Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
24 (2008), 323–332
www.emis.de/journals
ISSN 1786-0091

CERTAIN CLASS OF HARMONIC STARLIKE FUNCTIONS WITH SOME MISSING COEFFICIENTS

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Abstract. In this paper we have introduced a new class $J_H(\alpha, \beta, \gamma)$ of Harmonic Univalent functions in the unit disk $E = \{z; |z| < 1\}$ on the lines of [3] and [4], but with some missing coefficient. We have studied various properties such as coefficient estimates, extreme points, convolution and their related results.

1. Introduction

The class of functions of the form,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic univalent and normalized in the unit disc E, is denoted by S. The class K of convex functions and class S^* of starlike functions are two widely investigated subclasses of S.

A continuous function $f = u + iv$ defined in a domain $D \subseteq C$ is harmonic in D if u and v are real Harmonic in D. In any simply connected sub domain of D we can write,

$$f = h + \overline{g} \tag{1.1}$$

where h and g are analytic, h is called the analytic and g the coanalytic part of f. In this paper we have introduced a new class $J_H(\alpha, \beta, \gamma)$ of functions of the form (1.1) namely $f = h + \overline{g}$ that are Harmonic Univalent and sense preserving

2000 *Mathematics Subject Classification.* 30C45, 30C50.
Key words and phrases. Univalent functions, Harmonic functions, Starlike functions, and convolution.

in the unit disk E with $f(0) = f'(0) - 1 = 0$, where h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} a_{n+1} z^{n+1} \text{ and } g(z) = \sum_{n=1}^{\infty} b_{n+1} z^{n+1} \tag{1.2}$$

$\overline{J}_H$ is the subclass of J.

For $0 \leq \alpha < 1$, $\overline{J}_H$ denotes the subclass of J consisting of harmonic starlike functions of order α satisfying,

$$\frac{\partial}{\partial\theta}(\arg f(re^{i\theta})) \geq \alpha; \quad |z| = r < 1.$$

Clunie and Sheil - Small [3] and Jahangiri [4] studied Harmonic starlike functions of order α and Rosey et. al. [6] considered the Goodman-Ronning-Type harmonic univalent functions which satisfies the condition

$$\text{Re}\left\{(1 + e^{i\alpha})\frac{zf'}{f} - e^{i\alpha}\right\} \geq 0.$$

Definition. A function $f \in J_H(\alpha, \beta, \gamma)$ if it satisfies the condition

$$\text{Re}\left\{(1 + e^{i\alpha})\frac{zf'}{f} - \gamma e^{i\alpha}\right\} \geq \beta \tag{1.3}$$

$0 \leq \alpha < 1$, $0 \leq \beta < 1$, $\frac{1}{2} < \gamma \leq 1$ where

$$z' = \frac{\partial}{\partial\theta}(z = re^{i\theta}); \quad f'(z) = \frac{\partial}{\partial\theta} f(re^{i\theta})$$

α, β, γ and θ are real.

Let $\overline{J}_H$ denote a subclass of $J(\alpha, \beta, \gamma)$ consisting of functions $f = h + \overline{g}$ such that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} a_{n+1} z^{n+1} \text{ and } g(z) = \sum_{n=1}^{\infty} b_{n+1} z^{n+1}, \quad a_{n+1} \geq 0, b_{n+1} \geq 0 \tag{1.4}$$

2. Coefficient Estimates

Theorem 1. *Let $f = h + \overline{g}$, where h and g are given by (1.4). Furthermore let*

$$\sum_{n=2}^{\infty} \left\{ \frac{(2n - \beta - \gamma + 2)}{(2 - \beta - \gamma)} |a_{n+1}| + \frac{(2n + \beta + \gamma + 2)}{(2 - \beta - \gamma)} |b_{n+1}| \right\} \leq 2 \tag{2.1}$$

where $a_1 = 1$, $0 \leq \beta < 1$ and $\frac{1}{2} < \gamma \leq 1$. Then f is harmonic univalent in unit disc E and $f \in \overline{J}_H(\alpha, \beta, \gamma)$.

