SEPARABLE RANDOMIZATIONS OF MODELS

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Abstract. Every complete first order theory has a corresponding complete theory in continuous logic, called the randomization theory. It has two sorts, a sort for random elements of models of the first order theory, and a sort for events. In this paper we establish connections between properties of countable models of a first order theory and corresponding properties of separable models of the randomization theory. We show that the randomization theory has a prime model if and only if the first order theory has a prime model. And the randomization theory has the same number of separable homogeneous models as the first order theory has countable homogeneous models.

1. Introduction

This note is a follow-up to the paper [BK], which developed the randomization construction from [Ke] in the setting of continuous model theory (also called the theory of metric structures). A randomization of a first order structure $\mathcal{M}$ is a continuous prestructure $(\mathcal{K}, \mathcal{B})$ where $\mathcal{K}$ is a set of random elements of $\mathcal{M}$ and $\mathcal{B}$ is a set of events. For each first order theory T there is a corresponding continuous theory T^R whose models are exactly the completions of randomizations of models of T. In [BK], several results were obtained showing that a first order theory T has a property if and only if T^R has the corresponding property in continuous logic— T is ω-categorical iff T^R is separably categorical; T has a countable saturated model iff T^R has a separable saturated model; T is ω-stable iff T^R is ω-stable; T is stable iff T^R is stable.

In this paper we will obtain further results along this line, matching properties of countable models of T and separable models of T^R. We will show that T^R has a prime model if and only if T has a prime model, and that the number of separable homogeneous models of T^R is equal to the number of countable homogeneous models of T.

This paper is organized as follows. In Section 2 we review some notions we will need from the literature, including the key notion of a Borel randomization $(\mathcal{M}^{[0,1)}, \mathcal{L})$. Section 3 contains our results about prime models. In Section 4 we prepare the way for our main results about separable homogeneous models by studying strongly separable types and structures— those that are embeddable in Borel randomizations. Our results about separable homogeneous models are in Section 5.

I thank Uri Andrews, Itai Ben Yaacov, and Isaac Goldbring for valuable discussions related to this work.

Date: September 11, 2012.

2. Preliminaries

We refer to [BBHU] and [BU] for background in continuous model theory, and follow the notation of [BK]. We assume familiarity with the basic notions about continuous model theory as developed in [BBHU], including the notions of a theory, structure, pre-structure, completion, and model of a theory. In particular, the universe of a pre-structure is a pseudo-metric space, and the universe of a structure is a complete metric space. A prestructure is **separable** if its universe is separable as a topological space. Formulas have the quantifiers $\sup, \inf$, and take truth values in $[0,1]$.

We assume throughout that L is a finite or countable first order signature, and that T is a complete theory for L whose models have at least two elements. As in [BK], by a **countable model** of T we mean a model of T whose universe is either finite or of cardinality ω. A **tuple** is a finite sequence.

2.1. The theory T^R.

A randomization of a model $\mathcal{M}$ of T is a two-sorted continuous structure with a sort $\mathbf{K}$ whose elements are random elements of $\mathcal{M}$, and a sort $\mathbf{B}$ whose elements are events in an underlying probability space. The logic can express the probability that a first order formula holds for a tuple of random elements.

Formally, the **randomization signature** L^R is the two-sorted continuous signature with sorts $\mathbf{K}$ and $\mathbf{B}$, an n-ary function symbol $[\![\varphi(\cdot)]\!]$ of sort $\mathbf{K}^n \to \mathbf{B}$ for each first order formula φ of L with n free variables, a $[0,1]$-valued unary predicate symbol μ of sort $\mathbf{B}$ for probability, and the Boolean operations $\top, \bot, \sqcap, \sqcup, \neg$ of sort $\mathbf{B}$. The signature L^R also has distance predicates $d_{\mathbf{B}}$ of sort $\mathbf{B}$ and $d_{\mathbf{K}}$ of sort $\mathbf{K}$. In L^R, we use $\mathsf{B}, \mathsf{C}, \ldots$ for variables or parameters of sort $\mathbf{B}$, and $\mathsf{B} \doteq \mathsf{C}$ means $d_{\mathbf{B}}(\mathsf{B}, \mathsf{C}) = 0$.

A pre-structure for T^R will be a pair $\mathcal{P} = (\mathcal{K}, \mathcal{B})$ where $\mathcal{K}$ is the part of sort $\mathbf{K}$ and $\mathcal{B}$ is the part of sort $\mathbf{B}$. In this paper we will only need to consider pre-structures of a special kind— the Borel randomizations and their substructures. Borel randomizations are closely related to Boolean valued ultrapowers.

We let $\mathcal{L}$ be the family of Borel subsets of $[0,1)$, and let $([0,1), \mathcal{L}, \lambda)$ be the usual probability space where λ is the restriction of Lebesgue measure to $\mathcal{L}$. The phrase "almost all t" will mean "for all t in a set $\mathsf{B} \in \mathcal{L}$ of λ-measure one". Given a model $\mathcal{M}$ of T, we let $\mathcal{M}^{[0,1)}$ be the set of $\mathcal{L}$-measurable functions with countable range from $[0,1)$ into $\mathcal{M}$. Intuitively, an element of $\mathcal{M}^{[0,1)}$ is an experiment in which an element of $\mathcal{M}$ is chosen at random. The elements of $\mathcal{M}^{[0,1)}$ are called **random elements of** $\mathcal{M}$.

Definition 2.1. The **Borel randomization of** $\mathcal{M}$ is the pre-structure $(\mathcal{M}^{[0,1)}, \mathcal{L})$ for L^R whose universe of sort $\mathbf{K}$ is $\mathcal{M}^{[0,1)}$, whose universe of sort $\mathbf{B}$ is $\mathcal{L}$, whose measure μ is given by $\mu(\mathsf{B}) = \lambda(\mathsf{B})$ for each $\mathsf{B} \in \mathcal{L}$, and whose $[\![\psi(\cdot)]\!]$ functions are

$$[\![\psi(\vec{\mathbf{f}})]\!] = \{t \in [0,1) : \mathcal{M} \models \psi(\vec{\mathbf{f}}(t))\}.$$

(So $[\![\psi(\vec{\mathbf{f}})]\!] \in \mathcal{L}$ for each first order formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{f}}$ in $\mathcal{M}^{[0,1)}$). Its distance predicates are defined by

$$d_{\mathbf{B}}(\mathsf{B},\mathsf{C}) = \mu(\mathsf{B}\triangle\mathsf{C}), \quad d_{\mathbf{K}}(\mathbf{f},\mathbf{g}) = \mu([\![\mathbf{f} \neq \mathbf{g}]\!]),$$

where $\triangle$ is the symmetric difference operation.

Lemma 2.2. *$\mathcal{M}$ is countable if and only if the Borel randomization $(\mathcal{M}^{[0,1)},\mathcal{L})$ is separable.*

Proof. If $\mathcal{M}$ is countable, then Corollary 3.8 of [BK] shows that $(\mathcal{M}^{[0,1)},\mathcal{L})$ is separable. If $\mathcal{M}$ is uncountable, then the set C of constant functions from $[0,1)$ into M is an uncountable set of elements of $(\mathcal{M}^{[0,1)},\mathcal{L})$ such that the distance between any two elements of C is one, so $(\mathcal{M}^{[0,1)},\mathcal{L})$ is not separable. $\blacksquare_{2.2}$

A pre-structure $\mathcal{N} = (\mathcal{K},\mathcal{B})$ is **pre-complete** if the metric spaces obtained from $(\mathcal{K}, d_{\mathbf{K}})$ and $(\mathcal{B}, d_{\mathbf{B}})$ by identifying elements at distance zero are already complete. (In particular, every structure is pre-complete). The following basic facts are from [BK], Theorems 2.1 and 2.7, Proposition 2.2, and Theorem 2.9.

Fact 2.3. *There is a unique complete theory T^R for L^R, called the **randomization theory** of T, such that for each model $\mathcal{M}$ of T, $(\mathcal{M}^{[0,1)},\mathcal{L})$ is a pre-complete model of T^R.*

It follows that for each first order sentence φ, if $T \models \varphi$ then $T^R \models [\![\varphi]\!] \doteq \top$.

Fact 2.4. *Every pre-complete model $\mathcal{N} = (\mathcal{K},\mathcal{B})$ of T^R has perfect witnesses, i.e.,*

(i) *For each first order formula $\varphi(\vec{u}, v)$ and each $\vec{\mathbf{f}}$ in $\mathcal{K}^n$ there exists $\mathbf{g} \in \mathcal{K}$ such that*

$$[\![\varphi(\vec{\mathbf{f}},\mathbf{g})]\!] \doteq [\![((\exists v)\varphi)(\vec{\mathbf{f}})]\!];$$

(ii) *For each $\mathsf{B} \in \mathcal{B}$ there exist $\mathbf{f},\mathbf{g} \in \mathcal{K}$ such that $\mathsf{B} \doteq [\![\mathbf{f} = \mathbf{g}]\!]$.*

Remark 2.5. Let $\mathcal{N} = (\mathcal{K},\mathcal{B})$ be a pre-complete model of T^R and let $\mathbf{f},\mathbf{g} \in \mathcal{K}$ and $\mathsf{B} \in \mathcal{B}$. Then there is an element $\mathbf{h} \in \mathcal{K}$ that agrees with $\mathbf{f}$ on B and agrees with $\mathbf{g}$ on $\neg\mathsf{B}$, that is, $[\![\mathbf{h} = \mathbf{f}]\!] \sqcap \mathsf{B} \doteq \mathsf{B}$ and $[\![\mathbf{h} = \mathbf{g}]\!] \sqcap \neg\mathsf{B} \doteq \neg\mathsf{B}$.

