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A fixed point approach to the Hyers-Ulam stability of a functional equation in various normed spaces

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Abstract

Using direct method, Kenary (*Acta Universitatis Apulensis*, to appear) proved the Hyers-Ulam stability of the following functional equation

$$f(mx + ny) = \frac{(m+n)f(x+y)}{2} + \frac{(m-n)f(x-y)}{2}$$

in non-Archimedean normed spaces and in random normed spaces, where m, n are different integers greater than 1. In this article, using fixed point method, we prove the Hyers-Ulam stability of the above functional equation in various normed spaces.

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1. Introduction

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?" If the problem accepts a solution, then we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940. In the following year, Hyers [2] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [3] proved a generalization of Hyers' theorem for additive mappings. Furthermore, in 1994, a generalization of the Rassias' theorem was obtained by Găvruta [4] by replacing the bound $\varepsilon (||x||^p + ||y||^p)$ by a general control function $\varphi(x, y)$.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. In 1983, the Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [5] for mappings $f: X \rightarrow Y$, where X is a normed space and Y is a Banach space. In 1984, Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group and, in 2002, Czerwik [7] proved the Hyers-Ulam stability of the quadratic functional equation.

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [8-12]).

Using fixed point method, we prove the Hyers-Ulam stability of the following functional equation

$$f(mx + ny) = \frac{(m+n)f(x+\gamma)}{2} + \frac{(m-n)f(x-\gamma)}{2} \tag{1}$$

in various spaces, which was introduced and investigated in [13].

2. Preliminaries

In this section, we give some definitions and lemmas for the main results in this article.

A *valuation* is a function $|\cdot|$ from a field \mathbb{K} into $[0, \infty)$ such that, for all $r, s \in \mathbb{K}$, the following conditions hold:

- (a) $|r| = 0$ if and only if $r = 0$;
- (b) $|rs| = |r||s|$;
- (c) $|r + s| \leq |r| + |s|$.

A field \mathbb{K} is called a *valued field* if \mathbb{K} carries a valuation. The usual absolute values of \mathbb{R} and \mathbb{C} are examples of valuations.

In 1897, Hensel [14] has introduced a normed space which does not have the Archimedean property.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r + s| \leq \max\{|r|, |s|\}$$

for all $r, s \in \mathbb{K}$ then the function $|\cdot|$ is called a *non-Archimedean valuation* and the field is called a *non-Archimedean field*. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Definition 2.1. Let X be a vector space over a field \mathbb{K} with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is called a *non-Archimedean norm* if the following conditions hold:

- (a) $\|x\| = 0$ if and only if $x = 0$ for all $x \in X$;
- (b) $\|rx\| = |r| \|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;
- (c) the strong triangle inequality holds:

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 2.2. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X .

(a) The sequence $\{x_n\}$ is called a *Cauchy sequence* if, for any $\varepsilon > 0$, there is a positive integer N such that $\|x_n - x_m\| \leq \varepsilon$ for all $n, m \geq N$.

(b) The sequence $\{x_n\}$ is said to be *convergent* if, for any $\varepsilon > 0$, there are a positive integer N and $x \in X$ such that $\|x_n - x\| \leq \varepsilon$ for all $n \geq N$. Then the point $x \in X$ is called the *limit* of the sequence $\{x_n\}$, which is denote by $\lim_{n \rightarrow \infty} x_n = x$.

(c) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

It is noted that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1\}$$

for all $m, n \geq 1$ with $n > m$.

In the sequel (in random stability section), we adopt the usual terminology, notions, and conventions of the theory of random normed spaces as in [15].

Throughout this article (in random stability section), let Γ^+ denote the set of all probability distribution functions $F : \mathbb{R} \cup [-\infty, +\infty] \rightarrow [0, 1]$ such that F is left-continuous and nondecreasing on \mathbb{R} and $F(0) = 0, F(+\infty) = 1$. It is clear that the set $D^+ = \{F \in \Gamma^+ : lF(-\infty) = 1\}$, where $l^-f(x) = \lim_{t \rightarrow x^-} f(t)$, is a subset of Γ^+ . The set Γ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. For any $a \geq 0$, the element $H_a(t)$ of D^+ is defined by

$$H_a(t) = \begin{cases} 0 & \text{if } t \leq a, \\ 1 & \text{if } t > a. \end{cases}$$

We can easily show that the maximal element in Γ^+ is the distribution function $H_0(t)$.