Proof. We first observe that f is locally univalent and orientation preserving in unit disc E. This is because

$$\begin{aligned}
|h'(z)| &\geq 1-\sum_{n=2}^{\infty}(n+1)|a_{n+1}|r^n > 1-\sum_{n=2}^{\infty}(n+1)|a_{n+1}| \\
&\geq 1-\sum_{n=2}^{\infty}\frac{(2n-\beta-\gamma+2)}{(2-\beta-\gamma)}|a_{n+1}| \geq \sum_{n=2}^{\infty}\frac{(2n+\beta+\gamma+2)}{(2-\beta-\gamma)}|b_{n+1}| \\
&\geq \sum_{n=1}^{\infty}(n+1)|b_{n+1}| \geq \sum_{n=1}^{\infty}(n+1)|b_{n+1}|r^n \geq g'(z).
\end{aligned}$$

In order to show that f is univalent in E we show that $f(z_1) \neq f(z_2)$ whenever $z_1 \neq z_2$. Since E is simply connected and convex we have $z(\lambda) = (1-\lambda)z_1+\lambda z_2 \in E$ if $0 \leq \lambda \leq 1$ and if $z_1, z_2 \in E$ so that $z_1 \neq z_2$. Then we write,

$$f(z_2)-f(z_1) = \int_0^1 [(z_2-z_1)h'(z(t)) + \overline{(z_2-z_1)g'(z(t))}]dt.$$

Dividing by $z_2 - z_1 \neq 0$ and taking the real part we have,

$$\begin{aligned}
\operatorname{Re}\left\{\frac{f(z_2)-f(z_1)}{z_2-z_1}\right\} &= \int_0^1 \operatorname{Re}\left[h'(z(t)) + \frac{\overline{(z_2-z_1)}}{(z_2-z_1)}\overline{g'(z(t))}\right]dt \\
&> \int_0^1 \operatorname{Re}[h'(z(t)) - |g'(z(t))|]dt
\end{aligned} \tag{2.2}$$

on the other hand,

$$\begin{aligned}
\operatorname{Re}(h'(z)-|g'(z)|) &\geq \operatorname{Re} h'(z) - \sum_{n=1}^{\infty}(n+1)|b_{n+1}| \\
&\geq 1-\sum_{n=2}^{\infty}(n+1)|a_{n+1}| - \sum_{n=1}^{\infty}(n+1)|b_{n+1}| \\
&\geq 1-\sum_{n=2}^{\infty}\frac{(2n-\beta-\gamma+2)}{(2-\beta-\gamma)}|a_{n+1}| \\
&\qquad -\sum_{n=1}^{\infty}\frac{(2n+\beta+\gamma+2)}{(2-\beta-\gamma)}|b_{n+1}| \\
&\geq 0
\end{aligned}$$

using (2.1). This along with inequality (2.2) leads to the univalence of f. According to the condition (1.2), it suffices to show that (2.1) holds if

$$\operatorname{Re}\left\{\frac{(1+e^{i\alpha})(zh'(z)-z\overline{g'(z)})-\gamma e^{i\alpha}(h(z)+\overline{g(z)})}{h(z)+\overline{g(z)}}\right\} = \operatorname{Re}\frac{A(z)}{B(z)} \geq \beta$$

where $z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, $0 \leq r < 1$, $\frac{1}{2} < \gamma \leq 1$.

Let $A(z)=(1+e^{i\alpha})(zh'(z)-z\overline{g'(z)})-\gamma e^{i\alpha}(h(z)+\overline{g(z)})$ and $B(z)=h(z)+\overline{g(z)}$. Since $Re(w)\geq\beta$ if and only if $|\gamma-\beta+w|\geq|\gamma+\beta-w|$. It is enough to show that

