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ON ALMOST PRIME SUBMODULES*

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Abstract In this article, we define almost prime submodules as a new generalization of prime and weakly prime submodules of unitary modules over a commutative ring with identity. We study some basic properties of almost prime submodules and give some characterizations of them, especially for (finitely generated faithful) multiplication modules.

Key words Prime submodules; weakly prime submodules; almost prime submodules; multiplication modules

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1 Introduction

Throughout this article, we consider all rings as commutative rings with identities and all modules as unital. For any two submodules N and K of an R-module M, the residual of N by K is defined as the set $(N : K) = \{r \in R : rK \subseteq N\}$ which is clearly an ideal of R. In particular, the ideal $(0 : M)$ is called the annihilator of M and is denoted by Ann zero ideal of R. Let N be a submodule of M and I be an ideal of R. The residual submodule of N by I is defined as $(N : I) = \{m \in M : I(m) \subseteq N\}$. These two residual ideal and submodule were proved to be useful in studying many concepts of modules, see, for example, [7, 14, 17].

Recall that a proper submodule N of an R-module M is a prime submodule of M if, whenever $rm \in N$ for $r \in R$ and $m \in M$, then either $m \in N$ or $r \in (N : M)$. An integral module is one in which the zero submodule is prime. A proper submodule K is maximal in M if there is no proper submodule of M containing K properly. A local module is a module with unique maximal submodule. An R-module M is called a multiplication module provided that, for every submodule N of M, there exists an ideal I of R so that $N = IM$ (or equivalently, $N = (N : M)M$). Multiplication modules were studied extensively in [14, 15, 18] and [1]. An R-module M is called a cancellation module of R if, for all ideals I and J of R, $IM = JM$ implies that $I = J$, see [6, 8]. For example, cancellation modules of a ring R include free modules and finitely generated faithful multiplication modules over R.

A proper ideal P of a ring R is called weakly prime if, whenever $a, b \in R$ with $0 \neq ab \in P$, then $a \in P$ or $b \in P$. Weakly prime ideals were studied by Anderson and Smith in 2003, see [9].

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Recently in 2008, Anderson and Bataineh [10] studied a generalization of the class of weakly prime ideals, which is the class of n-almost prime ideals, where n is a positive integer. A proper ideal I of R is called n-almost prime if, for $a, b \in R$, $ab \in I - I^n$ implies $a \in I$ or $b \in I$. In particular, the 2-almost prime ideals are simply called almost prime ideals.

The class of prime submodules of modules was introduced and studied in 1992 as a generalization of the class of prime ideals of rings. Then, many generalizations of prime submodules were studied such as primary, classical prime, weakly prime and classical primary submodules, see [3, 12, 13, 16] and [11]. In this article, we study almost prime submodules as one of the generalizations of prime (and weakly prime) submodules. We generalize some basic properties of prime and weakly prime to almost prime submodules. In particular, we give characterizations of almost prime submodules in multiplication modules.

2 Some Properties of Almost Prime Submodules

Definition 2.1 Let M be an R-module and N be a proper submodule of M.

(1) N is called a weakly prime submodule of M if, whenever $r \in R$ and $m \in M$ such that $0 \neq rm \in N$, then either $m \in N$ or $r \in (N : M)$.

(2) N is called an almost prime submodule of M if, whenever $r \in R$ and $m \in M$ such that $rm \in N - (N : M)N$, then either $m \in N$ or $r \in (N : M)$.

Clearly, any prime submodule is weakly prime and any weakly prime submodule is almost prime. However, the converses are not necessarily true. For example, the zero submodule in a non-integral module is weakly prime, which is not prime. For an almost prime submodule which is not weakly prime, we consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{24}$ and the proper cyclic submodule N of M generated by $\overline{8}$. Then clearly $(N : M)N = N$ and so N is almost prime. In contrast, $\subset \overline{0} \neq 4(\overline{4}) \in N$ witis not prime, we consider the following.

