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# Some classes of analytic functions involving differential subordinations

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ABSTRACT

The main object of this paper is to apply the method of differential subordinations in order to obtain certain properties of some subclasses of analytic functions in the unit disc involving differential subordinations.



## 1. Introduction and definitions

Let $\mathscr{H}$ denote the class of functions analytic in the open unit disc $\mathscr{U}$. For a positive integer $n$ and $a \in \mathbb{C}$, let

$$H[a,n] := \left\{ f \in \mathscr{H} : \ f(z) = a + \sum_{k=n}^{\infty} a_k z^k \right\}$$

and

$$\mathscr{A} := \left\{ f \in \mathscr{H} : \ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \right\}.$$

The class $\mathscr{H}[a,n]$ has been studied by several researchers; for example, see [3]. Many of the subclasses of $\mathscr{A}$ and $\mathscr{H}[a,n]$ can be written very neatly in terms of subordination and convolution (or Hadamard products ). We recall these definitions.

If $f, g \in \mathscr{H}$, then the function $f$ is said to be *subordinate* to $g$, written as $f \prec g$ or $f(z) \prec g(z)$, $z \in \mathscr{U}$, if there exists a Schwarz function $\omega \in \mathscr{H}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathscr{U}$ such that $f(z) = g(\omega(z))$.

Next, for the functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

let $f * g$ denote the *convolution* (or *Hadamard product*) of $f$ and $g$ defined by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in \mathscr{U}).$$

Throughout the paper, unless otherwise stated, $g$ will denote a fixed function in $\mathscr{H}$ and $h$ will denote an analytic, convex, univalent function on $\mathscr{U}$ with $h(0) = 1$ and $\Re(h(z)) > 0$ for all $z$ in $\mathscr{U}$.

For any $\lambda$ $(0 \leqslant \lambda \leqslant 1)$, $\alpha$ $(\alpha \geqslant 0)$ and for all $z \in \mathscr{U}$, we define the following subclasses of $\mathscr{A}$.

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$$\mathcal{S}(g,h,\lambda) := \left\{ f \in \mathcal{A} : \frac{z(f*g)'(z) + \lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z) + \lambda z(f*g)'(z)} \prec h(z) \right\},$$
$$\mathcal{T}(g,h,\alpha) := \left\{ f \in \mathcal{A} : (1-\alpha)\frac{(f*g)(z)}{z} + \alpha(f*g)'(z) \prec h(z) \right\},$$
$$\mathcal{R}(g,h,\alpha) := \{ f \in \mathcal{A} : (f*g)'(z) + \alpha z(f*g)''(z) \prec h(z) \}.$$

Note that the function $g(z) = \frac{z}{(1-z)^c}, c \in \mathbb{R}$, these three classes reduce to the corresponding classes in [3]. Let

$$\mathcal{L}(z) = -log(1-z) = \int_0^z \frac{dt}{1-t} = z + \sum_{n=2}^{\infty} \frac{z^n}{n}$$

for all $z$ in $\mathcal{U}$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$, then

$$(\mathcal{L}*f)(z) = z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n. \tag{1}$$

For $g := \mathcal{L}$, we define classes

$$\mathcal{S}(h,\lambda) := \mathcal{S}(\mathcal{L},h,\lambda), \quad 0 \leqslant \lambda \leqslant 1,$$
$$\mathcal{T}(h,\alpha) := \mathcal{T}(\mathcal{L},h,\alpha), \quad \alpha \geqslant 0,$$
$$\mathcal{R}(h,\alpha) := \mathcal{R}(\mathcal{L},h,\alpha), \quad \alpha \geqslant 0.$$

According to (1), by taking derivatives, we find that

$$z(\mathcal{L}*f)'(z) = f(z) \tag{2}$$

and

$$z^2(\mathcal{L}*f)''(z) = zf'(z) - f(z). \tag{3}$$

In 1981, Miller and Mocanu [1] laid a foundation for the Theory of differential subordinations in the complex plane. In very simple language, a differential subordination in the complex plane is the generalization of a differential inequality on the real line. Later on, several researchers developed many applications and extensions of the theory of differential subordinations. For definitions and works of hundreds of researchers in this field, one may refer to an excellent monograph by Miller and Mocanu [2].

