}^r(i_l+n_l+j_l)}\left(\Pi_{l=1}^r\chi(j_l)\right)\left(\Pi_{l=1}^r\chi(i_l)\right)}{\left[ab\left(x+d\sum_{l=1}^rn_l\right)+\sum_{l=1}^r(bj_l+ai_l)\right]_q^s}\times q^{\sum_{l=1}^r(ai_l+bj_l+abdn_l)}.\end{aligned} \tag{2.2}$$

By the same method as (2.2), we get

$$\begin{aligned}&\frac{[a]_q^s}{[2]^r_{q^b}}\sum_{j_1,\ldots,j_r=0}^{db-1}(-1)^{\sum_{l=1}^rj_l}\left(\Pi_{l=1}^r\chi(j_l)\right)q^{a\sum_{l=1}^rj_l}l_{q^b,r}\left(s,ax+\frac{a}{b}\sum_{l=1}^rj_l|\chi\right)\\&=[a]_q^s[b]_q^s\sum_{j_1,\ldots,j_r=0}^{db-1}\sum_{i_1,\ldots,i_r=0}^{da-1}\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{(-1)^{\sum_{l=1}^r(i_l+n_l+j_l)}\left(\Pi_{l=1}^r\chi(j_l)\chi(i_l)\right)}{\left[ab\left(x+d\sum_{l=1}^rn_l\right)+\sum_{l=1}^r(aj_l+bi_l)\right]_q^s}\times q^{\sum_{l=1}^r(aj_l+bi_l+abdn_l)}.\end{aligned} \tag{2.3}$$

Therefore, by (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *For $a,b\in\mathbb{N}$ with $a\equiv 1 \pmod 2$2 and $b\equiv 1 \pmod 2$, we have*

$$\begin{aligned}&[2]^r_{q^b}[b]_q^s\sum_{j_1,\ldots,j_r=0}^{da-1}(-1)^{\sum_{l=1}^rj_l}\left(\Pi_{l=1}^r\chi(j_l)\right)q^{b\sum_{l=1}^rj_l}l_{q^a,r}\left(s,bx+\frac{b}{a}\sum_{l=1}^rj_l|\chi\right)\\&=[2]^r_{q^a}[a]_q^s\sum_{j_1,\ldots,j_r=0}^{db-1}(-1)^{\sum_{l=1}^rj_l}\left(\Pi_{l=1}^r\chi(j_l)\right)q^{a\sum_{l=1}^rj_l}l_{q^b,r}\left(s,ax+\frac{a}{b}\sum_{l=1}^rj_l|\chi\right).\end{aligned}$$

From (1.4) and Theorem 2.1, we obtain the following theorem.

Theorem 2.2. *For $n\geqslant 0$ and $a,b\in\mathbb{N}$ with $a\equiv 1 \pmod 2$ and $b\equiv 1 \pmod 2$, we have*

$$\begin{aligned}&[2]^r_{q^b}[a]_q^n\sum_{j_1,\ldots,j_r=0}^{da-1}(-1)^{\sum_{l=1}^rj_l}\left(\Pi_{l=1}^r\chi(j_l)\right)q^{b\sum_{l=1}^rj_l}E^{(r)}_{n,\chi,q^a}\left(bx+\frac{b}{a}\sum_{l=1}^rj_l\right)\\&=[2]^r_{q^a}[b]_q^n\sum_{j_1,\ldots,j_r=0}^{db-1}(-1)^{\sum_{l=1}^rj_l}\left(\Pi_{l=1}^r\chi(j_l)\right)q^{a\sum_{l=1}^rj_l}E^{(r)}_{n,\chi,q^b}\left(ax+\frac{a}{b}\sum_{l=1}^rj_l\right).\end{aligned}$$

By (1.5), we easily get

$$E^{(r)}_{n,\chi,q}(x+y)=\left(q^{x+y}E^{(r)}_{\chi,q}+[x+y]_q\right)^n=\left(q^{x+y}E^{(r)}_{\chi,q}+q^x[y]_q+[x]_q\right)^n=\sum_{i=0}^{n}\binom{n}{i}q^{ix}\left(q^yE^{(r)}_{\chi,q}+[y]_q\right)^i[x]_q^{n-i}=\sum_{i=0}^{n}\binom{n}{i}q^{xi}E^{(r)}_{i,\chi,q}(y)[x]_q^{n-i}. \tag{2.4}$$