$$|A(z)+(1-\beta)B(z)|-|A(z)-(1+\beta)B(z)|\geq 0. \tag{2.3}$$

Substitute for $A(z)$ and $B(z)$ in (2.3) to yield

$$\begin{aligned}
&|(1-\beta)h(z)+(1+e^{i\alpha})zh'(z)-\gamma e^{i\alpha}h(z)\\
&\quad+\overline{(1-\beta)g(z)-(1+e^{i\alpha})zg'(z)-\gamma e^{i\alpha}g(z)}|\\
&\quad-|(1+\beta)h(z)-(1+e^{i\alpha})zh'(z)+\gamma e^{i\alpha}h(z)\\
&\quad+\overline{(1+\beta)g(z)+(1+e^{i\alpha})zg'(z)+\gamma e^{i\alpha}g(z)}|\\
&=|(2-\beta)z+ze^{i\alpha}(1-\gamma)-\sum_{n=2}^{\infty}[(2+n-\beta)+e^{i\alpha}(n+1-\gamma)]a_{n+1}z^{n+1}\\
&\quad-\sum_{n=1}^{\infty}\overline{[(n+\beta)+e^{i\alpha}(1+n+\gamma)]b_{n+1}z^{n+1}}|\\
&\quad-|\beta z+ze^{i\alpha}(1-\gamma)+\sum_{n=2}^{\infty}[(n-\beta)+e^{i\alpha}(1+n-\gamma)]a_{n+1}z^{n+1}\\
&\quad+\sum_{n=1}^{\infty}\overline{[(2+\beta+n)+e^{i\alpha}(1+n+\gamma)]b_{n+1}z^{n+1}}|\\
&\geq(3-\beta-\gamma)|z|-\sum_{n=2}^{\infty}(3+2n-\beta-\gamma)|a_{n+1}||z|^{n+1}\\
&\quad-\sum_{n=1}^{\infty}(2n+\beta+\gamma+1)|b_{n+1}||z|^{n+1}\\
&\quad-(\beta+\gamma-1)|z|-\sum_{n=2}^{\infty}(2n-\beta-\gamma+1)|a_{n+1}||z|^{n+1}\\
&\quad-\sum_{n=1}^{\infty}(3+2n+\beta+\gamma)|b_{n+1}||z|^{n+1}\\
&\geq 2(2-\beta-\gamma)|z|\left\{1-\sum_{n=2}^{\infty}\frac{(2n-\beta-\gamma+2)}{(2-\beta-\gamma)}|a_{n+1}||z|^{n}\right.\\
&\quad\left.-\sum_{n=1}^{\infty}\frac{(2n+\beta+\gamma+2)}{(2-\beta-\gamma)}|b_{n+1}||z|^{n}\right\}\\
&\geq 2(2-\beta-\gamma)|z|\left\{1-\left[\sum_{n=2}^{\infty}\frac{(2n-\beta-\gamma+2)}{(2-\beta-\gamma)}|a_{n+1}|\right.\right.
\end{aligned}$$

$$+\sum_{n=1}^{\infty}\frac{(2n+\beta+\gamma+2)}{(2-\beta-\gamma)}|b_{n+1}|\Bigg]\Bigg\}\geq 0.$$

By (2.1), the functions

$$f(z)=z+\sum_{n=2}^{\infty}\frac{2-\beta-\gamma}{2n-\beta-\gamma+2}x_{n+1}z^{n+1}+\sum_{n=1}^{\infty}\frac{2-\beta-\gamma}{2n+\beta+\gamma+2}\overline{y}_{n+1}\overline{z}^{n+1} \tag{2.4}$$

where

$$\sum_{n=2}^{\infty}|x_{n+1}|+\sum_{n=1}^{\infty}|y_{n+1}|=1$$

shows that the coefficient bound given by (2.1) is sharp. □

The function of the form (2.4) are in $\overline{J}_H(\alpha,\beta,\gamma)$ because

$$\sum_{n=2}^{\infty}\left\{\frac{(2n-\beta-\gamma+2)}{(2-\beta-\gamma)}|a_{n+1}|+\frac{(2n+\beta+\gamma+2)}{(2-\beta-\gamma)}|b_{n+1}|\right\}$$
$$=1+\sum_{n=2}^{\infty}|x_{n+1}|+\sum_{n=1}^{\infty}|y_{n+1}|=2$$

where $a_1=1$ and some coefficients are missing. The restriction placed in Theorem (1) on the module of the coefficients of f, enables us to conclude for arbitrary rotation of the coefficients of f that the resulting function would still be harmonic and univalent in $\overline{J}_H(\alpha,\beta,\gamma)$. The following theorem establishes that such coefficient bounds cannot be improved.