The main object of this paper is to apply the method of the differential subordinations in order to obtain several properties of the six new classes defined in this section.

## 2. Main lemmas

In view of (2) and (3), we immediately obtain the following characterizations.

**Lemma 1**

(i) $f \in \mathcal{S}(h,\lambda) \iff \frac{(1-\lambda)f(z)+\lambda zf'(z)}{(1-\lambda)(\mathcal{L}*f)(z)+\lambda f(z)} \prec h(z).$
(ii) $f \in \mathcal{T}(h,\alpha) \iff \frac{(1-\alpha)(\mathcal{L}*f)(z)+\alpha f(z)}{z} \prec h(z).$
(iii) $f \in \mathcal{R}(h,\alpha) \iff \frac{1}{z}[(1-\alpha)f(z) + \alpha zf'(z)] \prec h(z).$

An immediate consequence of Lemma 1(iii) is asserted by following.

**Corollary 1**

(i) $f \in \mathcal{R}(h,1) \iff f'(z) \prec h(z).$
(ii) $f \in \mathcal{R}(h,0) \iff \frac{f(z)}{z} \prec h(z).$

*The case* $h(z) = \frac{1+z}{1-z}$ *of Corollary 1(i), gives*

**Corollary 2.** $-2ln(1-z) - z \in \mathcal{R}(\frac{1+z}{1-z}, 1).$

**Lemma 2** [2, p. 81]. *Let* $\beta$ *and* $\gamma$ *be complex numbers with* $\beta \neq 0$. *Also suppose* $h$ *is convex in* $\mathcal{U}$, *with* $\Re\{\beta h(z) + \gamma\} > 0$. *If* $p$ *is analytic in* $\mathcal{U}$ *with* $p(0) = h(0)$ *then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

**Lemma 3** [2, p. 71]. *Let h be convex in $\mathcal{U}$, with $h(0) = a, \gamma \neq 0$ and $Re(\gamma) \geqslant 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z),$$

*then $p(z) \prec q(z) \prec h(z)$, where*

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt.$$

## 3. Main results

Making use of Lemma 2, we prove the following.

**Theorem 1.** *If $f, g \in \mathcal{A}$ and $f \in \mathcal{S}(g, h, \lambda)$, then $F_\mu(f) \in \mathcal{S}(g, h, \lambda)$, where*

$$F_\mu(f) = \frac{\mu+1}{z^\mu} \int_0^z t^{\mu-1} f(t) dt \quad (\mu \geqslant 0).$$

**Proof.** Let $f(z) = z + \sum_{n=2}^\infty b_n z^n$ and $g(z) = z + \sum_{n=2}^\infty c_n z^n$. Then

$$F_\mu(f)(z) = z + \sum_{n=2}^\infty \frac{b_n(\mu+1)}{\mu+n} z^n.$$

Therefore

$$(g * z(F_\mu(f))')(z) = z + \sum_{n=2}^\infty \frac{nb_n c_n(\mu+1)}{\mu+n} z^n.$$

So

$$(g * z(F_\mu(f))')(z) = z(g * F_\mu(f))'(z). \tag{4}$$

Also it easily follows that

$$z(F_\mu(f))'(z) + \mu F_\mu(f)(z) = (\mu+1) f(z).$$

Thus we can write

$$(g * z(F_\mu(f))')(z) + \mu(g * F_\mu(f))(z) = (\mu+1)(g * f)(z).$$

By using the relation (4), we have

$$z(g * (F_\mu(f))')(z) + \mu(g * F_\mu(f))(z) = (\mu+1)(g * f)(z). \tag{5}$$

On differentiating both sides of (5), we have

$$z(g * F_\mu(f))''(z) = (\mu+1)[(g * f)'(z) - (g * F_\mu(f))'(z)]. \tag{6}$$

Setting

$$p(z) = \frac{z(g * F_\mu(f))'(z) + \lambda z^2 (g * F_\mu(f))''(z)}{(1-\lambda)(g * F_\mu(f))(z) + \lambda z(g * F_\mu(f))'(z)}$$

and by using (5) and (6), we obtain

$$p(z) + \mu = (\mu+1) \frac{(1-\lambda)(g * f)(z) + \lambda z(g * f)'(z)}{(1-\lambda)(g * F_\mu(f))(z) + \lambda z(g * F_\mu(f))'(z)}. \tag{7}$$