Definition 2.3. [15] A function $T : [0, 1]^2 \rightarrow [0, 1]$ is a *continuous triangular norm* (briefly, a t -norm) if T satisfies the following conditions:

- (a) T is commutative and associative;
- (b) T is continuous;
- (c) $T(x, 1) = x$ for all $x \in [0, 1]$;
- (d) $T(x, y) \leq T(z, w)$ whenever $x \leq z$ and $y \leq w$ for all $x, y, z, w \in [0, 1]$.

Three typical examples of continuous t -norms are as follows: $T(x, y) = xy$, $T(x, y) = \max\{a + b - 1, 0\}$, and $T(x, y) = \min(a, b)$.

Definition 2.4. [16] A *random normed space* (briefly, *RN-space*) is a triple (X, μ, T) , where X is a vector space, T is a continuous t -norm, and $\mu : X \rightarrow D^+$ is a mapping such that the following conditions hold:

- (a) $\mu_x(t) = H_0(t)$ for all $x \in X$ and $t > 0$ if and only if $x = 0$;
- (b) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0, x \in X$ and $t \geq 0$;
- (c) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Definition 2.5. Let (X, μ, T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ (write $x_n \rightarrow x$ as $n \rightarrow \infty$) if $\lim_{n \rightarrow \infty} \mu_{x_n - x}(t) = 1$ for all $t > 0$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* in X if $\lim_{n \rightarrow \infty} \mu_{x_n - x_m}(t) = 1$ for all $t > 0$.

(3) The RN-space (X, μ, T) is said to be *complete* if every Cauchy sequence in X is convergent.

Theorem 2.1. [15] If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \rightarrow x$, then $\lim_{n \rightarrow \infty} \mu_{x_n}(t) = \mu_x(t)$.

Definition 2.6. [17] Let X be a real vector space. A function $N : X \times \mathbb{R} \rightarrow [0, 1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

- (N1) $N(x, t) = 0$ for $t \leq 0$;
- (N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;

- (N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
- (N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
- (N5) $N(x, \cdot)$ is a non-decreasing function of \mathbb{R} and $\lim_{t \rightarrow \infty} N(x, t) = 1$;
- (N6) for $x \neq 0$, $N(x, \cdot)$ is continuous on \mathbb{R} .

The pair (X, N) is called a fuzzy normed vector space. The properties of fuzzy normed vector space are given in [18].

Example 2.1. Let $(X, \|\cdot\|)$ be a normed linear space and $\alpha, \beta > 0$. Then

$$N(x, t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta \|x\|} & t > 0, x \in X \\ 0 & t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X .

Definition 2.7. [17] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{t \rightarrow \infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ in X and we denote it by $N - \lim_{t \rightarrow \infty} x_n = x$.

Definition 2.8. [17] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

Example 2.2. Let $N : \mathbb{R} \times \mathbb{R} \rightarrow [0, 1]$ be a fuzzy norm on \mathbb{R} defined by

$$N(x, t) = \begin{cases} \frac{t}{t+|x|} & t > 0 \\ 0 & t \leq 0 \end{cases}.$$

Then (\mathbb{R}, N) is a fuzzy Banach space.

We say that a mapping $f : X \rightarrow Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \rightarrow Y$ is continuous at each $x \in X$, then $f : X \rightarrow Y$ is said to be continuous on X [19].

Throughout this article, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

Definition 2.9. Let X be a set. A function $d : X \times X \rightarrow [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:

- (a) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
- (b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 2.2. [20,21] Let (X, d) be a complete generalized metric space and $J : X \rightarrow X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all non-negative integers n or there exists a positive integer n_0 such that

- (a) $d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \geq n$;
- (b) the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;
- (c) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^n x, y) < \infty\}$;

$$(d) \ d(y, y^*) \leq \frac{1}{1-L}d(y, Jy) \text{ for all } y \in Y.$$

3. Non-Archimedean stability of the functional equation (1)

In this section, using the fixed point alternative approach, we prove the Hyers-Ulam stability of the functional equation (1) in non-Archimedean normed spaces.