Proof. This follows easily from Facts 2.3 and 2.4. $\blacksquare_{2.5}$

The next fact is a reformulation of Theorem 3.2 in [AK].

Fact 2.6. *Suppose $\mathcal{M}$ is a model of T, $\mathcal{N} = (\mathcal{K},\mathcal{B}) \prec (\mathcal{M}^{[0,1)},\mathcal{L})$, and $\mathcal{N}$ is pre-complete. Then $\mathcal{N}$ is isomorphic to a pre-structure $\mathcal{P} = (\mathcal{J},\mathcal{L}) \prec (\mathcal{M}^{[0,1)},\mathcal{L})$.*

Fact 2.7. *(Strong quantifier elimination) Every formula Φ in the continuous language L^R is T^R-equivalent to a formula with the same free variables and no quantifiers of sort **K** or **B**.*

We will use $\mathcal{M}, \mathcal{H}$ to denote first order structures with universes M and H, and use $\mathcal{N}, \mathcal{P}$ to denote continuous structures or pre-structures. In both the first order and continuous settings, $\equiv$ denotes elementary equivalence. We extend the notions of embedding and elementary embedding to pre-structures in the natural way. Given continuous pre-structures $\mathcal{P}, \mathcal{N}$, we write $h : \mathcal{P} \subseteq \mathcal{N}$ if h is an injection from $\mathcal{P}$ into $\mathcal{N}$ which preserves the truth values of atomic formulas, and $h : \mathcal{P} \prec \mathcal{N}$ (h is an **elementary embedding**) if h preserves the truth values of all formulas. We write $\mathcal{P} \subseteq \mathcal{N}$ if $h : \mathcal{P} \subseteq \mathcal{N}$ where h is the identity map, and $\mathcal{P} \prec \mathcal{N}$ if $h : \mathcal{P} \prec \mathcal{N}$ where h is the identity map. By Fact 2.7, if $\mathcal{P}, \mathcal{N}$ are pre-models of T^R and $\mathcal{P} \subseteq \mathcal{N}$, then $\mathcal{P} \prec \mathcal{N}$.

We write $h\colon \mathcal{P} \cong \mathcal{N}$, and say that $\mathcal{P}$ is **isomorphic to** $\mathcal{N}$, if $h : \mathcal{P} \prec \mathcal{N}$ and every element of $\mathcal{N}$ is a distance 0 from some element of $h(\mathcal{P})$. It is clear that if $\mathcal{P} \subseteq \mathcal{N}, \mathcal{P} \prec \mathcal{N}$, or $\mathcal{P} \cong \mathcal{N}$, then the corresponding relation also holds between the completions of $\mathcal{P}$ and $\mathcal{N}$. Note that every pre-structure that is isomorphic to a structure is pre-complete. Conversely, if $\mathcal{P}$ is pre-complete and $\mathcal{N}$ is its completion, then $h\colon \mathcal{P} \cong \mathcal{N}$ where h is the mapping that identifies elements at distance zero from each other. In particular, for each $\mathcal{M} \models T$, the Borel randomization $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is isomorphic to its completion.

2.2. **First order types.** For each natural number n, $S_n(T)$ is the space of complete first order n-types of T. Since L is countable, $S_n(T)$ is a compact Polish space. For each model $\mathcal{M}$ of T and n-tuple $\vec{a}$ in M, the type $tp(\vec{a})$ of $\vec{a}$ (in $\mathcal{M}$) is the unique type $p \in S_n(T)$ such that

$$p = \{\varphi : \mathcal{M} \models \varphi(\vec{a})\}.$$

We say that p is **realized** in $\mathcal{M}$ if $p = tp(\vec{a})$ for some n-tuple $\vec{a}$ in M.

When $\vec{a}$ is a (finite or infinite) sequence of elements of M, $(\mathcal{M}, \vec{a})$ denotes the structure $\mathcal{M}$ with an extra constant symbol for each term of $\vec{a}$ (so $tp(\vec{a}) = tp(\vec{b})$ if and only if $(\mathcal{M}, \vec{a}) \equiv (\mathcal{M}, \vec{b})$). $\mathcal{M}$ is said to be **countable homogeneous** if $\mathcal{M}$ is countable and for every n and every pair of n-tuples $\vec{a}, \vec{b}$ which realize the same type in $\mathcal{M}$, for every c in $\mathcal{M}$ there exists d in $\mathcal{M}$ such that $(\vec{a}, c)$ and $(\vec{b}, d)$ realize the same type in $\mathcal{M}$.

Fact 2.8. *(See Morley and Vaught [MV] and Keisler and Morley [KM])*

(i) *Every countable model of T has a countable homogeneous elementary extension.*
(ii) *Any two countable homogeneous models of T that realize the same types are isomorphic.*
(iii) *If $\mathcal{M}$ is countable homogeneous and $tp(\vec{a}) = tp(\vec{b})$ in $\mathcal{M}$, then $(\mathcal{M}, \vec{a}) \cong (\mathcal{M}, \vec{b})$.*

2.3. **Continuous types.** For each n-tuple $\vec{\mathbf{f}}$ of elements in a continuous pre-structure $\mathcal{N}$, the type $tp(\vec{\mathbf{f}})$ of $\vec{\mathbf{f}}$ in $\mathcal{N}$ is the function p from formulas to $[0, 1]$ such that for each formula $\Phi(\vec{x})$, we have $\Phi(\vec{x})^p = \Phi(\vec{\mathbf{f}})^{\mathcal{N}}$.

By quantifier elimination (Fact 2.7), the n-types in T^R of sort $\mathbf{B}$ do not depend on the theory T at all, and can be identified with the n-types in the continuous theory of atomless measure algebras $(\mathcal{B}, \top, \bot, \sqcap, \sqcup, \neg, \mu(\cdot))$. Formally, we have

Remark 2.9. Let $\mathcal{N}, \mathcal{P}$ be models of T^R and let $\vec{\mathsf{B}}, \vec{\mathsf{C}}$ be tuples of sort $\mathbf{B}$ in $\mathcal{N}$ and $\mathcal{P}$ respectively. Then $tp(\vec{\mathsf{B}}) = tp(\vec{\mathsf{C}})$ if and only if $\mu(\tau(\vec{\mathsf{B}}))^{\mathcal{N}} = \mu(\tau(\vec{\mathsf{C}}))^{\mathcal{P}}$ for every Boolean term τ.

By Fact 2.4 (ii), in a model of T^R we can always replace an element of sort $\mathbf{B}$ by a term $[\![\mathbf{f} = \mathbf{g}]\!]$. Thus every type in T^R of sort $\mathbf{B}$ can be obtained from a type in T^R of sort $\mathbf{K}$. The space of continuous n-types in T^R with variables of sort $\mathbf{K}$ will be denoted by $S_n(T^R)$. For each pre-model $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ of T^R and n-tuple $\vec{\mathbf{f}}$ in $\mathcal{K}$, the type $tp(\vec{\mathbf{f}})$ of $\vec{\mathbf{f}}$ is the unique element $p \in S_n(T^R)$ such that for each first order formula $\varphi(\vec{v})$,

$$(\mu[\![\varphi(\vec{\mathbf{f}})]\!])^{\mathcal{N}} = (\mu[\![\varphi(\vec{v})]\!])^{p}.$$

We say that a type $p \in S_n(T^R)$ is **realized** in a pre-model $\mathcal{N}$ if we have $p = tp(\vec{\mathbf{f}})$ for some n-tuple $\vec{\mathbf{f}}$ in $\mathcal{N}$. By the Compactness Theorem, every type $p \in S_n(T^R)$ is realized in some separable model of T^R.

A continuous pre-structure $\mathcal{N}$ is ω-**homogeneous** if for every pair of n-tuples $\vec{\mathbf{f}}, \vec{\mathbf{g}}$ in $\mathcal{N}$ which realize the same type in $\mathcal{N}$, and every $\mathbf{h}$ in $\mathcal{N}$, there exists $\mathbf{k}$ in $\mathcal{N}$ such that $(\vec{\mathbf{f}}, \mathbf{h})$ and $(\vec{\mathbf{g}}, \mathbf{k})$ realize the same type in $\mathcal{N}$. We say that $\mathcal{N}$ is **separable homogeneous** if $\mathcal{N}$ is separable and ω-homogeneous.

The next fact shows that the types of T^R can be identified with probability measures on the space of types of T. Let $\mathfrak{R}(S_n(T))$ be the space of Borel probability measures on $S_n(T)$. Since $S_n(T)$ is a Polish space, every element of $\mathfrak{R}(S_n(T))$ is regular.

Fact 2.10. *([BK], Corollary 2.10) For every $p \in S_n(T^R)$ there is a unique measure $\nu_p \in \mathfrak{R}(S_n(T))$ such that for each formula $\varphi(\vec{v})$ of L,*

$$\nu_p(\{q \in S_n(T) : \varphi(\vec{v}) \in q\}) = (\mu[\![\varphi(\vec{v})]\!])^{p}.$$

Moreover, for each measure $\nu \in \mathfrak{R}(S_n(T))$ there is a unique $p \in S_n(T^R)$ such that $\nu = \nu_p$.