Theorem 2.2 Let M bh $\overline{4} \notin N$ and $4 \notin (N : M)$ and so N is not weakly prime. For a non trivial example of an almost prime submodule which e a nonzero local multiplication module with unique maximal submodule Q such that $(Q : M)Q = 0$. Then, every non-maximal proper submodule of M is weakly prime (and so almost prime), which is not prime.

Proof See [7].

Let M be an R-module and N be a submodule of M. Following [1], N is called idempotent in M if $(N : M)N = N$. Thus, any proper idempotent submodule of M is almost prime. If M is a multiplication R-module and $N = IM$, $K = JM$ are two submodules of M, then, the product NK of N and K is defined as $NK = (IM)(JM) = (IJ)M$, see [4]. In particular, we have

$$N^2 = NN = [(N : M)M][(N : M)M] = (N : M)^2M.$$

If further, M is a cancellation R-module, then using Lemma 3.4,

$$(N : M)N = ((N : M)N : M)M = (N : M)^2M = N^2.$$

So, in this particular case, a submodule N is idempotent in M if and only if $N = N^2$.

Following [5], a submodule N of an R-module M is called a pure submodule if $IN = N \cap IM$ for any ideal I of R. In [1], it was proved that if N is a pure submodule in a multiplication R-module M with pure annihilator, then N is idempotent in M and so is almost prime.

Theorem 2.3 Let M be an R-module and N be a proper submodule of M. Then, N is almost prime in M if and only if $N/(N:M)N$ is weakly prime in $M/(N:M)N$.

Proof Suppose that N is almost prime in M. Let $r \in R$ and $m \in M$, such that $\overline{0} \neq r(m+(N:M)N) \in N/(N:M)N$ in $M/(N:M)N$. Then, $rm \in N-(N:M)N$ and so either $m \in M$ or $r \in (N:M)$. Hence, either $m+(N:M)N \in N/(N:M)N$ or $r \in (N:M) = (N/(N:M)N : M/(N:M)N)$ and so $N/(N:M)N$ is weakly prime in $M/(N:M)N$. Conversely, assume that $N/(N:M)N$ is weakly prime in $M/(N:M)N$ and let $r \in R$ and $m \in M$ such that $rm \in N-(N:M)N$. Then, $\overline{0} \neq r(m+(N:M)N) \in N/(N:M)N$ and hence either $m+(N:M)N \in N/(N:M)N$ (and so $m \in N$) or $r \in (N/(N:M)N : M/(N:M)N) = (N:M)$.

We recall that, for any multiplicatively closed subset S of a ring R and any submodule N of an R-module M, we have $(N:M) \subseteq (S^{-1}N : S^{-1}M)$ where we consider $S^{-1}M$ as an R-module. We recall also that, if M is an R-module and $K \subseteq P \subset M$ are submodules of M, then P is a prime submodule of M if and only if P/K is a prime submodule of M/K.

Theorem 2.4 Let N be an almost prime submodule of an R-module M.

(1) If K is a submodule of M with $K \subseteq N$, then, N/K is an almost prime submodule of M/K.

(2) If S is a multiplicatively closed subset of R with $(N:M) \cap S = \phi$, then, $S^{-1}N$ is almost prime in $S^{-1}M$.

Proof (1) Let $r \in R$ and $m+K \in M/K$ such that $r(m+K) \in (N/K) - (N/K : M/K)N/K$. Then, $rm+K \in N/K - (N:M)N/K$ and so $rm \in N-(N:M)N$. As N is almost prime in M, either $m \in N$ or $r \in (N:M)$. Therefore, $m+K \in N/K$ or $r \in (N/K : M/K)$, and N/K is almost prime in M/K.