Throughout this section, let X be a non-Archimedean normed space and Y a complete non-Archimedean normed space. Assume that $|m| \neq 1$.

Lemma 3.1. *Let X and Y be linear normed spaces and $f : X \rightarrow Y$ a mapping satisfying (1). Then f is an additive mapping.*

Proof. Letting $y = 0$ in (1), we obtain

$$f(mx) = mf(x)$$

for all $x \in X$. So one can show that

$$f(m^n x) = m^n f(x)$$

for all $x \in X$ and all $n \in \mathbb{N}$. \square

Theorem 3.1. *Let $\zeta : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$|m|\zeta(x, y) \leq L\zeta(mx, my)$$

for all $x, y \in X$. If $f : X \rightarrow Y$ is a mapping satisfying $f(0) = 0$ and the inequality

$$\left\| f(mx + ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2} \right\| \leq \zeta(x, y) \tag{2}$$

for all $x, y \in X$, then there is a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{L\zeta(x, 0)}{|m| - |m|L}. \tag{3}$$

Proof. Putting $y = 0$ and replacing x by $\frac{x}{m}$ in (2), we have

$$\left\| mf\left(\frac{x}{m}\right) - f(x) \right\| \leq \zeta\left(\frac{x}{m}, 0\right) \leq \frac{L}{|m|}\zeta(x, 0) \tag{4}$$

for all $x \in X$. Consider the set

$$S := \{g : X \rightarrow Y; g(0) = 0\}$$

and the generalized metric d in S defined by

$$d(f, g) = \inf \{ \mu \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq \mu\zeta(x, 0), \forall x \in X \},$$

where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [[22], Lemma 2.1]).

Now, we consider a linear mapping $J : S \rightarrow S$ such that

$$Jh(x) := mh\left(\frac{x}{m}\right)$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \varepsilon$. Then we have

$$\|g(x) - h(x)\| \leq \varepsilon\zeta(x, 0)$$

for all $x \in X$ and so

$$\|Jg(x) - Jh(x)\| = \left\| mg\left(\frac{x}{m}\right) - mh\left(\frac{x}{m}\right) \right\| \leq |m|\varepsilon\zeta\left(\frac{x}{m}, 0\right) \leq |m|\varepsilon\frac{L}{|m|}\zeta(x, 0)$$

for all $x \in X$. Thus $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$. It follows from (4) that

$$d(f, Jf) \leq \frac{L}{|m|}.$$

By Theorem 2.2, there exists a mapping $A : X \rightarrow Y$ satisfying the following:

(1) A is a fixed point of J , that is,

$$A\left(\frac{x}{m}\right) = \frac{1}{m}A(x) \tag{5}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}.$$

This implies that A is a unique mapping satisfying (5) such that there exists $\mu \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \leq \mu\zeta(x, 0)$$

for all $x \in X$.

(2) $d(J^n f, A) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$\lim_{n \rightarrow \infty} m^n f\left(\frac{x}{m^n}\right) = A(x)$$

for all $x \in X$.

(3) $d(f, A) \leq \frac{d(f, Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, A) \leq \frac{L}{|m| - |m|L}.$$

This implies that the inequality (3) holds. By (2), we have

$$\begin{aligned} & \left\| m^n f\left(\frac{mx + ny}{m^n}\right) - \frac{m^n(m+n)f\left(\frac{x+y}{m^n}\right)}{2} - \frac{m^n(m-n)f\left(\frac{x-y}{m^n}\right)}{2} \right\| \\ & \leq |m|^n \zeta\left(\frac{x}{m^n}, \frac{y}{m^n}\right) \leq |m|^n \cdot \frac{L^n}{|m|^n} \zeta(x, y) \end{aligned}$$

for all $x, y \in X$ and $n \geq 1$ and so

$$\left\| A(mx + ny) - \frac{(m+n)A(x+y)}{2} - \frac{(m-n)A(x-y)}{2} \right\| = 0$$

for all $x, y \in X$.

