NONSTANDARD ARITHMETIC AND REVERSE MATHEMATICS

H. JEROME KEISLER

Abstract. We show that each of the five basic theories of second order arithmetic that play a central role in reverse mathematics has a natural counterpart in the language of nonstandard arithmetic. In the earlier paper [HKK1984] we introduced saturation principles in nonstandard arithmetic which are equivalent in strength to strong choice axioms in second order arithmetic. This paper studies principles which are equivalent in strength to weaker theories in second order arithmetic.

§1. Introduction. Reverse mathematics was introduced by H. Friedman and is developed extensively in the book of Simpson [Si1999]. It shows that many results in classical mathematics are equivalent to one of five basic theories in the language L_2 of second order arithmetic. These theories, from weakest to strongest, are called Recursive Comprehension (RCA_0), Weak Koenig Lemma (WKL_0), Arithmetical Comprehension (ACA_0), Arithmetical Transfinite Recursion (ATR_0), and Π^1_1 Comprehension (Π^1_1-CA_0). In this paper we find natural counterparts to each of these theories in the language ${}^*L_1$ of nonstandard arithmetic.

The language L_2 of second order arithmetic has a sort for the natural numbers and a sort for sets of natural numbers, while the language ${}^*L_1$ of nonstandard arithmetic has a sort for the natural numbers and a sort for the hyperintegers. In nonstandard analysis one often uses first order properties of hyperintegers to prove second order properties of integers. An advantage of this method is that the hyperintegers have more structure than the sets of integers. The method is captured by the Standard Part Principle (STP), a statement in the combined language $L_2 \cup {}^*L_1$ which says that a set of integers exists if and only if it is coded by a hyperinteger.

For each of the basic theories $T = \mathsf{WKL}_0, \mathsf{ACA}_0, \mathsf{ATR}_0, \Pi^1_1\text{-}\mathsf{CA}_0$ in the language L_2 of second order arithmetic, we will find a natural counterpart U in the language ${}^*L_1$ of nonstandard arithmetic, and prove that:

1) $U + \mathsf{STP} \vdash T$, and

2) $U + \mathsf{STP}$ is conservative with respect to T (that is, any sentence of L_2 provable from $U + \mathsf{STP}$ is provable from T.

We also get a result of this kind for the theory RCA_0, but with a weaker form of STP.

For instance, in the case that $T = \mathsf{WKL}_0$, the corresponding theory $U = {}^*\Sigma\mathsf{PA}$ in the language ${}^*L_1$ has the following axioms (stated formally in Section 3):

- Basic axioms for addition, multiplication,and exponentiation,
- The natural numbers form a proper initial segment of the hyperintegers,
- Induction for bounded quantifier formulas about hyperintegers,
- If there is a finite n such that $(z)_n$ is infinite, then there is a least n such that $(z)_n$ is infinite.

In the case that $T = \Pi^1_1\text{-}\mathsf{CA}_0$, our result proves a conjecture stated in the paper [HKK1984]. The missing ingredient was a nonstandard analogue of the Kleene normal form theorem for Σ^1_1 formulas, which is proved here in Section 8 and is also used for the case $T = \mathsf{ATR}_0$. In the case $T = \mathsf{WKL}_0$, our conservation result uses a self-embedding theorem of Tanaka [Ta1997].

The paper [Ke2005] is a companion to this paper which develops a framework for nonstandard reverse mathematics in the setting of higher order type theory. There is a close relationship between this paper and the paper [En2005] of Enayat. In the earlier papers [HKK1984] and [HK1986] we introduced saturation principles in nonstandard arithmetic which are equivalent in strength to strong choice axioms in second and higher order arithmetic. This paper studies principles which are equivalent in strength to weaker theories in second order arithmetic.

I thank Itay Ben Yaacov, Ali Enayat, Arnold Miller, and Steffen Lempp for helpful discussions in connection with this work.

§2. Preliminaries. We refer to [Si1999] for background in reverse mathematics and second order number theory, and to [CK1990] for background in model theory.

We begin with a brief review of the first order base theory $\Sigma\mathsf{PA}$ (Peano arithmetic with restricted induction) and the second order base theory RCA_0 (recursive comprehension).

The language L_1 of $\Sigma\mathsf{PA}$ is a first order language with variables $m, n, \ldots$, equality $=$, the order relation $<$, the constants $0, 1$, and the binary operations $+, \cdot$. For convenience in coding finite sequences of integers, we also include in the vocabulary of L_1 the symbols exp, p_n, and $(m)_n$ for the exponentiation function $exp(m,n) = m^n$, the function $p_n =$ the n-th prime, and the function $(m)_n =$ the largest $k \le m$ such that $(p_n)^k$ divides m. The theory $\Sigma\mathsf{PA}$ without the extra symbols exp, p_n, and $(m)_n$ is called $I\Sigma_1$ in the literature.

The language L_2 of RCA_0 is a two sorted language with the symbols of L_1 in the number sort N, variables $X, Y, \ldots$ of the set sort P, and the binary operation $\in$ of sort $N \times P$.