(2) Let $r \in R$, $s \in S$ and $m \in M$ such that $r(\frac{m}{s}) \in S^{-1}N - (S^{-1}N : S^{-1}M)S^{-1}N$. Then, $\frac{rm}{s} \in S^{-1}N - S^{-1}((N:M)N)$. Indeed, if $\frac{rm}{s} \in S^{-1}((N:M)N)$, then there is $t \in S$ such that

$$\frac{rm}{s} = \frac{r_1n_1 + r_2n_2 + \cdots + r_kn_k}{t} = r_1\frac{n_1}{t} + r_2\frac{n_2}{t} + \cdots + r_k\frac{n_k}{t},$$

where $r_i \in (N:M)$ and $n_i \in N$, $i = 1, 2, \cdots, k$. Thus, $\frac{rm}{s} \in (N:M)(S^{-1}N) \subseteq (S^{-1}N : S^{-1}M)S^{-1}N$, which is a contradiction. As $\frac{rm}{s} \in S^{-1}N$, there is $t \in S$, such that $trm \in N-(N:M)N$. Thus, $rm \in N$ because N is almost prime and $t \notin (N:M)$. Again, as $rm \in N-(N:M)N$, then, either $r \in (N:M) \subseteq (S^{-1}N : S^{-1}M)$ or $m \in N$ (and so $\frac{m}{s} \in S^{-1}N$). Therefore, $S^{-1}N$ is almost prime in $S^{-1}M$.

While in the prime submodules case, N is a prime submodule of M if and only if N/K is so in M/K, for any submodule $K \subseteq N$, the converse part may not be true in the case of almost prime submodules. For example, for any non almost prime submodule N of an R-module M, we verify that $0 = N/N$ is a weakly prime (and so almost prime) submodule of M/N. For another non-trivial example, we consider the ring $R = K[x,y]$, where K is a field and ideals $P = (x, y^2)$, $I = (x,y)^2$. Then, P/I is an almost prime submodule of an R-module R/I, while P is not so in R, see [10]. In the following theorem, we give other characterizations of almost prime submodules.

Theorem 2.5 Let M be an R-module and N be a proper submodule of M. The following are equivalent:

(1) N is an almost prime submodule.

(2) For $r \in R-(N:M)$, $(N:(r)) = N \cup ((N:M)N:(r))$.

(3) For $r \in R-(N:M)$, $(N:(r)) = N$ or $(N:(r)) = ((N:M)N:(r))$.

Proof (1) $\Rightarrow$ (2) Suppose that N is almost prime such that $r \notin (N:M)$. Let $m \in (N:(r))$ so that $rm \in N$. If $rm \notin (N:M)N$, then, N almost prime implies that $m \in N$. Suppose that $rm \in (N:M)N$. Then, $m \in ((N:M)N:(r))$ and so, $(N:(r)) \subseteq N \cup ((N:M)N:(r))$. The other containment holds for any submodule N.

(2) $\Rightarrow$ (3) It is well known that if a submodule is the union of two submodules, then it is equal to one of them.

(3) $\Rightarrow$ (1) Let $r \in R$ and $m \in M$ such that $rm \in N-(N:M)N$ and suppose $r \notin (N:M)$. By assumption, either $(N:(r)) = N$ or $(N:(r)) = ((N:M)N:(r))$. As $rm \notin (N:M)N$, then $m \notin ((N:M)N:(r))$ and so $m \in N$ as required.

3 Almost Prime Submodules of Multiplication Modules

In the following Theorem, we give a characterization of almost prime submodules in multiplication modules.

Theorem 3.1 Let M be a multiplication R-module and N be a proper submodule of M. Then, N is almost prime in M if and only if, for any ideal A of R and submodule K of M with $AK \subseteq N-(N:M)N$, we have $A \subseteq (N:M)$ or $K \subseteq N$.