We will sometimes use Fact 2.10 to build types of T^R.

Example 2.11. Let $p_0, p_1, \ldots$ be a finite or countable sequence of first-order types in $S_n(T)$ and let $\alpha_0, \alpha_1, \ldots$ be elements of $[0, 1]$ such that $\sum_i \alpha_i = 1$. Then there is a unique type $p \in S_n(T^R)$ such that $\nu_p(\{p_i\}) = \alpha_i$ for each i. We denote this type by

$$p = \sum_i \alpha_i p_i^{\star}.$$

Note that in this case the measure ν_p is purely atomic. In particular, for each first order type $q \in S_n(T)$, $q^{\star}$ is the type $p \in S_n(T^R)$ such that ν_p is the point mass at q, that is, $\nu_{q^{\star}}(\{q\}) = 1$.

Remark 2.12. (i) Suppose $\mathcal{N}$ is a model of T^R, and let $p = \sum_i \alpha_i p_i^{\star}$ be as in Example 2.11. If each type $p_i^{\star}$ is realized in $\mathcal{N}$, then p is realized in $\mathcal{N}$.

(ii) If $\mathcal{M} \models T$ and each type p_i is realized in $\mathcal{M}$, then the type $p = \sum_i \alpha_i p_i^\star$ is realized in $(\mathcal{M}^{[0,1)}, \mathcal{L})$. If $\vec{\mathbf{f}}$ is a tuple in $\mathcal{M}^{[0,1)}$, then $tp(\vec{\mathbf{f}}) = \sum_i \alpha_i p_i^\star$ where $\{\vec{a}_0, \vec{a}_1, \ldots\}$ is the range of $\vec{\mathbf{f}}$, and for each i, $p_i = tp(\vec{a}_i)$ and $\alpha_i = \lambda(\{t\colon \vec{\mathbf{f}}(t) = \vec{a}_i\})$.

(iii) In particular, if $\vec{\mathbf{f}}(t)$ has the constant value $\vec{a}$ for all $t \in [0,1)$, then $tp(\vec{\mathbf{f}}) = (tp(\vec{a}))^\star$.

Proof. (i) For each i, let $\vec{\mathbf{f}}_i$ realize the type $p_i^\star$ in $\mathcal{N}$. Using Remark 2.5 countably many times and taking a limit, we can obtain a family of pairwise disjoint events A_i in $\mathcal{N}$ such that $\mu(\mathsf{A}_i) = \alpha_i$ for each i, and a tuple $\vec{\mathbf{f}}$ in $\mathcal{N}$ such that for each i, $\vec{\mathbf{f}}$ agrees with $\vec{\mathbf{f}}_i$ on A_i. Then $\vec{\mathbf{f}}$ realizes $\sum_i \alpha_i p_i^\star$ in $\mathcal{N}$. (ii) and (iii) follow easily from the definitions involved. $\blacksquare_{2.12}$

3. Prime models

In this section we show that T has a prime model if and only if its randomization theory T^R has a prime model.

Let $\mathcal{N}$ be a first order or continuous structure with a countable signature and let U be the complete theory of $\mathcal{N}$. By definition, $\mathcal{N}$ is **prime** if $\mathcal{N}$ is elementarily embeddable in every model of U. We will call a pre-structure **prime** if its completion is prime. Following Theorem 13.4 in [BBHU], we will say that an n-type $p \in S_n(U)$ is **principal** if p is realized in every model of U.

We use the following results from the literature.

Fact 3.1. *(Vaught [Va].)*

(i) *A model $\mathcal{M}$ of T is prime if and only if $\mathcal{M}$ is countable and every type which is realized in $\mathcal{M}$ is principal.*
(ii) *Any two prime models of T are isomorphic.*
(iii) *T has a prime model if and only if every formula $\varphi(\vec{v})$ which is consistent with T belongs to a principal type.*
(iv) *A type in $S_n(T)$ is principal if and only if it contains a maximal consistent formula.*

Fact 3.2. *([BBHU], Corollary 13.7.) Let U be a complete continuous theory with a countable signature.*

(i) *A model $\mathcal{N}$ of U is prime if and only if $\mathcal{N}$ is separable and every type which is realized in $\mathcal{N}$ is principal.*
(ii) *Any two prime models of U are isomorphic.*

Lemma 3.3. *Let $\mathcal{M}$ be a countable model of T. Then $\mathcal{M}$ is prime if and only if $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is prime.*

Proof. Suppose first that $\mathcal{M}$ is not prime. By Fact 3.1 there is a tuple $\vec{a}$ in $\mathcal{M}$ and a countable model $\mathcal{H}$ of T such that the type of $\vec{a}$ in $\mathcal{M}$ is not realized in $\mathcal{H}$. One can then check that the type of the constant function at $\vec{a}$ in $\mathcal{M}^{[0,1)}$ is not realized in $(\mathcal{H}^{[0,1)}, \mathcal{L})$, so by Fact 3.2, $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is not prime.

Now suppose that $\mathcal{M}$ is prime. By Fact 3.2, it is enough to show that for every tuple $\vec{\mathbf{f}}$ in $\mathcal{M}^{[0,1)}$, the type p of $\vec{\mathbf{f}}$ in $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is realized in every model of T^R. By Remark 2.12, $p = \sum_i \alpha_i p_i^\star$, where $\alpha_i = \lambda(\{t\colon \vec{\mathbf{f}}(t) = \vec{a}_i\})$ and $p_i = tp(\vec{a}_i)$ in $\mathcal{M}$. Since $\mathcal{M}$ is prime, each type p_i is realized in every model of T. By Fact 3.1, each type p_i contains a maximal consistent formula. It follows that $p_i^\star$ is realized in every model of T^R. By Remark 2.12, p is realized in every model of T^R. $\blacksquare_{3.3}$

Theorem 3.4. *T has a prime model if and only if T^R has a prime model.*

Proof. Suppose T has a prime model $\mathcal{M}$. Then $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is prime by Lemma 3.3.

For the converse, suppose that T does not have a prime model, but T^R does have a prime model $\mathcal{N}$. We will arrive at a contradiction, completing the proof. By Fact 3.1, there is a formula $\varphi(\vec{v})$ which is consistent with T but does not belong to a principal type. Then $T \models (\exists \vec{v})\varphi(\vec{v})$. By Fact 2.3, $T^R \models [\![(\exists \vec{v})\varphi(\vec{v})]\!] \doteq \top$. By Fact 2.4, there is a tuple $\vec{\mathbf{f}}$ in $\mathcal{N}$ such that $\mathcal{N} \models \mu([\![\varphi(\vec{\mathbf{f}})]\!]) = 1$. Let $p = tp(\vec{\mathbf{f}})$. Then $\mu([\![\varphi(\vec{v})]\!])^p = 1$. By Fact 3.2, p is principal. By Remark 2.12, $p = \sum_i \alpha_i p_i^\star$ for some sequence of types $p_i \in S_n(T)$ and some sequence of numbers $\alpha_i \in [0,1]$ such that $\sum_i \alpha_i = 1$. Take an i such that $\alpha_i > 0$. Now consider an arbitrary countable model $\mathcal{H}$ of T. Since p is principal, p is realized by some tuple $\vec{\mathbf{g}}$ in $(\mathcal{H}^{[0,1)}, \mathcal{L})$. We have $\lambda(\{t\colon tp((\vec{\mathbf{g}})(t)) = p_i\}) = \alpha_i > 0$, so p_i is realized in $\mathcal{H}$. Thus p_i is realized in every countable model of T, and hence is a principal type. But since $\mu([\![\varphi(\vec{v})]\!])^p = 1$, $\mathcal{H} \models \varphi(\vec{\mathbf{g}}(t))$ for almost all t, so $\varphi(\vec{v}) \in p_i$. This is a contradiction, and completes the proof. $\blacksquare_{3.4}$

We now show that the randomization theory T^R cannot have a minimal prime model. This is a place where the model theory of T^R differs from first order model theory. We first state two easy lemmas.

For each finite or countable model $\mathcal{M}$ of T, and each λ-atomless σ-algebra $\mathcal{A} \subseteq \mathcal{L}$, let $(\mathcal{M}^{\mathcal{A}}, \mathcal{A})$ be the pre-structure where $\mathcal{M}^{\mathcal{A}}$ is the set of $\mathcal{A}$-measurable functions from $[0,1)$ into $\mathcal{M}$.

The following fact is well-known and easily proved.