On the other hand

$$mA\left(\frac{x}{m}\right) - A(x) = \lim_{n \rightarrow \infty} m^{n+1}f\left(\frac{x}{m^{n+1}}\right) - \lim_{n \rightarrow \infty} m^n f\left(\frac{x}{m^n}\right) = 0.$$

Therefore, the mapping $A : X \rightarrow Y$ is additive. This completes the proof. \square

Corollary 3.1. *Let $\theta \geq 0$ and p be a real number with $0 < p < 1$. Let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ and the inequality*

$$\left\| f(mx + ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2} \right\| \leq \theta(|x|^p + |y|^p) \tag{6}$$

for all $x, y \in X$. Then, for all $x \in X$,

$$A(x) = \lim_{n \rightarrow \infty} m^n f\left(\frac{x}{m^n}\right)$$

exists and $A : X \rightarrow Y$ is a unique additive mapping such that

$$\|f(x) - A(x)\| \leq \frac{|m|\theta||x|^p}{|m|^{\rho+1} - |m|^2} \tag{7}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.1 if we take

$$\zeta(x, y) = \theta(|x|^p + |y|^p)$$

for all $x, y \in X$. In fact, if we choose $L = |m|^{1-p}$, then we get the desired result. \square

Theorem 3.2. Let $\zeta : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $L < 1$ with

$$\frac{\zeta(mx, my)}{|m|} \leq L\zeta(x, y)$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ and (2). Then there is a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{\zeta(x, 0)}{|m| - |m|L}$$

Proof. The proof is similar to the proof of Theorem 3.1. \square

Corollary 3.2. Let $\theta \geq 0$ and p be a real number with $p > 1$. Let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ and (6). Then, for all $x \in X$

$$A(x) = \lim_{n \rightarrow \infty} \frac{f(m^n x)}{m^n}$$

exists and $A : X \rightarrow Y$ is a unique additive mapping such that

$$\|f(x) - A(x)\| \leq \frac{\theta||x|^p}{|m| - |m|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 if we take

$$\zeta(x, y) = \theta(|x|^p + |y|^p)$$

for all $x, y \in X$. In fact, if we choose $L = |2m|^{p-1}$, then we get the desired result. \square

Example 3.1. Let Y be a complete non-Archimedean normed space. Let $f : Y \rightarrow Y$ be a mapping defined by

$$f(z) = \begin{cases} z, & z \in \{mx + ny : \|mx + ny\| < 1\} \cap \{x - y : \|x - y\| < 1\} \\ 0, & \text{otherwise} \end{cases}$$

Then one can easily show that $f : Y \rightarrow Y$ satisfies (3.5) for the case $p = 1$ and that there does not exist an additive mapping satisfying (3.6).

4. Random stability of the functional equation (1)

In this section, using the fixed point alternative approach, we prove the Hyers-Ulam stability of the functional equation (1) in random normed spaces.

Theorem 4.1. *Let X be a linear space, (Y, μ, T) a complete RN-space and Φ a mapping from X^2 to D^+ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that there exists $0 < \alpha < \frac{1}{m}$ such that*

$$\Phi_{mx,my} \left(\frac{t}{\alpha} \right) \leq \Phi_{x,y}(t) \tag{8}$$

for all $x, y \in X$ and $t > 0$. Let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ and

$$\mu_{f(mx+ny) - \frac{(m+n)f(x+y) - (m-n)f(x-y)}{2}}(t) \geq \Phi_{x,y}(t) \tag{9}$$

for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$

$$A(x) := \lim_{n \rightarrow \infty} m^n f \left(\frac{x}{m^n} \right)$$

exists and $A : X \rightarrow Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t) \geq \Phi_{x,0} \left(\frac{(1-m\alpha)t}{\alpha} \right) \tag{10}$$

for all $x \in X$ and $t > 0$.