Proof Assume N is almost prime. Suppose that there is an ideal A of R and a submodule K of M such that $AK \subseteq N$ but $A \nsubseteq (N:M)$ and $K \nsubseteq N$. We prove that $AK \subseteq (N:M)N$. Let $a \in A-(N:M)$. Then, $aK \subseteq N$ implies that $K \subseteq (N:(a))$. As N is almost prime and $K \nsubseteq N$, then by Theorem 2.5, we get $K \subseteq ((N:M)N:(a))$ and so $aK \subseteq (N:M)N$. Next, suppose $a \in A \cap (N:M)$ and let $b \in (K:M)$. If $b \in (N:M)$, then, $ab \in (N:M)^2 \subseteq ((N:M)N:M)$ and so $abM \subseteq (N:M)N$. If $b \in (K:M)-(N:M)$, then $bM \subseteq K$ implies that $bAM \subseteq AK \subseteq N$. Therefore, $AM \subseteq (N:(b))$ with $AM \nsubseteq N$ (as $A \nsubseteq (N:M)$). Again, by Theorem 2.5, we have $AM \subseteq ((N:M)N:(b))$ and so $bAM \subseteq (N:M)N$. As b is arbitrary in $(K:M)$, then, $aK = a(K:M)M \subseteq (N:M)N$, as required. In the converse part, the condition that M is a multiplication R-module is not required. Suppose that $rm \in N-(N:M)N$ for $r \in R$ and $m \in M$. Then, $(r)(m) \subseteq N-(N:M)N$ and so either $(r) \subseteq (N:M)$ or $(m) \subseteq N$. Therefore, $r \in (N:M)$ or $m \in N$, and N is almost prime.

Recall that if N is a submodule of an R-module M, then the radical of N (denoted by M-rad$N$) is defined as the intersection of all prime submodules of M containing N. It is well known that, if M is a multiplication R-module, then $M\text{-rad}N = \sqrt{(N:M)}M$, where $\sqrt{(N:M)}$ denotes the radical of the ideal $(N:M)$ in R.

Theorem 3.2 Let M be a multiplication R-module. If N is a submodule of M, then, $N \subseteq M\text{-rad}((N:M)N)$. Moreover, if N is a prime submodule of M, then, $N = M\text{-rad}((N:M)N)$.

Proof As M is multiplication R-module, then,

$$M\text{-rad}((N:M)N) = \sqrt{((N:M)N:M)}M.$$

As $(N:M)^2 \subseteq ((N:M)N:M)$, then $(N:M) \subseteq \sqrt{((N:M)N:M)}$ and so

$$N = (N:M)M \subseteq \sqrt{((N:M)N:M)}M = M\text{-rad}((N:M)N).$$

Moreover, suppose N is prime in M. If $r \in \sqrt{((N:M)N:M)}$, then, $r^n \in ((N:M)N:M) \subseteq (N:M)$ for some integer n. As $(N:M)$ is prime in R, then $r \in (N:M)$ and so $\sqrt{((N:M)N:M)} \subseteq (N:M)$. Therefore, $\sqrt{((N:M)N:M)}M \subseteq (N:M)M = N$ and the required equality holds.

Lemma 3.3 Let M be a multiplication R-module. If N is an almost prime submodule of M, then, $\sqrt{((N:M)N:M)}N = (N:M)N$.

Proof We first note that $(N:M)^2 \subseteq ((N:M)N:M)$. Let $a \in \sqrt{((N:M)N:M)}$. If $a \in (N:M)$, then, $a(N:M) \subseteq (N:M)^2 \subseteq ((N:M)N:M)$ and so, $aN = a(N:M)M \subseteq ((N:M)N:M)M = (N:M)N$. Suppose $a \notin (N:M)$, then by Theorem 2.5, we have either $(N:(a)) = N$ or $(N:(a)) = ((N:M)N:(a))$. If $(N:(a)) = ((N:M)N:(a))$, then $aN \subseteq a(N:(a)) = a((N:M)N:(a)) \subseteq (N:M)N$. Suppose $(N:(a)) = N$ and let n be the smallest positive integer such that $a^n \in ((N:M)N:M)$. If $n = 1$, then $a \in (N:M)$, a contradiction. So, we assume $n \geq 2$. Then, $a^nM \subseteq (N:M)N \subseteq N$ with $a^kM \nsubseteq (N:M)N$ for every $k \leq n-1$. It follows that $a^{n-1}M \subseteq (N:(a)) \subseteq N - (N:M)N$. If $n = 2$, we also get a contradiction. If $n \geq 3$, then $a(a^{n-2}M) \subseteq N - (N:M)N$ and so either $a \in (N:M)$ or $a^{n-2}M \subseteq N$. Continuing this process, we conclude that $a \in (N:M)$, which is a contradiction. Therefore, $\sqrt{((N:M)N:M)}N \subseteq (N:M)N$. The other containment is always true.