Fact 3.5. *For each $\mathsf{B} \in \mathcal{L}$ such that $0 < \lambda(\mathsf{B}) < 1$ there is a λ-atomless σ-algebra $\mathcal{A} \subseteq \mathcal{L}$ such that $\mathcal{A}$ is independent of B with respect to λ.*

Proposition 3.6. *T^R does not have a minimal prime model. In fact, for every prime model $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ of T^R, and any element $\mathsf{B} \in \mathcal{B}$ such that $\mathcal{N} \models 0 < \mu(\mathsf{B}) < 1$, $\mathcal{N}$ has an elementary substructure which does not contain B.*

Proof. By Theorem 3.4, T has a prime model $\mathcal{M}$. By Lemma 3.3 and Fact 3.2 (ii), $\mathcal{N} \cong (\mathcal{M}^{[0,1)}, \mathcal{L})$. By Fact 3.5, there is a λ-atomless σ-algebra $\mathcal{A} \subseteq \mathcal{L}$ which is independent of B. By Lemma 3.3 (iii) and (iv), there is an elementary embedding h of $(\mathcal{M}^{[0,1)}, \mathcal{L})$ into $\mathcal{N}$, and the image of h is an elementary submodel of $\mathcal{N}$ that does not contain B. $\blacksquare_{3.6}$

4. Strongly Separable Models and Types

We recall the definition of strongly separable models from [BK].

Definition 4.1. A pre-model $\mathcal{N}$ of T^R is called **strongly separable** if $\mathcal{N}$ is elementarily embeddable in $(\mathcal{M}^{[0,1)}, \mathcal{L})$ for some countable model $\mathcal{M}$ of T.

A type $p \in S_n(T^R)$ is called **strongly separable** if p is realizable in some strongly separable model of T^R.

It is shown in [BK], Corollary 3.8 and Theorem 3.12, that every strongly separable model is separable.[1] By Remark 2.9, for every T, every n-type of sort $\mathbf{B}$ in T^R is realizable in a strongly separable model of T^R. (So the analogue of strong separability always holds for types of sort $\mathbf{B}$).

We now return to the construction $p = \sum_i \alpha_i p_i^\star$ introduced in Example 2.11.

Definition 4.2. Let $p \in S_n(T^R)$. We say that $p = \sum_i \alpha_i p_i^\star$ is a **nice decomposition** of p if

- $p_0, p_1, \ldots$ are pairwise distinct types in $S_n(T)$;
- $\alpha_i \in (0, 1]$ for each i;
- $\sum_i \alpha_i = 1$.

Given a model $\mathcal{M}$ of T, we say that a tuple $\vec{\mathbf{f}}$ in $\mathcal{M}^{[0,1)}$ is **nice** if:

- for each $s, t \in [0, 1)$, if $tp(\vec{\mathbf{f}}(s)) = tp(\vec{\mathbf{f}}(t))$ then $\vec{\mathbf{f}}(s) = \vec{\mathbf{f}}(t)$;
- for each $s \in [0, 1)$, $\lambda(\{t \colon \vec{\mathbf{f}}(t) = \vec{\mathbf{f}}(s)\}) > 0$.

Note that any two nice decompositions of an n-type $p \in S_n(T^R)$ are the same up to the ordering of the terms. That is, if $p = \sum_i \alpha_i p_i^\star$ and $p = \sum_k \beta_k q_k^\star$ are nice, then

$$\{\alpha_0 p_0^\star, \alpha_1 p_1^\star \ldots\} = \{\beta_0 q_0^\star, \beta_1 q_1^\star \ldots\}.$$

Also note that if $\vec{\mathbf{f}}$ is a nice n-tuple in $\mathcal{M}^{[0,1)}$, then the decomposition $tp(\vec{\mathbf{f}}) = \sum_i \alpha_i p_i^\star$ given in Remark 2.12 (ii) is a nice decomposition of $tp(\vec{\mathbf{f}})$.

The next lemma gives a characterization of strongly separable types. It is an upgrade of Lemma 3.9 in [BK].

Lemma 4.3. (i) *For every (countable or uncountable) model $\mathcal{M} \models T$ and every n-tuple $\vec{\mathbf{g}}$ in $\mathcal{M}^{[0,1)}$, the type $p = tp(\vec{\mathbf{g}})$ is strongly separable, and has a nice decomposition $p = \sum_i \alpha_i p_i^\star$ where*

$$\alpha_i = \lambda(\{t \colon tp(\vec{\mathbf{g}}(t)) = p_i\}).$$

(ii) *For every model $\mathcal{M} \models T$ and every n-tuple $\vec{\mathbf{g}}$ in $\mathcal{M}^{[0,1)}$, there is a nice n-tuple $\vec{\mathbf{f}}$ in $\mathcal{M}^{[0,1)}$ such that $tp(\vec{\mathbf{f}}) = tp(\vec{\mathbf{g}})$.*

(iii) *A type $p \in S_n(T^R)$ is strongly separable if and only if p has a nice decomposition $p = \sum_i \alpha_i p_i^\star$.*

[1] Example 3.10 in [BK] gives types and separable models of T^R that are not strongly separable.

Proof. (i) The range of $\vec{\mathbf{g}}$ is countable, and thus contained in some countable $\mathcal{H} \prec \mathcal{M}$. So p is realized in the strongly separable model $(\mathcal{H}^{[0,1)}, \mathcal{L})$, and thus is a strongly separable type. Let

$$\{p_0, p_1, \ldots\} = \{q \in S_n(T) \colon \lambda(\{t \colon tp(\vec{\mathbf{g}}(t)) = q\}) > 0\}.$$

This set is finite or countable because the range of $\vec{\mathbf{g}}$ is finite or countable. It follows that $p = \sum_i \alpha_i p_i^\star$ is a nice decomposition of p where

$$\alpha_i = \lambda(\{t \colon tp(\vec{\mathbf{g}}(t) = q\}).$$

(ii) Let $\{p_0, p_1, \ldots\}$ and $\{\alpha_0, \alpha_1, \ldots\}$ be as in the proof of part (i) above. Then the types $p_0, p_1, \ldots$ are pairwise distinct. Let $\mathsf{B} = \{t \colon (\exists m) tp(\vec{\mathbf{g}}(t)) = p_i\}$. Then B is Borel and $\lambda(\mathsf{B}) = 1$. For each i, pick a tuple $\vec{a}_i$ in M such that $tp(\vec{a}_i) = p_i$. Let $\vec{\mathbf{f}}$ be the n-tuple in $\mathcal{M}^{[0,1)}$ such that for each $t \in [0, 1)$,

$$\vec{\mathbf{f}}(t) = \begin{cases} \vec{a}_i & \text{if } t \in \mathsf{B} \text{ and } tp(\vec{\mathbf{g}}(t)) = p_i, \\ \vec{a}_0 & \text{if } t \notin \mathsf{B}. \end{cases}$$

Then $\vec{\mathbf{f}}$ has the required properties.

(iii) Suppose p has a nice decomposition $p = \sum_i \alpha_i p_i^\star$. By the Compactness Theorem, there is a countable model $\mathcal{M}$ of T such that for each i, the type p_i is realized by an n-tuple $\vec{a}_i$ in $\mathcal{M}$. Then for each i, the type $p_i^\star$ is realized by the n-tuple of constant functions with value $\vec{a}_i$ in $(\mathcal{M}^{[0,1)}, \mathcal{L})$. Hence by Remark 2.12, p is realized in $(\mathcal{M}^{[0,1)}, \mathcal{L})$ and hence is strongly separable.

Now suppose p is strongly separable. Then for some countable model $\mathcal{M}$ of T, there is an n-tuple $\vec{\mathbf{g}}$ that realizes p in $(\mathcal{M}^{[0,1)}, \mathcal{L})$. By (i), p has a nice decomposition. $\blacksquare_{4.3}$

Corollary 4.4. *Suppose $\mathcal{M}$ is a model of T and p is a strongly separable type of T^R with the nice decomposition $p = \sum_i \alpha_i p_i^\star$. Then p is realized in $(\mathcal{M}^{[0,1)}, \mathcal{L})$ if and only if p_i is realized in $\mathcal{M}$ for each i.*

Here is a characterization of strongly separable models.

Proposition 4.5. *A pre-model $\mathcal{N}$ of T^R is strongly separable if and only if $\mathcal{N}$ is separable and for each n, each n-type $p \in S_n(T^R)$ that is realized in $\mathcal{N}$ is strongly separable.*

Proof. Suppose $\mathcal{N}$ is strongly separable. Then $\mathcal{N}$ is separable by Lemma 3.2 in [BK], and by definition, every type that is realized in $\mathcal{N}$ is strongly separable.

For the other direction, suppose $\mathcal{N}$ is separable and each type $p \in S_n(T^R)$ that is realized in $\mathcal{N}$ is strongly separable. By Lemma 4.3, each type that is realized in $\mathcal{N}$ has a nice decomposition. Let $D = \{\mathbf{f}_0, \mathbf{f}_1, \ldots\}$ be a countable dense subset of $\mathcal{N}$. For each n, let C_n be the set of all n-types $q \in S_n(T)$ such that for some n-tuple $\vec{\mathbf{f}}$ in D, $q^\star$ occurs in a nice decomposition of $tp(\vec{\mathbf{f}})$. Since the nice decompositions of an n-type are unique up to the ordering of the terms, the set C_n is at most countable. By the Compactness Theorem and Fact 2.8, there is a countable homogeneous model $\mathcal{M}$ of T such that each type in $\bigcup_n C_n$ is realized in $\mathcal{M}$. Then by Remark 2.12 and Lemma 4.3, for each n, $p_n = tp(\mathbf{f}_0, \ldots, \mathbf{f}_{n-1})$ is realized in $(\mathcal{M}^{[0,1)}, \mathcal{L})$ by a nice n-tuple $(\mathbf{g}_0, \ldots, \mathbf{g}_{n-1})$ in $\mathcal{M}^{[0,1)}$.