When we write a formula $\varphi(\vec{v})$, it is understood that $\vec{v}$ is a tuple of variables that contains all the free variables of φ. If we want to allow additional free variables we write $\varphi(\vec{v}, \dots)$. The length of $\vec{v}$ is denoted by $|\vec{v}|$, and v_i is a typical element of $\vec{v}$. If $\vec{m} = \langle m_0, \dots, m_k \rangle$ then $(\vec{m})_n$ denotes the tuple $\langle (m_0)_n, \dots, (m_k)_n \rangle$.

The **bounded quantifiers** $(\forall n < t)$ and $(\exists n < t)$ are defined as usual, where t is a term of sort N. By a **bounded quantifier formula**, or Δ^0_0 formula, we mean a formula of L_2 built from atomic formulas using propositional connectives and bounded quantifiers. We put $\Sigma^0_0 = \Pi^0_0 = \Delta^0_0$, and

$$\Sigma^0_{k+1} = \{\exists m\, \theta : \theta \in \Pi^0_k\}, \quad \Pi^0_{k+1} = \{\forall m\, \theta : \theta \in \Sigma^0_k\}.$$

The **arithmetical formulas** are the formulas in the set $\bigcup_k \Sigma^0_k = \bigcup_k \Pi^0_k$. These are the formulas of L_2 with only first order quantifiers.

Definition 2.1. ***Axioms of*** ΣPA.

- *The **Basic Axioms**, a finite set of sentences giving the usual recursive rules for* $<, +, \cdot$, *exp*, p_n, *and* $(m)_n$.
- ***First Order*** Σ^0_1 ***Induction Scheme***

  $$[\varphi(0, \vec{n}) \wedge \forall m[\varphi(m, \vec{n}) \rightarrow \varphi(m+1, \vec{n})]] \rightarrow \forall m \varphi(m, \vec{n})$$

  where $\varphi(m, \vec{n})$ *is a* Σ^0_1 *formula of* L_1.

It is well known that in ΣPA, every primitive recursive relation can be defined in a canonical way by both a Σ^0_1 and a Π^0_1 formula, and that the primitive recursive relations are closed under bounded quantification.

A general L_2-structure has the form $\mathcal{M} = (\mathcal{N}, \mathcal{P})$ where $\mathcal{N}$ is an L_1-structure, called the **first order part of** $\mathcal{M}$, and $\mathcal{P}$ is a family of subsets of the universe of $\mathcal{N}$.

Definition 2.2. ***Axioms of*** RCA_0.

- *The **Basic Axioms** of* ΣPA.
- Σ^0_1 ***Induction:***

  $$[\varphi(0, \dots) \wedge \forall m[\varphi(m, \dots) \rightarrow \varphi(m+1, \dots)]] \rightarrow \forall m \varphi(m, \dots)$$

  where φ *is a* Σ^0_1 *formula of* L_2.
- Δ^0_1 ***Comprehension:***

  $$\forall m[\varphi(m, \dots) \leftrightarrow \psi(m, \dots)] \rightarrow \exists X \forall m\, [m \in X \leftrightarrow \varphi(m, \dots)]$$

  where φ *is a* Σ^0_1 *formula of* L_2 *in which* X *does not occur, and* ψ *a* Π^0_1 *formula of* L_2 *in which* X *does not occur.*

In RCA_0 one can define the notion of a binary tree as a set of numbers which code finite sequences of 1's and 2's with the natural ordering, as well as the notion of an infinite branch of a tree.

The **Weak Koenig Lemma**, WKL, is the L_2 statement that every infinite binary tree X has an infinite branch. The theory WKL_0 is the defined as

$$\mathsf{WKL}_0 = \mathsf{RCA}_0 + \mathsf{WKL}.$$

In this paper we will obtain several conservation results in the following sense.

DEFINITION 2.3. *Let T be a theory in a language L and T' be a theory in a language $L' \supseteq L$. We say that T' is* ***conservative with respect to*** *T if every sentence of L which is provable from T' is provable from T.*

This gives an upper bound on the strength of T'; if T' is conservative with respect to T, then any weakening of T' is also conservative with respect to T. The following characterization follows easily from the Löwenheim-Skolem theorem.

PROPOSITION 2.4. *Suppose $L \subseteq L'$ and L' is countable. A theory T' in L' is conservative with respect to a theory T in L if and only if every countable model M of T has an elementary extension that can be expanded to a model of T'.*

We will need the following conservation results of Friedman and Harrington (see [Si1999], Chapter IX) which show that $\Sigma\mathsf{PA}$ is the first order part of both RCA_0 and WKL_0.

PROPOSITION 2.5. *(i) For any model $\mathcal{N}$ of $\Sigma\mathsf{PA}$ there is a model $\mathcal{M} = (\mathcal{N}, \mathcal{P})$ of RCA_0 with first order part $\mathcal{N}$.*

(ii) For any countable model $\mathcal{M} = (\mathcal{N}, \mathcal{P})$ of RCA_0 there is a countable model $\mathcal{M}' = (\mathcal{N}, \mathcal{P}')$ of WKL_0 with the same first order part $\mathcal{N}$ and with $\mathcal{P}' \supseteq \mathcal{P}$.