In the following Theorem, we give a characterization of almost prime submodules in a kind of cancellation modules. We first need the following Lemma.

Lemma 3.4 Let N be a submodule of a finitely generated faithful multiplication (and so cancellation) R-module M. Then, we have $(IN:M) = I(N:M)$ for every ideal I of R.

Proof As M is multiplication R-module, then, $I(N:M)M = IN = (IN:M)M$. The result follows because M is a cancellation module.

Theorem 3.5 Let M be a finitely generated faithful multiplication R-module and N be a proper submodule of M. The following are equivalent

(1) N is almost prime in M.

(2) $(N:M)$ is almost prime in R.

(3) $N = QM$ for some almost prime ideal Q of R.

Proof (1) $\Rightarrow$ (2) Suppose N is almost prime and let $a, b \in R$ such that $ab \in (N:M) - (N:M)^2$. Then, $abM \subseteq N - (N:M)N$. Indeed, if $abM \subseteq (N:M)N$, then by Lemma (3.4), $ab \in ((N:M)N:M) = (N:M)^2$, a contradiction. Now, N is almost prime implies that either $a \in (N:M)$ or $bM \subseteq N$ (and so $b \in (N:M)$). Hence, $(N:M)$ is almost prime in R.

(2) $\Rightarrow$ (1) In this direction, we need M to be just a multiplication R-module. Let $r \in R$ and $m \in M$, such that $rm \in N-(N:M)N$. Then, $r((m):M) \subseteq ((rm):M) \subseteq (N:M)$. Moreover, $r((m):M) \nsubseteq (N:M)^2$ because otherwise, if $r((m):M) \subseteq (N:M)^2 \subseteq ((N:M)N:M)$, then, $r(m) = r((m):M)M \subseteq (N:M)N$, a contradiction. As $(N:M)$ is almost prime in R, then, either $r \in (N:M)$ or $((m):M) \subseteq (N:M)$. In the second case, we obtain $(m) = ((m):M)M \subseteq (N:M)M = N$ and so N is almost prime in M.

(2) $\Leftrightarrow$ (3) We choose $Q = (N:M)$.

The next Theorem is an almost prime version of Proposition (13) in [3]. First, we need the following lemma from [2].

Lemma 3.6 Let N be a submodule of a faithful multiplication R-module M and I be a finitely generated faithful multiplication ideal of R. Then,

(1) $N = (IN : I)$.

(2) If $N \subseteq IM$, then $(JN : I) = J(N : I)$ for any ideal J of R.

Theorem 3.7 Let N be a submodule of a faithful multiplication R-module M and I be a finitely generated faithful multiplication ideal of R. Then, N is an almost prime submodule of IM if and only if $(N : I)$ is almost prime in M.

Proof Suppose that N is almost prime in IM. Let $r \in R$ and $m \in M$, such that $rm \in (N : I) - ((N : I) : M)(N : I)$. Then, $rIm \subseteq N - (N : IM)N$. In fact, if $rIm \subseteq (N : IM)N$, then by Lemma 3.6,

$$rm \in ((N : IM)N : I) = (N : IM)(N : I) = ((N : I) : M)(N : I),$$

a contradiction. As N is almost prime in IM, then, $Im \subseteq N$ or $r \in (N : IM)$. If $Im \subseteq N$, then, $m \in (N : I)$. Suppose $r \in (N : IM)$, so that $rIM \subseteq N$. Then again by Lemma 3.6, $rM = r(IM : I) \subseteq (rIM : I) \subseteq (N : I)$, and so, $r \in ((N : I) : M)$. Therefore, $(N : I)$ is almost prime in M. Conversely, suppose that $(N : I)$ is almost prime in M. Let A be an ideal of R and K be a submodule of IM such that $AK \subseteq N - (N : IM)N$. Then, $A(K : I) \subseteq (AK : I) \subseteq (N : I)$. Moreover, if $A(K : I) \subseteq ((N : I) : M)(N : I) = (N : IM)(N : I)$, then,