Using the fact that $\mathcal{M}$ is countable homogeneous, we see that whenever $(\mathbf{g}_0, \ldots, \mathbf{g}_{n-1})$ is a nice n-tuple in $\mathcal{M}^{[0,1)}$, $(\mathbf{h}_0, \ldots, \mathbf{h}_n)$ is a nice $n+1$-tuple in $\mathcal{M}^{[0,1)}$, and $tp(\mathbf{g}_0, \ldots, \mathbf{g}_{n-1}) = tp(\mathbf{h}_0, \ldots, \mathbf{h}_{n-1})$, there exists $\mathbf{g}_n$ in $\mathcal{M}^{[0,1)}$ such that $(\mathbf{g}_0, \ldots, \mathbf{g}_n)$ is nice and $tp(\mathbf{g}_0, \ldots, \mathbf{g}_n) = tp(\mathbf{h}_0, \ldots, \mathbf{h}_n)$. It follows by induction that there is a single sequence $(\mathbf{g}_0, \mathbf{g}_1, \ldots)$ such that for each n, $(\mathbf{g}_0, \ldots, \mathbf{g}_{n-1})$ realizes p_n in $(\mathcal{M}^{[0,1)}, \mathcal{L})$. Therefore the mapping $\mathbf{f}_n \mapsto \mathbf{g}_n$ can be extended to an elementary embedding of $\mathcal{N}$ into $(\mathcal{M}^{[0,1)}, \mathcal{L})$. This shows that $\mathcal{N}$ is strongly separable. $\blacksquare_{4.5}$

The next result complements Theorem 3.12 in [BK].

Proposition 4.6. *The following are equivalent.*

(i) *$S_n(T)$ is countable for each n.*
(ii) *Every type in T^R is strongly separable.*
(iii) *There is no type p in T^R such that ν_p is an atomless measure.*

Proof. Assume (i), and let $p \in S_n(T^R)$. By the Compactness Theorem, p is realized in some separable model $\mathcal{N}$ of T^R. By (i) and Theorem 3.12 of [BK], every separable model of T^R is strongly separable, so p is strongly separable. Thus (i) implies (ii).

Assume (ii). By Fact 2.10, every type p in T^R has a nice decomposition $p = \sum_i \alpha_i p_i^\star$. Then $\alpha_0 > 0$, so $\{p_0^\star\}$ is an atom of the measure ν_p and hence ν_p is not atomless. Thus (ii) implies (iii).

Finally, assume that (i) fails, so there is an n such that $S_n(T)$ is uncountable. By the Cantor-Bendixson Theorem, the uncountable Polish space $S_n(T)$ contains a copy of the Cantor space $\{0,1\}^{\mathbb{N}}$. The Cantor space has an atomless Borel probability measure, and hence $S_n(T)$ has an atomless Borel probability measure ν. By Fact 2.10 there is a type $p \in S_n(T^R)$ such that $\nu = \nu_p$, so (iii) fails. Thus (iii) implies (i). $\blacksquare_{4.6}$

We conclude this section with a series of lemmas about strongly separable pre-structures that are countable. These lemmas will be needed in the next section.

Definition 4.7. Let $\mathcal{H}$ be a countable model of T. We say that $(\mathcal{K}, \mathcal{A})$ is a **countable part of** $(\mathcal{H}^{[0,1)}, \mathcal{L})$ if $(\mathcal{K}, \mathcal{A})$ is countable, $(\mathcal{K}, \mathcal{A}) \prec (\mathcal{H}^{[0,1)}, \mathcal{L})$, and $\mathbf{f}^{-1}(c) \in \mathcal{A}$ for each $\mathbf{f} \in \mathcal{K}$ and $c \in H$ (that is, $\mathbf{f}$ is $\mathcal{A}$-measurable). For each $t \in [0,1)$, we let $\mathcal{K}(t)$ be the substructure of $\mathcal{H}$ with universe $K(t) = \{\mathbf{f}(t) \colon \mathbf{f} \in \mathcal{K}\}$.

We will need the following result from [AK], Theorem 3.8.

Fact 4.8. *If $\mathcal{H}$ is a countable model of T and $(\mathcal{K}, \mathcal{A})$ is a countable part of $(\mathcal{H}^{[0,1)}, \mathcal{L})$, then the topological closure of $\mathcal{K}$ in $\mathcal{H}^{[0,1)}$ is equal to the set of all $\mathbf{f} \in \mathcal{H}^{[0,1)}$ such that $\mathbf{f}(t) \in K(t)$ for almost all t.*

Lemma 4.9. *If $\mathcal{H}$ is a countable model of T and $(\mathcal{K}, \mathcal{A})$ is a countable part of $(\mathcal{H}^{[0,1)}, \mathcal{L})$, then $\mathcal{K}(t) \prec \mathcal{H}$ for almost all t.*

Proof. We note that for each $a \in H$, the set

$$\{t\colon a \in K(t)\}$$

is Borel, because it is equal to the countable union of the sets $\mathbf{h}^{-1}(a)$ where $\mathbf{h} \in \mathcal{K}$. Since $(\mathcal{K}, \mathcal{A}) \prec (\mathcal{H}^{[0,1)}, \mathcal{L})$, for each first order formula $\varphi(\vec{u}, v)$ and each tuple $\vec{\mathbf{f}}$ in $\mathcal{K}$, we have

$$\mu([\![(\exists v)\varphi(\vec{\mathbf{f}}, v)]\!]) = \sup_{\mathbf{g}\in\mathcal{K}} \mu([\![\varphi(\vec{\mathbf{f}}, \mathbf{g})]\!]).$$

Therefore for each $\varphi(\vec{u}, v)$ and each $\vec{a} \in H^n$, for almost all t we have

$$\text{if } \vec{a} \in K(t)^n \text{ and } \mathcal{H} \models (\exists v)\varphi(\vec{a}, v) \text{ then } (\exists b \in K(t))\mathcal{H} \models \varphi(\vec{a}, b).$$

Since H is countable, it follows that for almost all t, for every formula $\varphi(\vec{u}, v)$ and every $\vec{a} \in H^n$ we have

$$\text{if } \vec{a} \in K(t)^n \text{ and } \mathcal{H} \models (\exists v)\varphi(\vec{a}, v) \text{ then } (\exists b \in K(t))\mathcal{H} \models \varphi(\vec{a}, b).$$

Then by the Tarski-Vaught condition, we have $\mathcal{K}(t) \prec \mathcal{H}$ for almost all t. $\blacksquare_{4.9}$

Lemma 4.10. *Suppose $\mathcal{H}$ is a countable model of T, $(\mathcal{K}, \mathcal{A})$ and $(\mathcal{K}', \mathcal{A}')$ are countable parts of $(\mathcal{H}^{[0,1)}, \mathcal{L})$, and $\mathcal{K}$ and $\mathcal{K}'$ have the same closure in $\mathcal{H}^{[0,1)}$. Then $K(t) = K'(t)$ for almost all t.*

Proof. We will show that for each $a \in H$, the statement [$a \in K(t)$ if and only if $a \in K'(t)$] holds for almost all t. Since H is countable, this will imply that $K(t) = K'(t)$ for almost all t.

Suppose that, on the contrary, there is an $a \in H$ and a Borel set B of positive measure such that for all $t \in \mathsf{B}$, $a \in K'(t) \setminus K(t)$. Then there is an element $\mathbf{f} \in \mathcal{K}'$ and a Borel set $\mathsf{C} \subseteq \mathsf{B}$ such that $\lambda(\mathsf{C}) > 0$ and for all $t \in \mathsf{C}$, $\mathbf{f}(t) = a$. Since $\mathcal{K}$ is dense in the closure of $\mathcal{K}'$, there is an element $\mathbf{g} \in \mathcal{K}$ such that $d_{\mathbf{K}}(\mathbf{f}, \mathbf{g}) < \lambda(\mathsf{C})$. But then there must exist $t \in \mathsf{C}$ such that $\mathbf{g}(t) = \mathbf{f}(t) = a$, so $a \in K(t)$. This contradiction completes the proof. $\blacksquare_{4.10}$

Our next lemma will use the infinitary logic $L_{\omega_1\omega}$ with countable conjunctions and disjunctions and finite quantifiers (for background see [Ke1]). For each $L_{\omega_1\omega}$ formula $\varphi(v_0, \ldots, v_{n-1})$, and each n-tuple $\vec{\mathbf{f}} \in \mathcal{K}^n$, we let

$$[\![\varphi(\vec{\mathbf{f}})]\!]^{\mathcal{K}} = \{t\colon \vec{\mathbf{f}}(t) \in \mathcal{K}(t)^n \text{ and } \mathcal{K}(t) \models \varphi(\mathbf{f}_0(t), \ldots, \mathbf{f}_{n-1}(t))\}.$$

Lemma 4.11. *Suppose $\mathcal{H}$ is a countable model of T and $(\mathcal{K}, \mathcal{A})$ is a countable part of $(\mathcal{H}^{[0,1)}, \mathcal{L})$. Then for each tuple $\vec{\mathbf{f}}$ in $\mathcal{K}$ and each $L_{\omega_1\omega}$ formula $\varphi(\vec{v})$, the set $[\![\varphi(\mathbf{f})]\!]^{\mathcal{K}}$ belongs to the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$,*