COROLLARY 2.6. *For any countable model $\mathcal{N}$ of $\Sigma\mathsf{PA}$ there is a countable model $\mathcal{M} = (\mathcal{N}, \mathcal{P})$ of WKL_0 with first order part $\mathcal{N}$.*

COROLLARY 2.7. WKL_0 *is conservative with respect to* $\Sigma\mathsf{PA}$*.*

§3. The theory $^*\Sigma\mathsf{PA}$. In this section we define a weak theory $^*\Sigma\mathsf{PA}$ of nonstandard arithmetic. The language $^*L_1$ has all the symbols of L_1 plus a new hyperinteger sort $^*N$ with variables $x, y, \ldots$. In sort $^*N$, $^*L_1$ has the symbols $=, <, 0, 1, +, \cdot, exp, p_y, (x)_y$ corresponding to the symbols of L_1. The sort N has the variables $k, m, n, p, q, \ldots$, and sort $^*N$ has the variables $u, v, w, x, y, z, \ldots$. The universe of sort N is to be interpreted as a subset of the universe of sort $^*N$. Terms built from variables of sort N are also of sort N. Variables and terms of sort N are allowed in argument places of sort $^*N$. Terms which contain at least one variable of sort $^*N$ are also of sort $^*N$. For example, $x + n$ is a term of sort $^*N$, and

$\exists n\, y < x + n$ is a formula. We introduce the predicate symbol S for the standard integers, and for each term t we write $S(t)$ for $\exists n\, n = t$.

We now build a hierarchy of formulas beginning with the stars of bounded quantifier formulas and applying quantifiers over variables of sort N.

DEFINITION 3.1. *In* $^*L_1$, *a* ***bounded quantifier*** *is an expression of the form* $(\forall s < t)$ *or* $(\exists s < t)$ *where* s *is a variable,* t *is a term, and if* s *has sort* N *then* t *has sort* N. *(Thus we do not count* $(\forall n < x)$ *as a bounded quantifier).*

An ***internal bounded quantifier formula, or*** Δ_0^S ***formula****, is a formula of* $^*L_1$ *built from atomic formulas using connectives and bounded quantifiers. We put* $\Pi_0^S = \Sigma_0^S = \Delta_0^S$, *and*

$$\Pi_{k+1}^S = \{\exists n\varphi : \varphi \in \Sigma_k^S\}, \qquad \Sigma_{k+1}^S = \{\forall n\varphi : \varphi \in \Pi_k^S\}.$$

A formula is S***-arithmetical*** *if it belongs to* $\bigcup_k \Sigma_k^S = \bigcup_k \Pi_k^S$.

In the above definition, The S-prefix indicates that the outer quantifiers are over standard integers.

DEFINITION 3.2. ***Axioms of*** $^*\Sigma\mathsf{PA}$.

- *The* ***Basic Axioms*** *of* $\Sigma\mathsf{PA}$, *but with variables of sort* $^*N$.
- ***Proper Initial Segment:***

$$\forall n\exists x(x = n),$$
$$\forall n\forall x[x < n \to S(x)],$$
$$\exists y\forall n(n < y).$$

- ***Internal Induction:***

$$[\varphi(0, \vec{u}) \wedge \forall x[\varphi(x, \vec{u}) \to \varphi(x + 1, \vec{u})]] \to \forall x\varphi(x, \vec{u})$$

  where $\varphi(x, \vec{u})$ *is a* Δ_0^S *formula.*
- ***Finiteness:***

$$\forall z[S((z)_0) \wedge \forall m[S((z)_m) \to S((z)_{m+1})] \to \forall mS((z)_m)].$$

We let $^*\Delta\mathsf{PA}$ *be the theory whose axioms are all the axioms of* $^*\Sigma\mathsf{PA}$ *except the Finiteness Axiom.*

An $^*L_1$ structure will be a structure of the form $(\mathcal{N}, {}^*\mathcal{N})$ where $\mathcal{N}$ is a substructure of $^*\mathcal{N}$. The usual rules for terms and equality hold. Thus for every term t of sort N we always have $\exists nt = n$, which we abbreviate as $S(t)$.

We rely heavily on the convention that $k, \dots, q$ are variables of sort N and $u, \dots, z$ are variables of sort $^*N$. For example, $^*\Sigma\mathsf{PA} \vdash \exists y\forall n(n < y)$ but $^*\Sigma\mathsf{PA} \vdash \neg\exists p\forall n(n < p)$. We say that x is **finite** if $S(x)$, and x is **infinite** otherwise. We sometimes use H, K for parameters of sort $^*N$ which are infinite.

We now establish some elementary facts in ${}^*\Delta\mathsf{PA}$. The following lemma (and its dual with existential quantifiers) will be used without explicit mention.

LEMMA 3.3. *For every formula $\varphi(x,\dots)$ of ${}^*L_1$ and term t of sort N,*
(i) ${}^*\Delta\mathsf{PA} \vdash (\forall x < t)\,\varphi(x,\dots) \leftrightarrow (\forall n < t)\varphi(n,\dots)$.
(ii) ${}^*\Delta\mathsf{PA} \vdash [\neg S(H) \wedge (\forall x < H)\varphi(x,\dots)] \rightarrow \forall n\varphi(n,\dots)$.