Theorem 2. *Let $f=h+\overline{g}$, be so that h and g are*

$$h(z)=z-\sum_{n=2}^{\infty}a_{n+1}z^{n+1};\quad g(z)=\sum_{n=1}^{\infty}b_{n+1}z^{n+1} \tag{2.5}$$

Then $f(z)\in\overline{J}_H(\alpha,\beta,\gamma)$ if and only if

$$\sum_{n=2}^{\infty}\left\{\frac{(2n-\beta-\gamma+2)}{(2-\beta-\gamma)}|a_{n+1}|+\frac{(2n+\beta+\gamma+2)}{(2-\beta-\gamma)}|b_{n+1}|\right\}\leq 2 \tag{2.6}$$

where $a_1=1, 0\leq\beta<1, \frac{1}{2}<\gamma\leq 1$ and some coefficients are missing.

Proof. The "if" part follows from theorem [1] upon noting that if the analytic part h and co-analytic part g of $f\in\overline{J}_H$ are of the form (2.5) then $f\in\overline{J}_H$.

For the "only if" part, we show that $f(z)\notin\overline{J}_H$ if the condition (2.6) does not hold. Note that a necessary and sufficient condition for $f=h+\overline{g}$ given by (2.5) to be in $\overline{J}_H$ is that

$$\operatorname{Re}\left\{(1+e^{i\alpha})z\frac{f'(z)}{f(z)}-\gamma e^{i\alpha}\right\}\geq\beta.$$

This is equivalent to

$$\operatorname{Re}\left\{\frac{(1+e^{i\alpha})(zh'(z)-z\overline{g'(z)})-\gamma e^{i\alpha}(h(z)-\overline{g(z)})}{h(z)+\overline{g(z)}}-\beta\right\}$$

$$=\operatorname{Re}\left\{\frac{(2-\beta-\gamma)z-\sum\limits_{n=2}^{\infty}(2n-\beta-\gamma+2)|a_{n+1}|z^{n+1}}{z-\sum\limits_{n=2}^{\infty}|a_{n+1}|z^{n+1}+\sum\limits_{n=1}^{\infty}|b_{n+1}|\overline{z}^{\ n+1}}\right.$$

$$\left.-\frac{\sum\limits_{n=1}^{\infty}(2n+\beta+\gamma+2)|b_{n+1}|\overline{z}^{n+1}}{z-\sum\limits_{n=2}^{\infty}|a_{n+1}|z^{n+1}+\sum\limits_{n=1}^{\infty}|b_{n+1}|\overline{z}^{\ n+1}}\right\}.$$

The above condition must hold for all values of z, $|z|=r<1\geq 0$. Choosing the values of z along +ve real axis where $0\leq z=r<1$, we must have

(2.7)

$$\frac{(2-\beta-\gamma)-\sum\limits_{n=2}^{\infty}(2n-\beta-\gamma+2)|a_{n+1}|r^n-\sum\limits_{n=1}^{\infty}(2n+\beta+\gamma+2)|b_{n+1}|r^n}{1-\sum\limits_{n=2}^{\infty}|a_{n+1}|r^n+\sum\limits_{n=1}^{\infty}|b_{n+1}|r^n}$$

If the condition (2.6) does not hold then the numerator in (2.7) is negative for r sufficiently close to 1. Thus, there exists $z_0=r_0$ in $(0,1)$ for which the quotient in (2.7) is negative. This contradicts the required condition for $f\in\overline{J}_H$ and hence the required result. □

3. Extreme Points

We obtain the extreme points of the closed convex hulls of $\overline{J}_H$, denoted by $CLCH\overline{J}_H$.