Proof. We argue by induction on the complexity of the formula φ. When φ is a first order formula, we have

$$[\![\varphi(\vec{\mathbf{f}})]\!]^{\mathcal{K}} \in \mathcal{A}.$$

Suppose the result holds for all subformulas of φ. If $\varphi = \bigwedge_k \psi_k$, then

$$[\![\varphi(\vec{\mathbf{f}})]\!]^{\mathcal{K}} = [\![\bigwedge_k \psi_k(\vec{\mathbf{f}})]\!]^{\mathcal{K}} = \bigcap_k [\![\psi_k(\vec{\mathbf{f}})]\!]^{\mathcal{K}},$$

which is a countable intersection of sets in $\sigma(\mathcal{A})$ and hence belongs to $\sigma(\mathcal{A})$. If $\varphi = \neg\psi$, then

$$[\![\varphi(\vec{\mathbf{f}})]\!]^{\mathcal{K}} = [\![\neg\psi(\vec{\mathbf{f}})]\!]^{\mathcal{K}} = \neg[\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{K}} \in \sigma(\mathcal{A}).$$

Finally, if $\varphi = (\exists v)\psi$, then

$$[\![\varphi(\vec{\mathbf{f}})]\!]^{\mathcal{K}} = [\![(\exists v)\psi(\vec{\mathbf{f}}, v)]\!]^{\mathcal{K}} = \bigcup_{\mathbf{g}\in\mathcal{K}} [\![\psi(\vec{\mathbf{f}}, \mathbf{g})]\!]^{\mathcal{K}},$$

which is a countable union of sets in $\sigma(\mathcal{A})$ and hence belongs to $\sigma(\mathcal{A})$. $\blacksquare_{4.11}$

5. Separable Homogeneous Models

In this section we will show that for each complete first order theory T, the number of separable homogeneous models of T^R is equal to the number of countable homogeneous models of T, up to isomorphism. The hard part will be to prove Theorem 5.4, which shows that the strongly separable homogeneous models of T^R are exactly the Borel randomizations of countable homogeneous models of T, up to isomorphism.

Lemma 5.1. *$\mathcal{M}$ is countable homogeneous if and only if $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is separable homogeneous.*

Proof. Suppose first that $\mathcal{M}$ is countable homogeneous. By Lemma 2.2, $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is separable. Let $\vec{\mathbf{f}}, \vec{\mathbf{g}}$ realize the same n-type p in $(\mathcal{M}^{[0,1)}, \mathcal{L})$, let $\mathbf{h} \in \mathcal{M}^{[0,1)}$, and let q be the $(n+1)$-type of $(\vec{\mathbf{f}}, \mathbf{h})$. By Lemma 4.3, p has a nice decomposition $p = \sum_i \alpha_i p_i^\star$ such that for each i, $\alpha_i = \lambda(\mathsf{A}_i) = \lambda(\mathsf{B}_i)$ where

$$\mathsf{A}_i = \{t \colon tp(\vec{\mathbf{f}}(t)) = p_i\}, \quad \mathsf{B}_i = \{t \colon tp(\vec{\mathbf{g}}(t)) = p_i\}.$$

Also q has a nice decomposition $q = \sum_j \beta_j q_j^\star$. By grouping the q_j's that contain p_i together for each i, we can write the nice decomposition of q as $q = \sum_i (\sum_j \beta_{ij} q_{ij}^\star)$ where $p_i \subseteq q_{ij}$ for each (i,j). Then for each (i,j) we have $\beta_{ij} = \lambda(\mathsf{C}_{ij})$, where

$$\mathsf{C}_{ij} = \{t \colon tp(\vec{\mathbf{f}}(t), \mathbf{h}(t)) = q_{ij}\} \subseteq \mathsf{A}_i.$$

Note that each of the sets $\mathsf{A}_i, \mathsf{B}_i, \mathsf{C}_{ij}$ belongs to $\mathcal{L}$. For each i, we may partition the set B_i into a union $\mathsf{B}_i = \bigcup_j \mathsf{D}_{ij}$ of Borel sets D_{ij} such that $\beta_{ij} = \lambda(\mathsf{D}_{ij})$. Each of the unions $\bigcup_i \mathsf{A}_i, \bigcup_i \mathsf{B}_i,, \bigcup_i(\bigcup_j \mathsf{C}_{ij})$, and $\bigcup_i(\bigcup_j \mathsf{D}_{ij})$ has measure one. Each of the sets C_{ij} has positive measure, so the type q_{ij} is realized in $\mathcal{M}$. Since $\mathcal{M}$ is countable homogeneous, for each (i,j) and each tuple $\vec{c} \in M^n$ such that $tp(\vec{c}) = p_i$, we may choose an element $d = d(\vec{a}, i, j) \in M$ such that $tp(\vec{c}, d) = q_{ij}$. Let $\mathbf{k}$ be the almost surely unique element of $\mathcal{M}^{[0,1)}$ such that for each (i,j) and $t \in \mathsf{D}_{ij}$, $\mathbf{k}(t) = d(\vec{\mathbf{g}}(t), i, j)$. Then $(\vec{\mathbf{g}}(t), \mathbf{k}(t))$ realizes

q_{ij} for all $t \in \mathsf{D}_{ij}$, and hence $(\vec{\mathbf{g}}, \mathbf{k})$ realizes q. This shows that $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is separable homogeneous.

Now suppose $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is separable homogeneous. $\mathcal{M}$ is countable by Lemma 2.2. Let $\vec{a}, \vec{b}$ be tuples in $\mathcal{M}$ such that $p = tp(\vec{a}) = tp(\vec{b})$, and let $c \in M$. Let $\vec{\mathbf{f}}, \vec{\mathbf{g}}, \mathbf{h}$ be the constant functions in $\mathcal{M}^{[0,1)}$ with values $\vec{a}, \vec{b}, c$ respectively. By Lemma 4.3, $p^\star = tp(\vec{\mathbf{f}}) = tp(\vec{\mathbf{g}})$. Since $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is separable homogeneous, there exists $\mathbf{k}$ in $\mathcal{M}^{[0,1)}$ such that $tp(\vec{\mathbf{f}}, \mathbf{h}) = tp(\vec{\mathbf{g}}, \mathbf{k})$. Let $q = tp(\vec{a}, c)$. Then $q^\star = tp(\vec{\mathbf{f}}, \mathbf{h}) = tp(\vec{\mathbf{g}}, \mathbf{k})$. Therefore $q = tp(\vec{\mathbf{f}}(t), \mathbf{h}(t)) = tp(\vec{\mathbf{g}}(t), \mathbf{k}(t)) = tp(\vec{b}, \mathbf{k}(t))$ for almost all t, and hence there exists $d \in M$ with $tp(\vec{b}, d) = q$. Thus $\mathcal{M}$ is countable homogeneous. $\blacksquare_{5.1}$

Lemma 5.2. *Two separable homogeneous continuous structures that realize the same types are isomorphic.*

Proof. Let $\mathcal{N}, \mathcal{P}$ be separable homogeneous continuous structures that realize the same types. By a back and forth argument, there are dense sequences $(\mathbf{f}_0, \mathbf{f}_1, \ldots)$, $(\mathbf{g}_0, \mathbf{g}_1, \ldots)$ in $\mathcal{N}, \mathcal{P}$ respectively that realize the same types. By density, there is an isomorphism from $\mathcal{N}$ onto $\mathcal{P}$ which sends each $\mathbf{f}_i$ to $\mathbf{g}_i$. $\blacksquare_{5.2}$

The following result characterizes the set of strongly separable types that are realized in a given separable homogeneous model. It is a converse of Remark 2.12, and should be compared with Corollary 4.4.

Proposition 5.3. *Suppose $\mathcal{N}$ is a separable homogeneous model of T^R, and p is a strongly separable type with the nice decomposition $p = \sum_i \alpha_i p_i^\star$. Then p is realized in $\mathcal{N}$ if and only if $p_i^\star$ is realized in $\mathcal{N}$ for each i.*

Proof. By Remark 2.12, for every model $\mathcal{N}$ of T^R, if $p_i^\star$ is realized in $\mathcal{N}$ for each i then p is realized in $\mathcal{N}$.