Making use of the logarithmic differentiation on both sides of (7) and multiplying the resulting equation by $z$, we have

$$p(z) + \frac{zp'(z)}{p(z)+\mu} = \frac{z(f * g)'(z) + \lambda z^2 (f * g)''(z)}{(1-\lambda)(f * g)(z) + \lambda z(f * g)'(z)}. \tag{8}$$

Since $f \in \mathcal{S}(g, h, \lambda)$, we get

$$\frac{z(f * g)'(z) + \lambda z^2 (f * g)''(z)}{(1-\lambda)(f * g)(z) + \lambda z(f * g)'(z)} \prec h(z). \tag{9}$$

By applying Lemma 2, (8) and (9) it follows that $p(z) \prec h(z)$. This proves that $F_\mu(f) \in \mathcal{S}(g, h, \lambda)$. $\square$

**Remark.** Theorem 1 is a generalization of the corresponding result in [3, Theorem 2].

**Corollary 3.** *For* $f \in \mathscr{A}$, $\mathscr{S}(f,h,\lambda) \subset \mathscr{S}(F_\mu(f),h,\lambda)$.

**Proof.** If $g \in \mathscr{S}(f,h,\lambda)$, then $f \in \mathscr{S}(g,h,\lambda)$. According to Theorem 1, $F_\mu(f) \in \mathscr{S}(g,h,\lambda)$, or $g \in \mathscr{S}(F_\mu(f),h,\lambda)$. □

Letting $g(z) = \mathscr{L}(z)$, Theorem 1 yields

**Corollary 4.** *If* $f \in \mathscr{S}(h,\lambda)$ *then* $F_\mu(f) \in \mathscr{S}(h,\lambda)$.

**Theorem 2.** *Let* $f,g \in \mathscr{A}$ *and* $F_\mu$ *be an integral operator as defined in Theorem 1. If*

$$F_\mu(f) \in \mathscr{T}(g,h,\alpha) \cap \mathscr{R}(g,h,\alpha), \quad \mu \geqslant 1,$$

*then* $F_\mu(f) \in \mathscr{T}(g,q,\alpha)$, *where*

$$q(z) = \frac{\mu}{z^\mu}\int_0^z h(t)t^{\mu-1}dt.$$

**Proof.** If

$$p(z) = \frac{(1-\alpha)(F_\mu(f)*g)(z) + \alpha z(F_\mu(f)*g)'(z)}{z},$$

then

$$p(z) + \frac{zp'(z)}{\mu} = \frac{\mu-1}{\mu}\left[(1-\alpha)\frac{(F_\mu(f)*g)(z)}{z} + \alpha(F_\mu(f)*g)'(z)\right] + \frac{1}{\mu}\left[(F_\mu(f)*g)'(z) + \alpha z(F_\mu(f)*g)''(z)\right]. \tag{10}$$

Since $F_\mu(f) \in \mathscr{T}(g,h,\alpha) \cap \mathscr{R}(g,h,\alpha)$ and $h$ is convex function, from (10) we have

$$p(z) + \frac{zp'(z)}{\mu} \prec h(z).$$

By using the Lemma 3, we have

$$p(z) = \frac{(1-\alpha)(F_\mu(f)*g)(z)}{z} + \alpha(F_\mu(f)*g)'(z) \prec q(z),$$

where

$$q(z) = \frac{\mu}{z^\mu}\int_0^z h(t)t^{\mu-1}dt.$$

The proof is completed. □

**Corollary 5.** *For* $f \in \mathscr{A}$, $\mathscr{R}(F_\mu(f),h,\alpha) \cap \mathscr{T}(F_\mu(f),h,\alpha) \subset \mathscr{T}(F_\mu(f),q,\alpha)$ *where*

$$q(z) = \frac{\mu}{z^\mu}\int_0^z h(t)t^{\mu-1}dt.$$

**Proof.** Let $g \in \mathscr{R}(F_\mu(f),h,\alpha) \cap \mathscr{T}(F_\mu(f),h,\alpha)$. Then $F_\mu(f) \in \mathscr{T}(g,q,\alpha)$, by Theorem 2. □

Letting $h(z) = \frac{1+Az}{1+Bz}$ and $\mu = 1$ in Corollary 5, we get the following corollary.