$$AK = A(IK : I) = A(K : I)I \subseteq (N : IM)(N : I)I = (N : IM)N,$$

a contradiction. As $(N : I)$ is almost prime in M, then either $A \subseteq ((N : I) : M) = (N : IM)$ or $(K : I) \subseteq (N : I)$, which implies that $K = (K : I)I \subseteq (N : I)N = N$. Hence, N is almost prime in IM.

In the following theorem, we give a new characterization of weakly prime submodules of finitely generated faithful multiplication modules using the definition of multiplication of submodules defined in the paragraph preceding Theorem 2.3.

Theorem 3.8 Let M be a finitely generated faithful multiplication R-module and P be a proper submodule of M. Then, P is weakly prime in M if and only if whenever N and K are submodules of M such that $0 \neq NK \subseteq P$, then, either $N \subseteq P$ or $K \subseteq P$.

Proof Suppose that P is weakly prime. We have $N = (N : M)M$ and $K = (K : M)M$, and so $NK = (N : M)(K : M)M$. Suppose $0 \neq NK \subseteq P$, but $N \nsubseteq P$ and $K \nsubseteq P$. Then, $(N : M) \nsubseteq (P : M)$ and $(K : M) \nsubseteq (P : M)$. As $(P : M)$ is weakly prime by Proposition (13) in [3], then, either $(N : M)(K : M) \nsubseteq (P : M)$ or $(N : M)(K : M) = 0$. In the first case, we have $NK = (N : M)(K : M)M \nsubseteq (P : M)M = P$, a contradiction. If $(N : M)(K : M) = 0$, then, $NK = 0M = 0$ and also we get a contradiction. Therefore, either $N \subseteq P$ or $K \subseteq P$. Conversely, to prove that P is weakly prime in M, it is enough by Proposition (13) in [3] to prove that $(P : M)$ is weakly prime in R. Let $r_1, r_2 \in R$, such that $0 \neq r_1r_2 \in (P : M)$, but $r_1 \notin (P : M)$ and $r_2 \notin (P : M)$. Let $N = (r_1)M$ and $K = (r_2)M$. Then, $0 \neq NK = (r_1)(r_2)M \subseteq P$. Indeed, if $NK = (r_1)(r_2)M = 0$, then, $(r_1r_2) \subseteq \mathrm{Ann}(M) = 0$, which is a contradiction. By assumption, either $(r_1)M = N \subseteq P$ or $(r_2)M = K \subseteq P$ and so, either $r_1 \in (P : M)$ or $r_2 \in (P : M)$, a contradiction. Therefore, $(P : M)$ is weakly prime in R and so P is weakly prime in M.

Corollary 3.9 Let P be a proper submodule of a finitely generated faithful multiplication R-module M. Then, P is weakly prime if and only if whenever $m_1, m_2 \in M$, $0 \neq m_1m_2 \in P$ implies $m_1 \in P$ or $m_2 \in P$.

The previous Theorem can be generalized to almost prime submodules.

Theorem 3.10 Let M be a finitely generated faithful multiplication R-module and P be a proper submodule of M. Then, P is almost prime in M if and only if whenever N and K are submodules of M such that $NK \subseteq P - (P : M)P$, then, either $N \subseteq P$ or $K \subseteq P$.

Proof By Theorem 3.5, P is almost prime in M if and only if $(P : M)$ is almost prime in R. As $(P : M)^2 = ((P : M)P : M)$ by Lemma 3.4, then, the proof is similar to that of Theorem 3.8.

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