For other direction, let $\vec{\mathbf{f}}$ realize $p = \sum_i \alpha_i p_i^\star$ in $\mathcal{N}$. Fix i. If $\alpha_i = 1$, then $p = p_i^\star$, so $\vec{\mathbf{f}}$ realizes $p_i^\star$ and we are done. Suppose $\alpha_i < 1$. Let $\beta = \min(\alpha_i, 1 - \alpha_i)$. Then $0 < \beta$, and there are disjoint events B, C of measure β in $\mathcal{N}$ such that $\vec{\mathbf{f}}$ realizes p_i everywhere in B and nowhere in C. Formally, this means that in $\mathcal{N}$, $\mathsf{B} \sqcap \mathsf{C} = \bot$, $\mu(\mathsf{B}) = \beta$, $\mu(\mathsf{C}) = \beta$,

$$\inf\{\mu(\mathsf{B} \sqcap [\![\varphi(\vec{\mathbf{f}})]\!]) : \varphi \in p_i\}) = \beta,$$

and

$$\inf\{\mu(\mathsf{C} \sqcap [\![\varphi(\vec{\mathbf{f}})]\!]) : \varphi \in p_i\}) = 0.$$

By Fact 2.7 (quantifier elimination), $tp(\mathsf{B}) = tp(\mathsf{C})$ in $\mathcal{N}$. Therefore by separable homogeneity, there is a tuple $\vec{\mathbf{g}}$ in $\mathcal{N}$ such that $tp(\vec{\mathbf{f}}, \mathsf{B}) = tp(\vec{\mathbf{g}}, \mathsf{C})$. Then $\vec{\mathbf{g}}$ realizes p_i everywhere on C. By Remark 2.5, there is a tuple $\vec{\mathbf{h}}$ in $\mathcal{N}$ such that $\vec{\mathbf{h}}$ agrees with $\vec{\mathbf{g}}$ on C and agrees with $\vec{\mathbf{f}}$ on $\neg\mathsf{C}$. Then $\vec{\mathbf{h}}$ realizes p_i on a set of measure $\alpha_i + \beta = \min(1, 2\alpha_i)$. By repeating this process finitely many times, we obtain a tuple in $\mathcal{N}$ that realizes p_i on an event of measure one, and thus realizes $p_i^\star$ in $\mathcal{N}$. $\blacksquare_{5.3}$

Our next theorem shows that the converse of Lemma 5.1 holds, and thus gives a characterization of strongly separable homogeneous models of T^R.

Theorem 5.4. *$\mathcal{N}$ is a strongly separable homogeneous model of T^R if and only if $\mathcal{N}$ is isomorphic to $(\mathcal{M}^{[0,1)},\mathcal{L})$ for some countable homogeneous $\mathcal{M}\models T$.*

Proof. The "if" direction follows from Lemma 5.1. For the other direction, we assume that $\mathcal{N}$ is a strongly separable homogeneous model of T^R.

Since $\mathcal{N}$ is strongly separable, there is a countable model $\mathcal{H}$ of T such that $\mathcal{N}$ is elementarily embeddable in $(\mathcal{H}^{[0,1)},\mathcal{L})$. By Fact 2.8, we may take $\mathcal{H}$ to be countable homogeneous. By Fact 2.6, $\mathcal{N}$ is isomorphic to a pre-structure $\mathcal{P}=(\mathcal{J},\mathcal{L})\prec(\mathcal{H}^{[0,1)},\mathcal{L})$. Since $\mathcal{N}\cong\mathcal{P}$, $\mathcal{P}$ is pre-complete. $\mathcal{P}$ is also separable homogeneous. To prove the theorem, it suffices to show that $\mathcal{P}=\cong(\mathcal{M}^{[0,1)},\mathcal{L})$ for some countable homogeneous $\mathcal{M}\prec\mathcal{H}$.

Our plan is to use the lemmas in Section 4 to show that for almost all t, $\mathcal{K}(t)$ is isomorphic to a fixed homogeneous model $\mathcal{M}\prec\mathcal{H}$, and then show that $\mathcal{P}\cong(\mathcal{M}^{[0,1)},\mathcal{L})$. To do this we establish a series of claims.

Claim 1. (Zero–one Law) For every $L_{\omega_1\omega}$ sentence φ, either $[\![\varphi]\!]^{\mathcal{K}}\doteq\top$ or $[\![\varphi]\!]^{\mathcal{K}}\doteq\bot$.

Proof of Claim 1: We first note that $\sigma(\mathcal{A})\subseteq\mathcal{L}$, so by Lemma 4.11 we have $[\![\varphi]\!]^{\mathcal{K}}\in\mathcal{L}$. Suppose Claim 1 fails. Then for some $L_{\omega_1\omega}$ sentence φ, $0<\lambda([\![\varphi]\!]^{\mathcal{K}})<1$. Hence there are two events $\mathsf{A},\mathsf{B}\in\mathcal{L}$ such that $\mathsf{A}\subseteq[\![\varphi]\!]^{\mathcal{K}}$, $\mathsf{B}\subseteq\neg[\![\varphi]\!]^{\mathcal{K}}$, and $0<\lambda(\mathsf{A})=\lambda(\mathsf{B})$. By Fact 2.7, A and B have the same type in $\mathcal{P}$. Since $\mathcal{P}$ is separable homogeneous, $(\mathcal{P},\mathsf{A})$ and $(\mathcal{P},\mathsf{B})$ are separable homogeneous and realize the same types. Then by Lemma 5.2, there is an automorphism h of $\mathcal{P}$ such that $h(\mathsf{A})=\mathsf{B}$. Let $(\mathcal{K}',\mathcal{A}')$ be the image of $(\mathcal{K},\mathcal{A})$ under h. Then $(\mathcal{K}',\mathcal{A}')$ is also countable and dense in $\mathcal{P}$, and $(\mathcal{K}',\mathcal{A}')\prec\mathcal{P}$. By Lemma 4.10, $K(t)=K'(t)$ for almost all t. But then by Lemma 4.11,

$$\mathsf{B}=h(\mathsf{A})\subseteq h([\![\varphi]\!]^{\mathcal{K}})=[\![\varphi]\!]^{\mathcal{K}'}=[\![\varphi]\!]^{\mathcal{K}},$$

contradicting the assumption that $\mathsf{B}\subseteq\neg[\![\varphi]\!]^{\mathcal{K}}$. This proves Claim 1.

Claim 2. There is a Borel set E such that $\lambda(\mathsf{E})=1$ and for all $s,t\in\mathsf{E}$, $\mathcal{K}(s)$ and $\mathcal{K}(t)$ realize the same types.

Proof of Claim 2: By Fact 4.9, there is a Borel set E_1 such that $\lambda(\mathsf{E}_1)=1$ and $\mathcal{K}(t)\prec\mathcal{H}$ for all $t\in\mathsf{E}_1$. For each type $q\in S_n(T)$, the $L_{\omega_1\omega}$ sentence $\varphi_q=(\exists\vec{v})\bigwedge q$ holds in a structure $\mathcal{K}(t)$ if and only if q is realized in $\mathcal{K}(t)$. By Claim 1, for each type q, either $\mathcal{K}(t)\models\varphi_q$ for almost all t, or $\mathcal{K}(t)\models\neg\varphi_q$ for almost all t. Moreover, if φ_q holds in $\mathcal{K}(t)$ for some $t\in\mathsf{E}_1$, then q is realized in $\mathcal{H}$. Since $\mathcal{H}$ is countable, the set

$$Q=\{q\in\bigcup_n S_n(T)\colon(\exists t\in\mathsf{E}_1)\,\mathcal{K}(t)\models\varphi_q\}$$

is countable. Hence there is a Borel set $\mathsf{E}\subseteq\mathsf{E}_1$ such that $\lambda(\mathsf{E})=1$ and for each $q\in Q$, either $\mathcal{K}(t)\models\varphi_q$ for all $t\in\mathsf{E}$, or $\mathcal{K}(t)\models\neg\varphi_q$ for all $t\in\mathsf{E}$. Then E satisfies the requirements for Claim 2.

Claim 3. For almost every $t \in \mathsf{E}$, $\mathcal{K}(t)$ is countable homogeneous.

Proof of Claim 3: It is sufficient to prove the following for each n and each pair $\vec{a}, \vec{b} \in H^n$ such that $tp(\vec{a}) = tp(\vec{b})$ in $\mathcal{H}$, and each $c \in H$:

(1) *For almost all t, if $\vec{a}, \vec{b} \in K(t)^n$ and $c \in K(t)$ then there exists $d \in K(t)$ such that $tp(\vec{a}, c) = tp(\vec{b}, d)$.*

Fix $\vec{a}, \vec{b}, c$ such that $tp(\vec{a}) = tp(\vec{b})$ in $\mathcal{H}$. Let A be the Borel set of all $t \in [0,1)$ such that $\vec{a}, \vec{b} \in K(t)^n$ and $c \in K(t)$. If $\lambda(\mathsf{A}) = 0$, then (1) is trivial, so we assume $\lambda(\mathsf{A}) > 0$. Then there is a partition $\mathsf{A} = \bigcup_m \mathsf{B}_m$ of A into Borel sets, such that for each m there is a pair $\vec{\mathbf{f}}_m, \vec{\mathbf{g}}_m \in \mathcal{K}^n$ and an element $\mathbf{h}_m \in \mathcal{K}$ with $\vec{\mathbf{f}}_m(t) = \vec{a}$, $\vec{\mathbf{g}}_m(t) = \vec{b}$, and $\mathbf{h}_m(t) = c$ for all $t \in \mathsf{B}_m$. Fix m, and let $\vec{\mathbf{e}}_m$ be the n-tuple that agrees with $\vec{\mathbf{g}}_m$ on B_m and agrees with $\vec{\mathbf{f}}_m$ elsewhere. By Remark 2.5, $\vec{\mathbf{e}}_m$ belongs to $\mathcal{J}^n$. We have $tp(\vec{\mathbf{f}}_m(t)) = tp(\vec{\mathbf{e}}_m(t))$ for all $t \in [0,1)$. Hence for each first order formula $\varphi(\vec{u})$,

$$\lambda(\mathsf{B}_m \sqcap [\![\varphi(\vec{\mathbf{f}}_m)]\!]) = \lambda(\mathsf{B}_m \sqcap [\![\varphi(\vec{\mathbf{e}}_m)]\!])$$

and

$$\lambda((\neg\mathsf{B}_m) \sqcap [\![\varphi(\vec{\mathbf{f}}_m)]\!]) = \lambda((\neg\mathsf{B}_m) \sqcap [\![\varphi(\vec{\mathbf{e}}_m)]\!]),$$

so $tp(\vec{\mathbf{f}}_m, \mathsf{B}_m) = tp(\vec{\mathbf{e}}_m, \mathsf{B}_m)$. Since $\mathcal{P}$ is separable homogeneous, there is an element $\mathbf{k}_m \in \mathcal{J}$ such that

$$tp(\vec{\mathbf{f}}_m, \mathbf{h}_m, \mathsf{B}_m) = tp(\vec{\mathbf{e}}_m, \mathbf{k}_m, \mathsf{B}_m).$$