Theorem 3. *$f(z)\in CLCH\overline{J}_H$ if and only if,*

$$f(z)=\sum_{n=2}^{\infty}(x_{n+1}h_{n+1}+y_{n+1}g_{n+1}) \tag{3.1}$$

where $h_1(z)=z$;

$$h_{n+1}(z)=z-\frac{(2-\beta-\gamma)}{(2n-\beta-\gamma+2)}z^{n+1};\quad n=2,3,4,\cdots$$

$$g_{n+1}(z)=z+\frac{(2-\beta-\gamma)}{(2n+\beta+\gamma+2)}z^{n+1};\quad n=1,2,3,\cdots$$

$$\sum_{n=2}^{\infty}(x_{n+1}+y_{n+1})=1;\quad x_{n+1}\geq 0 \textit{ and } y_{n+1}\geq 0.$$

In particular, the extreme points of $\overline{J}_H$, *are* $\{h_{n+1}\}$ *and* $\{g_{n+1}\}$.

Proof. For function f of the form (3.1) we have,

$$f(z) = \sum_{n=2}^{\infty} (x_{n+1}h_{n+1} + y_{n+1}g_{n+1})$$

$$f(z) = \sum_{n=2}^{\infty} (x_{n+1} + y_{n+1})z - \sum_{n=2}^{\infty} \frac{(2-\beta-\gamma)}{(2n-\beta-\gamma+2)} x_{n+1}z^{n+1} + \sum_{n=1}^{\infty} \frac{(2-\beta-\gamma)}{(2n+\beta+\gamma+2)} y_{n+1}\overline{z}^{n+1}$$

Then

$$\sum_{n=2}^{\infty} \frac{(2n-\gamma-\beta+2)}{(2-\beta-\gamma)} \left(\frac{(2-\beta-\gamma)}{2n-\gamma-\beta+2)} x_{n+1} \right) + \sum_{n=1}^{\infty} \frac{(2n+\beta+\gamma+2)}{(2-\beta-\gamma)} \left(\frac{(2-\beta-\gamma)}{2n+\beta+\gamma+2} y_{n+1} \right)$$

$$\sum_{n=2}^{\infty} x_{n+1} + \sum_{n=1}^{\infty} y_{n+1} = 1 - x_1 \leq 1$$

and so $f(z) \in CLCH\overline{J}_H$.

Conversely, suppose that $f(z) \in CLCH\overline{J}_H$. Set

$$x_{n+1} = \frac{(2n-\gamma-\beta+2)}{(2-\beta-\gamma)} |a_{n+1}|; \quad n = 2, 3, 4, \dots$$

and

$$y_{n+1} = \frac{(2n+\gamma+\beta+2)}{(2-\beta-\gamma)} |b_{n+1}|; \quad n = 1, 2, 3, 4, \dots$$

Then note that by theorem (2), $0 \leq x_{n+1} \leq 1, n = 2, 3, 4, \dots$ and $0 \leq y_{n+1} \leq 1, n = 1, 2, 3, \dots$.

Consequently, we obtain

$$f(z) = \sum_{n=2}^{\infty} (x_{n+1}h_{n+1} + y_{n+1}g_{n+1}).$$

Using Theorem 2 it is easily seen that $\overline{J}_H$ is convex and closed and so

$$CLCH\overline{J}_H = \overline{J}_H.$$

□

4. Covolution Result

For harmonic functions,

$$f(z) = z - \sum_{n=2}^{\infty} a_{n+1} z^{n+1} + \sum_{n=1}^{\infty} b_{n+1} \overline{z}^{n+1}$$

$$G(z) = z - \sum_{n=2}^{\infty} A_{n+1} z^{n+1} + \sum_{n=1}^{\infty} B_{n+1} \overline{z}^{n+1}$$

we define the convolution of f and G as,

$$\begin{aligned}(f * G)(z) &= f(z) * G(z) \\ &= z - \sum_{n=2}^{\infty} a_{n+1} A_{n+1} z^{n+1} + \sum_{n=1}^{\infty} b_{n+1} B_{n+1} \overline{z}^{n+1}\end{aligned} \tag{4.1}$$