PROOF. This follows from the Basic and Proper Initial Segment Axioms. ⊣

LEMMA 3.4. (Δ^S_0*-comprehension) For each Δ^S_0 formula $\varphi(m,\vec{u})$ in which y does not occur,*

$${}^*\Delta\mathsf{PA} \vdash \exists y\forall m\,[(y)_m > 0 \leftrightarrow \varphi(m,\vec{u})].$$

PROOF. Work in ${}^*\Delta\mathsf{PA}$. Pick an infinite H. By Internal Induction, there exists $y < (p_H)^H$ such that $(\forall x < H)(y)_x < 2$ and

$$(\forall x < H)[(y)_x > 0 \leftrightarrow \varphi(x,\vec{u})].$$

Then

$$\forall m\,[(y)_m > 0 \leftrightarrow \varphi(m,\vec{u})].$$

⊣

LEMMA 3.5. (Δ^S_1*-comprehension) Let $\varphi(x,\vec{u})$ be a Σ^S_1 formula, and $\psi(x,\vec{u})$ be a Π^S_1 formula, in which y does not occur. Then*

$${}^*\Delta\mathsf{PA} \vdash \forall m\,[\varphi(m,\vec{u}) \leftrightarrow \psi(m,\vec{u})] \rightarrow \exists y\forall m\,[(y)_m > 0 \leftrightarrow \varphi(m,\vec{u})].$$

PROOF. Work in ${}^*\Delta\mathsf{PA}$. Let $\varphi(x,\vec{u})$ be $\exists k\varphi'(x,k,\vec{u})$ and $\psi(x,\vec{u})$ be $\forall k\psi'(x,k,\vec{u})$ where $\varphi',\psi' \in \Delta^S_0$. Pick an infinite H. Assume that

$$\forall m\,[\varphi(m,\vec{u}) \leftrightarrow \psi(m,\vec{u})].$$

Then

$$\forall m\,[\exists k\,\varphi'(m,k,\vec{u}) \leftrightarrow (\exists z < H)[\varphi'(m,z,\vec{u}) \wedge (\forall v \le z)\psi'(m,v,\vec{u})]].$$

Therefore by Δ^S_0-comprehension, there exists y such that

$$\forall m\,[(y)_m > 0 \leftrightarrow \exists k\,\varphi'(m,k,\vec{u})].$$

⊣

Given a Δ^S_0 formula $\varphi(x,\vec{u})$ of ${}^*L_1$, the **bounded minimum** operator $(\mu x < y)\,\varphi(x,\vec{u})$ equals the least $x < y$ such that $\varphi(x,\vec{u})$ if there is one, and equals y otherwise. The formal definition is

$$z = (\mu x < y)\,\varphi(x,\vec{u}) \leftrightarrow [(\forall x < z)\neg\varphi(x,\vec{u}) \wedge [(z < y \wedge \varphi(z,\vec{u})) \vee z = y]],$$

where z is a new variable. Note that if $\varphi(x,\vec{u})$ is a Δ^S_0 formula, then so is $z = (\mu x < y)\,\varphi(x,\vec{u})$.

LEMMA 3.6. *(i) For each Δ_0^S formula $\varphi(x, \vec{u})$,*

$$^*\Delta\mathsf{PA} \vdash \exists! z\, z = (\mu x < y)\varphi(x, \vec{u}).$$

(ii) For each Δ_0^S formula $\psi(v, x, \vec{u})$,

$$^*\Delta\mathsf{PA} \vdash \exists z (\forall v < w)(z)_v = (\mu x < y)\psi(v, x, \vec{u}).$$

PROOF. (i) Take a new variable v which does not occur in $\varphi(x, \vec{u})$ and prove

$$(\exists! z \leq v)\, z = (\mu x < v)[x \leq y \wedge \varphi(x, \vec{u})]$$

by internal induction on v. (It is easily seen that this is equivalent to a Δ_0^S formula.)

(ii) Prove by internal induction on w that

$$(p_w)^y < H \rightarrow (\exists z < H^w)(\forall v < w)(z)_v = (\mu x < y)\psi(v, x, \vec{u}),$$

and take H so that $(p_w)^y < H$. ⊣

This lemma allows us to treat $(\mu x < y)\varphi(x, \vec{u})$ as a term. If $\varphi(x, \vec{u})$ and $\psi(x, \vec{u})$ are Δ_0^S formulas, then $\psi((\mu x < y)\varphi(x, \vec{u}), \vec{u})$ is the Δ_0^S formula

$$(\exists z \leq y)\, [z = (\mu x < y)\varphi(x, \vec{u}) \wedge \psi(z, \vec{u})].$$

We may use the bounded minimum operator to introduce new notation in the usual way. For example, we write $y \restriction w$ for

$$(\mu x < y)(\forall v < w)\, (x)_v = (y)_v.$$

$y \restriction w$ is the code of the first w terms of the sequence coded by y, and for each Δ_0^S formula $\psi(x, \vec{u})$, $\psi(y \restriction w, \vec{u})$ is the Δ_0^S formula

$$(\exists z \leq y)\, [z = y \restriction w \wedge \psi(z, \vec{u})].$$

We use the vector notation $\vec{x} = \vec{y} \restriction w$ to mean that $x_i = y_i \restriction w$ for each i.