Proof. Putting $y = 0$ in (9) and replacing x by $\frac{x}{m}$, we have

$$\mu_{mf(\frac{x}{m}) - f(x)}(t) \geq \Phi_{\frac{x}{m},0}(t) \tag{11}$$

for all $x \in X$ and $t > 0$. Consider the set

$$S^* := \{g : X \rightarrow Y; g(0) = 0\}$$

and the generalized metric d^* in S^* defined by

$$d^*(f, g) = \inf_{u \in (0, +\infty)} \{ \mu_{g(x)-h(x)}(ut) \geq \Phi_{x,0}(t), \forall x \in X, t > 0 \},$$

where $\inf \emptyset = +\infty$. It is easy to show that (S^*, d^*) is complete (see [[22], Lemma 2.1]).

Now, we consider a linear mapping $J : S^* \rightarrow S^*$ such that

$$Jh(x) := mh \left(\frac{x}{m} \right)$$

for all $x \in X$.

First, we prove that J is a strictly contractive mapping with the Lipschitz constant $m\alpha$. In fact, let $g, h \in S^*$ be such that $d^*(g, h) < \varepsilon$. Then we have

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \Phi_{x,0}(t)$$

for all $x \in X$ and $t > 0$ and so

$$\begin{aligned} \mu_{Jg(x)-Jh(x)}(m\alpha\varepsilon t) &= \mu_{mg(\frac{x}{m})-mh(\frac{x}{m})}(m\alpha\varepsilon t) = \mu_{g(\frac{x}{m})-h(\frac{x}{m})}(\alpha\varepsilon t) \\ &\geq \Phi_{\frac{x}{m},0}(\alpha t) \\ &\geq \Phi_{x,0}(t) \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d^*(g, h) < \varepsilon$ implies that $d^*(Jg, Jh) < m\alpha\varepsilon$. This means that

$$d^*(Jg, Jh) \leq m\alpha d(g, h)$$

for all $g, h \in S^*$. It follows from (11) that

$$d^*(f, Jf) \leq \alpha.$$

By Theorem 2.2, there exists a mapping $A : X \rightarrow Y$ satisfying the following:

(1) A is a fixed point of J , that is,

$$A\left(\frac{x}{m}\right) = \frac{1}{m}A(x) \tag{12}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$\Omega = \{h \in S^* : d^*(g, h) < \infty\}.$$

This implies that A is a unique mapping satisfying (12) such that there exists $u \in (0, \infty)$ satisfying

$$\mu_{f(x)-A(x)}(ut) \geq \Phi_{x,0}(t)$$

for all $x \in X$ and $t > 0$.

(2) $d^*(J^n f, A) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$\lim_{n \rightarrow \infty} m^n f\left(\frac{x}{m^n}\right) = A(x)$$

for all $x \in X$.

(3) $d^*(f, A) \leq \frac{d^*(f, Jf)}{1-m\alpha}$ with $f \in \Omega$, which implies the inequality

$$d^*(f, A) \leq \frac{\alpha}{1-m\alpha}$$

and so

$$\mu_{f(x)-A(x)}\left(\frac{\alpha t}{1-m\alpha}\right) \geq \Phi_{x,0}(t)$$

for all $x \in X$ and $t > 0$. This implies that the inequality (10) holds.

On the other hand

$$\mu_{m^n f\left(\frac{mx+ny}{m^n}\right) - \frac{m^n(m+n)f\left(\frac{x+y}{m^n}\right) - m^n(m-n)f\left(\frac{x-y}{m^n}\right)}{2}}(t) \geq \Phi_{\frac{x}{m^n}, \frac{y}{m^n}}\left(\frac{t}{m^n}\right)$$

for all $x, y \in X$, $t > 0$ and $n \geq 1$ and so, from (8), it follows that

$$\Phi_{\frac{x}{m^n}, \frac{y}{m^n}}\left(\frac{t}{m^n}\right) \geq \Phi_{x,y}\left(\frac{t}{m^n \alpha^n}\right).$$

Since

$$\lim_{n \rightarrow \infty} \Phi_{x,y}\left(\frac{t}{m^n \alpha^n}\right) = 1$$

for all $x, y \in X$ and $t > 0$, we have

$$\mu_{A(mx+ny) - \frac{(m+n)A(x+y)}{2} - \frac{(m-n)A(x-y)}{2}}(t) = 1$$

for all $x, y \in X$ and $t > 0$.