**Corollary 6.** $\mathscr{R}(F_1(f),\frac{1+Az}{1+Bz},\alpha) \cap \mathscr{T}(F_1(f),\frac{1+Az}{1+Bz},\alpha) \subset \mathscr{T}(F_1(f),q,\alpha)$, *where* $A \in \mathbb{C}$, $B \in [-1,0]$, $A \neq B$ *and*

$$q(z) = \frac{1}{Bz}\left[\left(1-\frac{A}{B}\right)ln(1+Bz) + Az\right].$$

*We make use of Lemma 3 in the following.*

**Theorem 3**

(i) *For* $\alpha > 0, \mathscr{T}(g,h,\alpha) \subset \mathscr{T}(g,h,0)$.
(ii) *For* $\alpha > \gamma \geqslant 0, \mathscr{T}(g,h,\alpha) \subset \mathscr{T}(g,h,\gamma)$.

**Proof.** (**i**). Let $f \in \mathscr{T}(g,h,\alpha)$. Then

$$\frac{1-\alpha}{z}(f*g)(z)+\alpha(f*g)'(z) \prec h(z). \tag{11}$$

Suppose $p(z)=\frac{(f*g)(z)}{z}$. We have

$$\alpha zp'(z)=\alpha(f*g)'(z)+(1-\alpha)\frac{(f*g)(z)}{z}-p(z)$$

or

$$p(z)+\alpha zp'(z)=(1-\alpha)\frac{(f*g)(z)}{z}+\alpha(f*g)'(z). \tag{12}$$

By applying Lemma 3, (11) and (12) we have $p(z) \prec h(z)$. Then $\frac{(f*g)(z)}{z} \prec h(z)$, or $f\in\mathscr{T}(g,h,0)$.

(**ii**). If $\gamma=0$, then it reduces to part (i). Now let $\gamma>0$ and $z\in\mathscr{U}$. If $f\in\mathscr{T}(g,h,\alpha)$, then

$$(1-\alpha)\frac{(f*g)(z)}{z}+\alpha(f*g)'(z)\in h(\mathscr{U}).$$

Also according to part (i), $\frac{(f*g)(z)}{z}\in h(\mathscr{U})$. Moreover

$$(1-\gamma)\frac{(f*g)(z)}{z}+\gamma(f*g)'(z)=\left(1-\frac{\gamma}{\alpha}\right)\frac{(f*g)(z)}{z}+\frac{\gamma}{\alpha}\left[(1-\alpha)\frac{(f*g)(z)}{z}+\alpha(f*g)'(z)\right].$$

Since $\frac{\gamma}{\alpha}<1$ and $h(\mathscr{U})$ is convex,

$$(1-\gamma)\frac{(f*g)(z)}{z}+\gamma(f*g)'(z)\in h(\mathscr{U}).$$

This proves that $f\in\mathscr{T}(g,h,\gamma)$. □

**Remark.** Letting $g(z)=\frac{z}{(1-z)^c}$, Theorem 3 reduces to the corresponding result in [3, Theorem 10].

**Theorem 4.** *Let $h$ be analytic, univalent and convex in $\mathscr{U}$, with $h(0)=1,\Re\{h(z)+\gamma\}>0, z\in\mathscr{U},\gamma\in\mathbb{C},\Re\{\gamma\}<0$, $f\in\mathscr{A}, 0\leqslant\lambda\leqslant 1$ and*

$$\frac{\gamma(1-\lambda)f(z)+(1+\gamma\lambda)zf'(z)+\lambda z^2f''(z)}{(1-\lambda+\gamma\lambda)f(z)+\lambda zf'(z)+\gamma(1-\lambda)(\mathscr{L}*f)(z)} \prec h(z). \tag{13}$$