Let us write $\forall^\infty x\, \varphi(x, \vec{u})$ for $\forall x[\neg S(x) \rightarrow \varphi(x, \vec{u})]$ and $\exists^\infty x\, \varphi(x, \vec{u})$ for $\exists x\, [\neg S(x) \wedge \varphi(x, \vec{u})]$.

LEMMA 3.7. *(Overspill) Let $\varphi(x, \vec{u})$ be a Δ_0^S formula. In $^*\Delta\mathsf{PA}$, $\forall n \varphi(n, \vec{u}) \rightarrow \exists^\infty x\, \varphi(x, \vec{u})$, and $\forall^\infty x\, \varphi(x, \vec{u}) \rightarrow \exists n \varphi(n, \vec{u})$.*

PROOF. We prove the first statement. Assume $\forall n \varphi(n, \vec{u})$. Pick an infinite H. If $\varphi(H, \vec{u})$ we may take $x = H$. Assume $\neg\varphi(H, \vec{u})$. By Lemma 3.6 we may take $z = (\mu y < H)\, \neg\varphi(y, \vec{u})$. Then $\neg S(z)$. Let $x = z - 1$. We have $x < z$, so $\varphi(x, \vec{u})$. But S is closed under the successor function, so $\neg S(x)$. ⊣

LEMMA 3.8. *(Internal Induction in S) Let $\varphi(x, \vec{u})$ be a Δ_0^S formula. In $^*\Delta\mathsf{PA}$,*

$$[\varphi(0, \vec{u}) \wedge \forall m[\varphi(m, \vec{u}) \rightarrow \varphi(m + 1, \vec{u})]] \rightarrow \forall m\, \varphi(m, \vec{u}).$$

Proof. Assume $\varphi(0,\vec{u}) \wedge \forall m[\varphi(m,\vec{u}) \rightarrow \varphi(m+1,\vec{u})]$. By internal induction, for each k we have $\forall x[x < k \rightarrow \varphi(x,\vec{u})]$. Therefore $\forall m\, \varphi(m,\vec{u})$. ⊣

Proposition 3.9. *(Σ_1^S Induction in S) In* $^*\Delta\mathsf{PA}$*, the Finiteness Axiom is equivalent to the* Σ_1^S *Induction scheme*

$$\varphi(0,\vec{u}) \wedge \forall m[\varphi(m,\vec{u}) \rightarrow \varphi(m+1,\vec{u})] \rightarrow \forall m \varphi(m,\vec{u}) \tag{1}$$

where $\varphi(m,\vec{u})$ *is a* Σ_1^S *formula.*

Proof. The Finiteness Axiom follows from (1) where $\varphi(m,z)$ is the formula $\exists n\, n = (z)_m$. Let $\varphi(m,\vec{u})$ be $\exists n \psi(m,n,\vec{u})$ where ψ is a Δ_0^S formula. Pick an infinite H. By Lemma 3.6 we may take z such that

$$(\forall x < H)\, (z)_x = (\mu y < H)\psi(x,y,\vec{u}).$$

Then for all m, we have

$$\exists n\, \psi(m,n,\vec{u}) \leftrightarrow S((z)_m).$$

The formula (1) now follows from the Finiteness Axiom. ⊣

Theorem 3.10. $^*\Sigma\mathsf{PA} \vdash \Sigma\mathsf{PA}$.

Proof. The basic axioms of $\Sigma\mathsf{PA}$ follow from the basic axioms of $^*\Sigma\mathsf{PA}$. The Σ_1^0-Induction scheme follows from the Σ_1^S-Induction scheme of Proposition 3.9 with the parameters $\vec{u}$ in S. ⊣

In the next result, $S(y \restriction k)$ denotes the Σ_1^S formula $\exists p\, (p = y \restriction k)$.

Proposition 3.11.

$$^*\Sigma\mathsf{PA} \vdash\ S(y \restriction n) \leftrightarrow (\forall m < n)\, S((y)_m).$$

Proof. Work in $^*\Sigma\mathsf{PA}$. If $S(y \restriction n)$ and $m < n$, then $(y)_m \leq y \restriction n$, so $S(y_m)$ by the Proper Initial Segment Axiom. For the converse, assume $(\forall m < n)\, S((y)_m)$. We prove by Σ_1^S induction in S (Lemma 3.9) that for all m,

$$m \leq n \rightarrow S(y \restriction m).$$

It is trivial that $S(y \restriction 0)$. Assume $m \leq n \rightarrow S(y \restriction m)$. If $m \geq n$ then $m+1 \leq n \rightarrow S(y \restriction m+1)$ is trivially true. Suppose $m < n$. Then $S(y \restriction m)$, $S(p_m)$, and $S((y)_m)$. We have

$$y \restriction (m+1) = (y \restriction m) \cdot p_m^{(y)_m},$$

so $S(y \restriction (m+1))$. This completes the induction. ⊣

§4. Standard Parts. We now combine the languages L_2 and ${}^*L_1$ into a common language $L_2 \cup {}^*L_1$, and introduce the notions of a standard set and a standard function. They will provide a link between hyperintegers and sets. The language $L_2 \cup {}^*L_1$ has an integer sort N, a set sort P, and a hyperinteger sort ${}^*N$. $L_2 \cup {}^*L_1$ has all the symbols of L_2 and ${}^*L_1$. In this language, it will make sense to ask whether a formula of ${}^*L_1$ implies a formula of L_2.