On the other hand

$$\begin{aligned} A(mx) - mA(x) &= \lim_{n \rightarrow \infty} m^n f\left(\frac{x}{m^{n-1}}\right) - m \lim_{n \rightarrow \infty} m^n f\left(\frac{x}{m^n}\right) \\ &= m \left[\lim_{n \rightarrow \infty} m^{n-1} f\left(\frac{x}{m^{n-1}}\right) - \lim_{n \rightarrow \infty} m^n f\left(\frac{x}{m^n}\right) \right] \\ &= 0. \end{aligned}$$

Thus the mapping $A : X \rightarrow Y$ is additive. This completes the proof. \square

Corollary 4.1. *Let X be a real normed space, $\theta \geq 0$ and let p be a real number with $p > 1$. Let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ and*

$$\mu_{f(mx+ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2}}(t) \geq \frac{t}{t + \theta (||x||^p + ||y||^p)} \tag{13}$$

for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$,

$$A(x) = \lim_{n \rightarrow \infty} m^n f\left(\frac{x}{m^n}\right)$$

exists and $A : X \rightarrow Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t) \geq \frac{m^p(1 - m^{1-p})t}{m^p(1 - m^{1-p})t + \theta ||x||^p} \tag{14}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 4.1 if we take

$$\Phi_{x,y}(t) = \frac{t}{t + \theta (||x||^p + ||y||^p)}$$

for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = m^{-p}$, then we get the desired result. \square

Theorem 4.2. *Let X be a linear space, (Y, μ, T) a complete RN-space and Φ a mapping from X^2 to D^+ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that for some $0 < \alpha < m$*

$$\Phi_{\frac{x}{m}, \frac{y}{m}}(t) \leq \Phi_{x,y}(\alpha t)$$

for all $x, y \in X$ and $t > 0$. Let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ and

$$\mu_{f(mx+ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2}}(t) \geq \Phi_{x,y}(t)$$

for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$,

$$A(x) := \lim_{n \rightarrow \infty} \frac{f(m^n x)}{m^n}$$

exists and $A : X \rightarrow Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t) \geq \Phi_{x,0}((m - \alpha)t) \tag{15}$$

for all $x \in X$ and $t > 0$.

Proof. The proof is similar to the proof of Theorem 4.1. \square

Corollary 4.2. Let X be a real normed space, $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ and

$$\mu_{f(mx+ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2}}(t) \geq \frac{t}{t + \theta (||x||^p + ||y||^p)}$$

for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$,

$$A(x) = \lim_{n \rightarrow \infty} \frac{f(m^n x)}{m^n}$$

exists and $A : X \rightarrow Y$ is a unique additive mapping such that

$$\mu_{f(x) - A(x)}(t) \geq \frac{(m - m^p)t}{(m - m^p)t + \theta ||x||^p}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 4.2 if we take

$$\Phi_{x,y}(t) = \frac{t}{t + \theta (||x||^p + ||y||^p)}$$

for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = m^p$, then we get the desired result. \square

Example 4.1. Let (Y, μ, T) be a normed complete RN-space. Let $f : Y \rightarrow Y$ be a mapping defined by

$$f(z) = \begin{cases} z, & z \in \{mx + ny : ||mx + ny|| < 1\} \cap \{x - y : ||x - y|| < 1\} \\ 0, & \text{otherwise} \end{cases}$$

Then one can easily show that $f : Y \rightarrow Y$ satisfies (4.6) for the case $p = 1$ and that there does not exist an additive mapping satisfying (4.7).

5. Fuzzy stability of the functional equation (1)

Throughout this section, using the fixed point alternative approach, we prove the Hyers-Ulam stability of the functional equation (1) in fuzzy normed spaces.