Then for each first order formula $\psi(\vec{u}, v)$,

$$\lambda(\mathsf{B}_m \sqcap [\![\psi(\vec{\mathbf{f}}_m, \mathbf{h}_m)]\!])) = \lambda(\mathsf{B}_m \sqcap [\![\psi(\vec{\mathbf{e}}_m, \mathbf{k}_m)]\!]).$$

Hence for each $\psi(\vec{u}, v) \in tp(\vec{a}, c)$,

$$\lambda(\mathsf{B}_m) = \lambda(\mathsf{B}_m \sqcap [\![\psi(\vec{b}, \mathbf{k}_m)]\!]),$$

and thus $tp(\vec{a}, c) = tp(\vec{b}, \mathbf{k}_m(t))$ for almost all $t \in \mathsf{B}_m$. Moreover, since $\mathcal{K}$ is dense in $\mathcal{J}$, we see from Fact 4.8 that $\mathbf{k}_m(t) \in K(t)$ for almost all $t \in \mathsf{B}_m$. This proves (1) and Claim 3.

Claim 4. There is a countable homogeneous model $\mathcal{M} \prec \mathcal{H}$ such that $\mathcal{K}(t) \cong \mathcal{M}$ for almost all t.

Proof of Claim 4: By Claims 2 and 3, there is a Borel set $\mathsf{E}' \subseteq \mathsf{E}$ such that $\lambda(\mathsf{E}') = 1$ and for all $s, t \in \mathsf{E}'$, $\mathcal{K}(s)$ and $\mathcal{K}(t)$ are countable homogeneous models that realize the same types. By Fact 2.8, $\mathcal{K}(s) \cong \mathcal{K}(t)$ for all $s, t \in \mathsf{E}'$. This proves Claim 4.

We will construct an isomorphism $h\colon (\mathcal{M}^{[0,1)}, \mathcal{L}) \cong \mathcal{P}$. If $\mathcal{H}$ is finite, the theorem holds because T^R is separably categorical, so we may assume H is countably infinite. Arrange the elements of H in a list of length ω. This gives us a listing of M and of each $K(t)$. To construct h we proceed inductively over the elements of M. For each $t \in [0,1)$ we will pick enumerations $M = \{a_0(t), a_1(t), \ldots\}$ and $K(t) = \{b_0(t), b_1(t), \ldots\}$

as follows. When $t \notin \mathsf{E}'$, $a_m(t)$ is the m-th element of M and $b_m(t)$ is the m-th element of $K(t)$. When $t \in \mathsf{E}'$ we proceed inductively on m. We assume that $a_0(t), \ldots, a_{2m-1}(t)$ and $b_0(t), \ldots, b_{2m-1}(t)$ have already been constructed so that

$$tp(a_0(t), \ldots, a_{2m-1}(t)) = tp(b_0(t), \ldots, b_{2m-1}(t))$$

in $\mathcal{H}$. We take $a_{2m}(t)$ to be the first element of $M \setminus \{a_0(t), \ldots, a_{2m-2}(t)\}$, and take $b_{2m}(t)$ to be the first element of $K(t)$ such that

$$tp(a_0(t), \ldots, a_{2m-1}(t), a_{2m}(t)) = tp(b_0(t), \ldots, b_{2m-1}(t), b_{2m}(t)).$$

We then take $b_{2m+1}(t)$ to be the first element of $K(t) \setminus \{b_1(t), \ldots, b_{2m-1}(t)\}$, and take $a_{2m+1}(t)$ to be the first element of M such that

$$tp(a_0(t), \ldots, a_{2m-1}(t), a_{2m}(t), a_{2m+1}(t)) = tp(b_0(t), \ldots, b_{2m-1}(t), b_{2m}(t), b_{2m+1}(t)).$$

This procedure can always be carried out because $\mathcal{M}$ and $\mathcal{K}(t)$ are countable homogeneous and realize the same types. The construction guarantees that for each t, $M = \{a_0(t), a_1(t), \ldots\}$ and $K(t) = \{b_0(t), b_1(t), \ldots\}$, and for each $t \in \mathsf{E}'$, the mapping $a_m(t) \mapsto b_m(t)$ is an isomorphism from $\mathcal{M}$ onto $\mathcal{K}(t)$. Because $\mathcal{K}$ is a countable set of $\mathcal{L}$-measurable functions and $K(t) = \{\mathbf{f}(t) \colon \mathbf{f} \in \mathcal{K}\}$, we see by induction that for each m the functions $a_m(t)$ and $b_m(t)$ are $\mathcal{L}$-measurable functions of t.

For each $\mathbf{f} \in \mathcal{M}^{[0,1)}$ let $h(\mathbf{f})$ be the unique function $\mathbf{g} \colon [0,1) \to H$ such that for each t and m, $\mathbf{f}(t) = a_m(t)$ if and only if $\mathbf{g}(t) = b_m(t)$. Since $\mathcal{P}$ is the closure of $\mathcal{K}$, we see from Fact 4.8 that h becomes a bijection from $\mathcal{M}^{[0,1)}$ onto $\mathcal{P}$ when we identify elements that are at distance 0 from each other. Moreover, for each first order formula φ and tuple $\vec{\mathbf{f}}$ in $\mathcal{M}^{[0,1)}$,

$$[\![\varphi(\vec{\mathbf{f}})]\!] \doteq [\![\varphi(h(\vec{\mathbf{f}}))]\!].$$

Therefore $h \colon (\mathcal{M}^{[0,1)}, \mathcal{L}) \cong \mathcal{P}$. $\blacksquare_{5.4}$

Corollary 5.5. *The mapping*

$$\Theta \colon \mathcal{M} \mapsto \textit{completion of } (\mathcal{M}^{[0,1)}, \mathcal{L})$$

is a bijection from the set of isomorphism types of countable homogeneous models of T onto the set of isomorphism types of strongly separable homogeneous models of T^R, and this mapping preserves elementary embeddability.

Proof. By Lemma 5.1, Θ maps countable homogeneous models to strongly separable homogeneous models. If $\Theta(\mathcal{M}) \cong \Theta(\mathcal{H})$, then $\Theta(\mathcal{M})$ and $\Theta(\mathcal{H})$ realize the same types, so by Corollary 4.4, $\mathcal{M}$ and $\mathcal{H}$ realize the same types, and by Fact 2.8, $\mathcal{M} \cong \mathcal{H}$. Thus Θ is one-to-one up to isomorphism. It is clear that Θ preserves elementary embeddability. Theorem 5.4 shows that Θ is onto. $\blacksquare_{5.5}$

Example 5.6. Baldwn and Lachlan [BL] showed that if T is ω_1-categorical but not ω-categorical, then all the countable models of T are countable homogeneous and form an elementary chain of length $\omega + 1$. Corollary 5.5 shows that in that case, the strongly separable homogeneous models of T^R also form an elementary chain of length $\omega + 1$.

Corollary 5.7. *Let κ be the number of countable homogeneous models of T. Then T^R has exactly κ separable homogeneous models, and exactly κ strongly separable homogeneous models.*

Proof. By Theorem 5.4, T^R has exactly κ strongly separable homogeneous models. Suppose first that T has countably many complete types. Then by Propositions 4.5 and 4.6, every separable model of T^R is strongly separable, so T has exactly κ separable homogeneous models.

Now suppose that T has uncountably many complete types. Then for some n, $S_n(T)$ is uncountable. $S_n(T)$ is an uncountable Polish space, and every uncountable Polish space has cardinality 2^ω, so T has 2^ω complete types. By Fact 2.8, every type $p \in S_n(T)$ is realized in some countable homogeneous model of T. Moreover, every countable model realizes only countably many complete types. Since the signature is countable, T has at most 2^ω countable models. T^R also has a countable signature, so T^R has at most 2^ω countable pre-models, and hence at most 2^ω separable models. It follows that T has exactly 2^ω countable homogeneous models, so $\kappa = 2^\omega$. Then T^R has exactly 2^ω strongly separable homogeneous models, and exactly 2^ω separable homogeneous models. $\blacksquare_{5.7}$

The paper [KM] gives an example of a complete first order theory T with exactly m countable homogeneous models for each positive integer m. By the above corollary, such a theory has exactly m separable homogeneous models.

In this Section we have used nice decompositions of strongly separable types to establish a tight connection between the countable homogeneous models of a first order theory and the strongly separable homogeneous models of the corresponding randomization theory. An open question is whether one can obtain results in this spirit about arbitrary types and arbitrary separable homogeneous models of the randomization theory.

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