An $L_2 \cup {}^*L_1$ structure will have the form $(\mathcal{M}, {}^*\mathcal{N})$ where $\mathcal{M} = (\mathcal{N}, \mathcal{P})$ is an L_2 structure and $(\mathcal{N}, {}^*\mathcal{N})$ is a ${}^*L_1$ structure.

We first define the notion of a standard set, which formalizes a construction commonly used in nonstandard analysis. In the following we work in the language $L_2 \cup {}^*L_1$ and assume the axioms of ${}^*\Delta\mathsf{PA}$.

DEFINITION 4.1. *We say that X is the **standard set of** x and that x is a **lifting of** X, and write $X = st(x)$, if $\forall n\, [n \in X \leftrightarrow (x)_n > 0]$. $\vec{X} = st(\vec{x})$ means that $X_i = st(x_i)$ for each i.*

Thus $X = st(x)$ means that X is the set of all finite n such that p_n divides x. We now introduce the Standard Part Principle, which says that every set has a lifting, and every hyperinteger has a standard set. Later on we will introduce several theories that have the Standard Part Principle as an axiom.

DEFINITION 4.2. *The **Upward Standard Part Principle** is the statement that every set has a lifting, formally, $\forall X \exists x\, X = st(x)$.*

*The **Downward Standard Part Principle** is the statement that every hyperinteger has a standard set, formally, $\forall x \exists X\, X = st(x)$.*

*The **Standard Part Principle** (STP) is the conjunction of the Upward and the Downward Standard Part Principles.*

In nonstandard analysis, STP often allows one to obtain results about sets of type $P(N)$ by reasoning about hyperintegers of type ${}^*N$.

The STP is related to H. Friedman's notion of a standard system, as generalized by Enayat [En2005]. Given a model $(\mathcal{N}, {}^*\mathcal{N})$ of ${}^*\Delta\mathsf{PA}$, Enayat defined the **standard system of** ${}^*\mathcal{N}$ **relative to** $\mathcal{N}$ by

$$SSy_{\mathcal{N}}({}^*\mathcal{N}) = \{st(x) : x \in {}^*\mathcal{N}\}.$$

In an $L_2 \cup {}^*L_1$-structure $(\mathcal{N}, \mathcal{P}, {}^*\mathcal{N})$,
the STP says that $\mathcal{P} = SSy_{\mathcal{N}}({}^*\mathcal{N})$,
the Upward STP says that $\mathcal{P} \subseteq SSy_{\mathcal{N}}({}^*\mathcal{N})$,
the Downward STP says that $\mathcal{P} \supseteq SSy_{\mathcal{N}}({}^*\mathcal{N})$.

The Standard Part Principles may also be formulated using functions instead of sets. Let us say that X is a **total function** if $\forall m \exists! n\, (m, n) \in X$. We let $f, g, \ldots$ range over total functions and write $f(m) = n$ instead of $(m, n) \in f$.

We say that x is **near-standard**, in symbols $ns(x)$, if $\forall n\, S((x)_n)$. Note that $ns(x)$ is a Π^S_2 formula. We employ the usual convention for relativized quantifiers, so that $\forall^{ns} x\, \psi$ means $\forall x[ns(x) \rightarrow \psi]$ and $\exists^{ns} x\, \psi$ means $\exists x[ns(x) \wedge \psi]$. We write

$$x \approx y \text{ if } ns(x) \wedge \forall n\, (x)_n = (y)_n.$$

We write $f = {}^o x$, and say f is the **standard function of** x and that x is a **lifting of** f, if

$$ns(x) \wedge \forall n f(n) = (x)_n.$$

Thus $st(x)$ is a set and ${}^o x$ is a function. The following is easily checked.

Proposition 4.3. *Assume the axioms of* $^*\Delta\mathsf{PA}$ *and that a set exists if and only if its characteristic function exists.*

(i) The Upward STP *holds if and only if* $\forall f \exists x\, f = {}^o x$, *that is, every total function has a lifting.*

(ii) The Downward STP *holds if and only if* $\forall^{ns} x \exists f\, f = {}^o x$, *that is, every near-standard hyperinteger has a standard function.*

We next show that for liftings of functions one can restrict attention to hyperintegers less than a given infinite hyperinteger H.

Lemma 4.4. *In* $^*\Delta\mathsf{PA}$, *suppose that* x *is near-standard and* H *is infinite. Then*

(i) If $x \approx y$ *then* $ns(y)$ *and* $y \approx x$.