From (2.4), we note that

$$\begin{aligned}\sum_{j_1,\dots,j_r=0}^{da-1}(-1)^{\sum_{l=1}^r j_l}q^{b\sum_{l=1}^r j_l}(\Pi_{l=1}^r\chi(j_l))E^{(r)}_{n,\chi,q^a}\left(bx+\frac{b}{a}\sum_{l=1}^r j_l\right) &= \sum_{j_1,\dots,j_r=0}^{da-1}(-1)^{\sum_{l=1}^r j_l}q^{b\sum_{l=1}^r j_l}(\Pi_{l=1}^r\chi(j_l))\\ &\quad\times\sum_{i=0}^n\binom{n}{i}q^{ib\sum_{l=1}^r j_l}E^{(r)}_{i,\chi,q^a}(bx)\left[\frac{b(j_1+\cdots+j_r)}{a}\right]_{q^a}^{n-i}\\ &= \sum_{j_1,\dots,j_r=0}^{da-1}(-1)^{\sum_{l=1}^r j_l}q^{b\sum_{l=1}^r j_l}(\Pi_{l=1}^r\chi(j_l))\\ &\quad\times\sum_{i=0}^n\binom{n}{i}q^{(n-i)b\sum_{l=1}^r j_l}E^{(r)}_{n-i,\chi,q^a}(bx)\left[\frac{b}{a}\sum_{l=1}^r j_l\right]_{q^a}^{i}\\ &= \sum_{i=0}^n\binom{n}{i}\left(\frac{[b]_q}{[a]_q}\right)^i E^{(r)}_{n-i,\chi,q^a}(bx)\\ &\quad\times\sum_{j_1,\dots,j_r=0}^{da-1}(-1)^{\sum_{l=1}^r j_l}q^{b\sum_{l=1}^r(n-i+1)j_l}[j_1+\cdots+j_r]^i_{q^b}\\ &= \sum_{i=0}^n\binom{n}{i}\left(\frac{[b]_q}{[a]_q}\right)^i E^{(r)}_{n-i,\chi,q^a}(bx)S^{(r)}_{n,i,q^b}(ad|\chi),\end{aligned}\tag{2.5}$$

where

$$S^{(r)}_{n,i,q}(a|\chi)=\sum_{j_1,\dots,j_r=0}^{a-1}(-1)^{j_1+\cdots+j_r}(\Pi_{l=1}^r\chi(j_l))q^{\sum_{l=1}^r j_l(n-i+1)}\left[\sum_{l=1}^r j_l\right]_q^i.\tag{2.6}$$

From (2.5) and (2.6), we can derive the following equation.

$$[2]^r_{q^b}[a]^n_q\sum_{j_1,\dots,j_r=0}^{da-1}(-1)^{\sum_{l=1}^r j_l}q^{b\sum_{l=1}^r j_l}(\Pi_{l=1}^r\chi(j_l))E^{(r)}_{n,\chi,q^a}\left(bx+\frac{b}{a}\sum_{l=1}^r j_l\right)=[2]^r_{q^b}\sum_{i=0}^n\binom{n}{i}[a]^{n-i}_q[b]^i_qE^{(r)}_{n-i,\chi,q^a}(bx)S^{(r)}_{n,i,q^b}(ad|\chi).\tag{2.7}$$

By the same method as (2.7), we get

$$[2]^r_{q^a}[b]^n_q\sum_{j_1,\dots,j_r=0}^{db-1}(-1)^{\sum_{l=1}^r j_l}q^{a\sum_{l=1}^r j_l}(\Pi_{l=1}^r\chi(j_l))E^{(r)}_{n,\chi,q^b}\left(ax+\frac{a}{b}\sum_{l=1}^r j_l\right)=[2]^r_{q^a}\sum_{i=0}^n\binom{n}{i}[b]^{n-i}_q[a]^i_qE^{(r)}_{n-i,\chi,q^b}(ax)S^{(r)}_{n,i,q^a}(bd|\chi).\tag{2.8}$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

Theorem 2.3. *For $n \geqslant 0$ and $a,b\in\mathbb{N}$ with $a\equiv 1 \pmod 2$ and $b\equiv 1 \pmod 2$, we have*

$$[2]^r_{q^b}\sum_{i=0}^n\binom{n}{i}[a]^{n-i}_q[b]^i_qE^{(r)}_{n-i,\chi,q^a}(bx)S^{(r)}_{n,i,q^b}(ad|\chi)=[2]^r_{q^a}\sum_{i=0}^n\binom{n}{i}[b]^{n-i}_q[a]^i_qE^{(r)}_{n-i,\chi,q^b}(ax)S^{(r)}_{n,i,q^a}(bd|\chi).$$

Remark. It is not difficult to show that

$$\begin{aligned}&e^{[x]_q u}\sum_{m_1,\dots,m_r=0}^{\infty}q^{\sum_{l=1}^r m_l}(-1)^{\sum_{l=1}^r m_l}(\Pi_{l=1}^r\chi(m_l))e^{\left[y+\sum_{l=1}^r m_l\right]_q q^x(u+v)}\\ &\quad= e^{-[x]_q u}\sum_{m_1,\dots,m_r=0}^{\infty}q^{\sum_{l=1}^r m_l}(-1)^{\sum_{l=1}^r m_l}(\Pi_{l=1}^r\chi(m_l))e^{\left[x+y+\sum_{l=1}^r m_l\right]_q(u+v)}.\end{aligned}\tag{2.9}$$

Thus, by (2.9), we get

$$\sum_{k=0}^m\binom{m}{k}q^{kx}E^{(r)}_{n+k,\chi,q}(y)[x]^{m-k}_q=\sum_{k=0}^n\binom{n}{k}q^{-kx}E^{(r)}_{m+k,\chi,q}(x+y)[-x]^{n-k}_q.\tag{2.10}$$

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