Theorem 4. *For* $0 \leq \beta < 1$ *let* $f(z) \in \overline{J}_H(\alpha, \beta, \gamma)$ *and* $G(z) \in \overline{J}_H(\alpha, \beta, \gamma)$. *Then*

$$f(z) * G(z) \in \overline{J}_H(\alpha, \beta, \gamma).$$

Proof. Let

$$f(z) = z - \sum_{n=2}^{\infty} |a_{n+1}| z^{n+1} + \sum_{n=1}^{\infty} |b_{n+1}| \overline{z}^{n+1} \text{ be in } \overline{J}_H(\alpha, \beta, \gamma)$$

and

$$G(z) = z - \sum_{n=2}^{\infty} |A_{n+1}| z^{n+1} + \sum_{n=1}^{\infty} |B_{n+1}| \overline{z}^{n+1} \text{ be in } \overline{J}_H(\alpha, \beta, \gamma)$$

Obviously, the coefficients of f and G must satisfy condition similar to the inequality (2.6). So for the coefficients of $f * G$ we can write

$$\begin{aligned}&\sum_{n=2}^{\infty} \left[\frac{(2n - \beta - \gamma + 2)}{(2 - \beta - \gamma)} |a_{n+1} A_{n+1}| + \frac{(2n + \beta + \gamma + 2)}{(2 - \beta - \gamma)} |b_{n+1} B_{n+1}| \right] \\ &\qquad \leq \sum_{n=2}^{\infty} \left[\frac{(2n - \beta - \gamma + 2)}{(2 - \beta - \gamma)} |a_{n+1}| + \frac{(2n + \beta + \gamma + 2)}{(2 - \beta - \gamma)} |b_{n+1}| \right]\end{aligned}$$

The right side of this inequality is bounded by 2 because $f \in \overline{J}_H(\alpha, \beta, \gamma)$. By the same token, we then conclude that

$$f(z) * G(z) \in \overline{J}_H(\alpha, \beta, \gamma).$$

□

Finally, we show that $f \in \overline{J}_H(\alpha, \beta, \gamma)$, is closed under convex combination of its members.

Theorem 5. *The family* $\overline{J}_H(\alpha, \beta, \gamma)$ *is closed under convex combination.*

Proof. For $i = 1, 2, 3\ldots$ let $f_i \in \overline{J}_H(\alpha,\beta,\gamma)$ where f_i is given by,

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{i(n+1)}| z^{n+1} + \sum_{n=1}^{\infty} |b_{i(n+1)}| \overline{z}^{n+1}$$

Then by (2.6),

$$\sum_{n=2}^{\infty} \left[\frac{(2n-\beta-\gamma+2)}{(2-\beta-\gamma)} |a_{i(n+1)}| + \frac{(2n+\beta+\gamma+2)}{(2-\beta-\gamma)} |b_{i(n+1)}| \leq 2 \right]. \tag{4.2}$$

For $\sum\limits_{i=1}^{\infty} t_i = 1; 0 \leq t_i \leq 1$, the convex combination of f_i may be written as,

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left[\sum_{i=1}^{\infty} t_i |a_{i(n+1)}| \right] z^{n+1} + \sum_{n=1}^{\infty} \left[\sum_{i=1}^{\infty} t_i |b_{i(n+1)}| \right] \overline{z}^{n+1}.$$

Then by (4.2)

$$\sum_{n=2}^{\infty} \left[\frac{(2n-\beta-\gamma+2)}{(2-\beta-\gamma)} \sum_{i=1}^{\infty} t_i |a_{i(n+1)}| + \frac{(2n+\beta+\gamma+2)}{(2-\beta-\gamma)} \sum_{i=1}^{\infty} t_i |b_{i(n+1)}| \right]$$
$$\sum_{i=1}^{\infty} t_i \left[\sum_{n=2}^{\infty} \frac{(2n-\beta-\gamma+2)}{(2-\beta-\gamma)} |a_{i(n+1)}| + \frac{(2n+\beta+\gamma+2)}{(2-\beta-\gamma)} |b_{i(n+1)}| \right]$$
$$\leq 2 \sum_{i=1}^{\infty} t_i = 2.$$

This is the condition required by (2.6) and so,

$$\sum_{i=1}^{\infty} t_i f_i(z) \in \overline{J}_H(\alpha,\beta,\gamma).$$

□

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