(ii) $(\exists y < H)\, x \approx y$.

Proof. (i) Suppose $x \approx y$. We have $ns(x)$, so for each n, $S((x)_n)$ and $(y)_n = (x)_n$, and hence $S((y)_n)$. Therefore $ns(y)$, and $y \approx x$ follows trivially.

(ii) By Overspill there is an infinite K such that $K^K < H$. By Lemma 3.8,

$$\forall n\, [n < K \wedge (\exists y < K^n)(\forall m < n)\, (y)_m = (x)_m].$$

By Overspill,

$$\exists^\infty u\, [u < K \wedge (\exists y < K^u)(\forall v < u)\, (y)_v = (x)_v].$$

Then $y < H \wedge y \approx x$. ⊣

We now define a lifting map from formulas of L_2 to formulas of $^*L_1$.

Definition 4.5. *Let* $\varphi(\vec{m}, \vec{X})$ *be a formula in* L_2, *where* $\vec{m}, \vec{X}$ *contain all the variables of* φ, *both free and bound. The* ***lifting*** $\overline{\varphi}(\vec{m}, \vec{x})$ *is defined as follows, where* $\vec{x}$ *is a tuple of variables of sort* $^*N$ *of the same length as* $\vec{X}$.

- *Replace each subformula* $t \in X_i$, *where* t *is a term, by* $(x_i)_t > 0$.
- *Replace each quantifier* $\forall X_i$ *by* $\forall x_i$, *and similarly for* $\exists$.

It is clear that if φ is a Δ^0_0 formula of L_2, then $\overline{\varphi}$ is a Δ^S_0 formula of ${}^*L_1$, and if φ is an arithmetical formula then $\overline{\varphi}$ is an S-arithmetical formula. The following lemma on liftings of formulas will be used many times.

LEMMA 4.6. *(i) For each arithmetical formula* $\varphi(\vec{m},\vec{X})$ *of* L_2*,*

$$^*\mathsf{\Delta PA} \vdash st(\vec{x}) = \vec{X} \to [\varphi(\vec{m},\vec{X}) \leftrightarrow \overline{\varphi}(\vec{m},\vec{x})].$$

(ii) For each formula $\varphi(\vec{m},\vec{X})$ *of* L_2*,*

$$^*\mathsf{\Delta PA} + \mathsf{STP} \vdash st(\vec{x}) = \vec{X} \to [\varphi(\vec{m},\vec{X}) \leftrightarrow \overline{\varphi}(\vec{m},\vec{x})].$$

PROOF. In the case that φ is atomic, the lemma follows from the definitions involved. The general case is then proved by induction on the complexity of φ, using STP at the quantifier steps in part (ii). ⊣

§5. The theory $^*\mathsf{WKL_0}$**.** In this section we define the theory $^*\mathsf{WKL_0}$ in the language $L_2 \cup {}^*L_1$, and show that $^*\mathsf{WKL_0}$ implies WKL_0 and $^*\mathsf{WKL_0}$ is conservative with respect to WKL_0. A consequence of this result is that $^*\Sigma\mathsf{PA}$ is conservative with respect to $\Sigma\mathsf{PA}$.

DEFINITION 5.1. *In the language* $L_2 \cup {}^*L_1$*, the theory* $^*\mathsf{WKL_0}$ *is defined by*

$$^*\mathsf{WKL_0} = {}^*\Sigma\mathsf{PA} + \mathsf{STP}.$$

PROPOSITION 5.2. *Every model of* $^*\Sigma\mathsf{PA}$ *has a unique expansion to a model of* $^*\mathsf{WKL_0}$*.*

PROOF. Given a model $(\mathcal{N}, {}^*\mathcal{N})$ of $^*\Sigma\mathsf{PA}$, the unique expansion to a model of $^*\mathsf{WKL_0}$ is obtained by taking $\mathcal{P} = \{st(x) : x \in {}^*\mathcal{N}\}$. ⊣

PROPOSITION 5.3. *Let* $(\mathcal{N}, \mathcal{N}')$ *be a model of* $^*\Sigma\mathsf{PA}$ *and let* $^*\mathcal{N}$ *be an end extension of* $\mathcal{N}'$ *which satisfies Internal Induction.*

(i) $(\mathcal{N}, {}^*\mathcal{N})$ *is a model of* $^*\Sigma\mathsf{PA}$*.*

(ii) If $(\mathcal{M}, \mathcal{N}')$ *is a model of* $^*\mathsf{WKL_0}$*, then* $(\mathcal{M}, {}^*\mathcal{N})$ *is a model of* $^*\mathsf{WKL_0}$*.*

PROOF. (i) It is clear that the axioms of $^*\mathsf{\Delta PA}$ hold in $(\mathcal{N}, {}^*\mathcal{N})$. The Finiteness Axiom in $(\mathcal{N}, {}^*\mathcal{N})$ follows from Lemma 4.4 and the Finiteness Axiom in $(\mathcal{N}, \mathcal{N}')$.