*Then $f\in\mathscr{S}(h,\lambda)$.*

**Proof.** Let

$$p(z)=\frac{(1-\lambda)f(z)+\lambda zf'(z)}{(1-\lambda)(\mathscr{L}*f)(z)+\lambda f(z)}.$$

Then

$$p(z)+\gamma=\frac{(1-\lambda+\gamma\lambda)f(z)+\lambda zf'(z)+\gamma(1-\lambda)(\mathscr{L}*f)(z)}{(1-\lambda)(\mathscr{L}*f)(z)+\lambda f(z)}. \tag{14}$$

Making use of the logarithmic differentiation on both sides of relation (14), multiplying the resulting equation by $z$, and by using the relation (2) we have

$$\frac{zp'(z)}{p(z)+\gamma}=\frac{\gamma(1-\lambda)f(z)+(1+\gamma\lambda)zf'(z)+\lambda z^2f''(z)}{(1-\lambda+\gamma\lambda)f(z)+\lambda zf'(z)+\gamma(1-\lambda)(\mathscr{L}*f)(z)}-p(z)$$

or

$$p(z)+\frac{zp'(z)}{p(z)+\gamma}=\frac{\gamma(1-\lambda)f(z)+(1+\gamma\lambda)zf'(z)+\lambda z^2f''(z)}{(1-\lambda+\gamma\lambda)f(z)+\lambda zf'(z)+\gamma(1-\lambda)(\mathscr{L}*f)(z)}. \tag{15}$$

By applying Lemma 2, (13) and (15) we have $p(z) \prec h(z)$. So $f\in\mathscr{S}(h,\lambda)$, by Lemma 1($i$). □

By putting $\lambda=\gamma=0$ in Theorem 4, we get the following result.

**Corollary 7.** *If $\frac{zf'(z)}{f(z)} \prec h(z)$, then $\frac{f(z)}{(\mathscr{L}*f)(z)} \prec h(z)$.*

**Theorem 5.** *$f\in\mathscr{T}(h,\alpha)$ if and only if*

$$\frac{1}{z}\left[(1-\alpha)\int_0^z \frac{f(t)}{t}dt + \alpha f(z)\right] \prec h(z), \quad \alpha \geqslant 0.$$

**Proof.** By using the definition of $\mathscr{L}$, it is easy to verify that

$$(\mathscr{L} * f)(z) = \int_0^z \frac{f(t)}{t}dt. \tag{16}$$

Also, $f \in \mathscr{T}(h,\alpha)$ if and only if

$$(1-\alpha)\frac{(\mathscr{L} * f)(z)}{z} + \alpha(\mathscr{L} * f)'(z) \prec h(z). \tag{17}$$

The conclusion follows from (2),(16) and (17). □

Letting $\alpha = 0$ and $\alpha = 1$, respectively, in Theorem 5, we obtain

**Corollary 8.** *If $f \in \mathscr{A}$, then*

(i) $f \in \mathscr{T}(h,0) \iff \frac{1}{z}\int_0^z \frac{f(t)}{t}dt \prec h(z).$

(ii) $f \in \mathscr{T}(h,1) \iff \frac{f(z)}{z} \prec h(z).$

*By putting $h(z) = \frac{1+z}{1-z}$ in Corollary 8, we obtain*

**Corollary 9.**

(i) $\frac{z(1+2z-z^2)}{(1-z)^2} \in \mathscr{T}\left(\frac{1+z}{1-z}, 0\right).$

(ii) $\frac{z(1+z)}{1-z} \in \mathscr{T}\left(\frac{1+z}{1-z}, 1\right).$

**Theorem 6.** *If $f \in \mathscr{T}(h,\alpha)$ and*

$$(1-\alpha)(f(z) - zf'(z)) - \alpha z^2 f''(z) = 0, \tag{18}$$

*then $f \in \mathscr{R}(h,\alpha)$.*

**Proof.** Setting

$$\frac{1}{z}\left[(1-\alpha)f(z) + \alpha z f'(z)\right] = \frac{1}{z}\left[(1-\alpha)\int_0^z \frac{f(t)}{t}dt + \alpha f(z)\right]$$

and taking the derivative we obtain (18). Now by applying Lemma 1(iii) and Theorem 5 the proof is complete. □

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