In the rest of the article, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

Theorem 5.1. Let $\phi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $L < 1$ with

$$\varphi\left(\frac{x}{m}, \frac{y}{m}\right) \leq \frac{L}{m} \varphi(x, y)$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ and

$$N\left(f(mx + ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2}, t\right) \geq \frac{t}{t + \varphi(x, y)} \tag{16}$$

for all $x, y \in X$ and all $t > 0$. Then the limit

$$A(x) := N - \lim_{n \rightarrow \infty} m^n f\left(\frac{x}{m^n}\right)$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(m - mL)t}{(m - mL)t + L\varphi(x, 0)}$$

for all $x, y \in X$ and all $t > 0$.

Proof. Putting $y = 0$ in (16) and replacing x by $\frac{x}{m}$, we have

$$N\left(mf\left(\frac{x}{m}\right) - f(x), t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, 0\right)}$$

for all $x \in X$ and $t > 0$. Consider the set

$$S^{**} := \{g : X \rightarrow Y, g(0) = 0\}$$

and the generalized metric d^{**} in S^{**} defined by

$$d^{**}(f, g) = \inf \left\{ \mu \in \mathbb{R}^+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}, \quad \forall x \in X, t > 0 \right\},$$

where $\inf \emptyset = +\infty$. It is easy to show that (S^{**}, d^{**}) is complete (see [[22], Lemma 2.1]).

Now, we consider a linear mapping $J : S^{**} \rightarrow S^{**}$ such that

$$Jg(x) := mg\left(\frac{x}{m}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 4.1. \square

Corollary 5.1. Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ and

$$N\left(f(mx + ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2}, t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{17}$$

for all $x, y \in X$ and all $t > 0$. Then

$$A(x) := N - \lim_{n \rightarrow \infty} m^n f\left(\frac{x}{m^n}\right)$$

exists for each $x \in X$ and defines an additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(m^{p+1} - m^2)t}{(m^{p+1} - m^2)t + m\theta\|x\|^p} \tag{18}$$

Proof. The proof follows from Theorem 5.1 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = m^{1-p}$ and we get the desired result. \square

Theorem 5.2. Let $\phi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $L < 1$ with

$$\varphi(mx, my) \leq mL\varphi(x, y)$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ and

$$N\left(f(mx + ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2}, t\right) \geq \frac{t}{t + \phi(x, y)}$$

for all $x, y \in X$ and all $t > 0$. Then the limit

$$R(x) := N - \lim_{n \rightarrow \infty} \frac{f(m^n x)}{m^n}$$

exists for each $x \in X$ and defines an additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(m - mL)t}{(m - mL)t + \varphi(x, 0)}$$

for all $x, y \in X$ and all $t > 0$.

Proof. The proof is similar to that of the proofs of Theorems 4.1 and 5.1. \square

Corollary 5.2. Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ and

$$N\left(f(mx + ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2}, t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}$$

for all $x, y \in X$ and all $t > 0$. Then the limit

$$A(x) := N - \lim_{n \rightarrow \infty} \frac{f(m^n x)}{m^n}$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(m - m^p)t}{(m - m^p)t + \theta\|x\|^p}.$$

Proof. The proof follows from Theorem 5.2 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y, z \in X$. Then we can choose $L = m^{p-1}$ and we get the desired result. \square

Example 5.1. Let (Y, N) be a normed fuzzy Banach space. Let $f : Y \rightarrow Y$ be a mapping defined by

$$f(z) = \begin{cases} z, & z \in \{mx + ny : \|mx + ny\| < 1\} \cap \{x - y : \|x - y\| < 1\} \\ 0, & \text{otherwise} \end{cases}.$$

Then one can easily show that $f : Y \rightarrow Y$ satisfies (5.2) for the case $p = 1$ and that there does not exist an additive mapping satisfying (5.3).

6. Conclusion

We linked here five different disciplines, namely, the random normed spaces, non-Archimedean normed spaces, fuzzy normed spaces, functional equations, and fixed point theory. We established the Hyers-Ulam stability of the functional equation (1) in various normed spaces by using fixed point method.

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Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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