(ii) STP in $(\mathcal{N}, {}^*\mathcal{N})$ follows from Lemma 4.4 and STP in $(\mathcal{N}, \mathcal{N}')$. ⊣

THEOREM 5.4. $^*\mathsf{WKL_0} \vdash \mathsf{WKL}_0$*.*

PROOF. Work in $^*\mathsf{WKL_0}$. We first prove Σ^0_1 Induction. For future reference, we note that this part of the proof will not use the Downward STP.

Let $\psi(m, \vec{n}, \vec{X})$ be a Σ^0_1 formula of L_2. Suppose that

$$\psi(0, \vec{n}, \vec{X}) \wedge \forall m[\psi(m, \vec{n}, \vec{X}) \to \psi(m+1, \vec{n}, \vec{X})].$$

By the Upward STP the tuple $\vec{X}$ has a lifting $\vec{x}$. By Lemma 4.6 (i),

$$\overline{\psi}(0,\vec{n},\vec{x}) \wedge \forall m\,[\overline{\psi}(m,\vec{n},\vec{x}) \rightarrow \overline{\psi}(m+1,\vec{n},\vec{x})].$$

Then by Proposition 3.9, $\forall m \overline{\psi}(m,\vec{n},\vec{x})$. By Lemma 4.6 (i) again, we have $\forall m \psi(m,\vec{n},\vec{X})$.

We next prove Δ^0_1-comprehension. Assume that

$$\forall m[\exists k \varphi(m,k,\vec{Y}) \leftrightarrow \forall k \psi(m,k,\vec{Y})]$$

where φ and ψ are Δ^0_0 formulas. By the Upward STP there is a lifting $\vec{y}$ of $\vec{Y}$. By Lemma 4.6,

$$\forall m\,[\exists k\,\overline{\varphi}(m,k,\vec{y}) \leftrightarrow \forall k\,\overline{\psi}(m,k,\vec{y})].$$

By Δ^S_1-comprehension (Lemma 3.5), there exists x such that

$$\forall m\,[(x)_m > 0 \leftrightarrow \exists k\,\overline{\varphi}(m,k,\vec{y})].$$

By the Downward STP there is a set $X = st(x)$. By Lemma 4.6,

$$\forall m[m \in X \leftrightarrow \exists k \varphi(m,k,\vec{Y})].$$

Finally, we prove WKL. Let $\psi(n)$ be the formula

$$(\forall m < n)[(n)_m < 3 \wedge (\forall k < m)[(n)_k = 0 \rightarrow (n)_m = 0]].$$

$\psi(n)$ says that n codes a finite sequence of 1's and 2's. Write $m \lhd n$ if

$$\psi(m) \wedge \psi(n) \wedge (\exists k < n) m = n \restriction k].$$

This says the sequence coded by m is an initial segment of the sequence coded by n. $\psi(n)$ and $m \lhd n$ are Δ^0_0 formulas, and their stars ${}^*\psi(u)$ and $v \,{}^*\!\lhd\, u$ are in Δ^S_0. Suppose that T codes an infinite binary tree, that is,

$$\forall m \exists n[m < n \wedge n \in T]$$

and

$$\forall n[n \in T \rightarrow \psi(n) \wedge (\forall m < n)[m \lhd n \rightarrow m \in T]].$$

By the axioms of RCA_0 (already proved), there is a function f such that for each k, $f(k)$ is the k-th element of T. Then

$$\forall k\,[k \leq f(k) \wedge f(k) \in T].$$

By STP, T has a lifting x and f has a lifting y. Then $\forall k\, S((y)_k)$. By Lemma 4.6,

$$\forall n[(x)_n > 0 \rightarrow {}^*\psi(n) \wedge (\forall m < n)[m\, {}^*\!\lhd n \rightarrow (x)_m > 0]]$$

and

$$\forall k\,[k \leq (y)_k \wedge (x)_{(y)_k} > 0].$$

Therefore

$\forall k\,[k \leq t \wedge (x)_t > 0 \wedge {}^*\psi(t) \wedge (\forall m < t)[m\, {}^*\!\lhd\, t \rightarrow (x)_m > 0]]$ where $t = (y)_k$.

By Overspill,

$\exists^\infty w\,[w \le t \wedge (x)_t > 0 \wedge {}^*\psi(t) \wedge (\forall u < t)[u\, {}^*\!\lhd\, t \to (x)_u > 0]]$ where $t = (y)_w$.

Then $ns(t)$, and by STP there exists $g = {}^o t$. It follows that g codes an infinite sequence of 1's and 2's, and each finite initial segment of g belongs to T. Thus g codes an infinite branch of T. ⊣

We now use a result of Tanaka [Ta1997] to show that $^*\mathsf{WKL}_0$ is conservative with respect to WKL_0.

DEFINITION 5.5. *By an* ω***-model*** *we mean an* L_2*-structure* $\mathcal{M} = (\mathcal{N}, \mathcal{P})$ *such that* $\mathcal{N}$ *is the standard model of arithmetic.*

THEOREM 5.6. *(Tanaka [Ta1997]). For every countable model* $\mathcal{M} = (\mathcal{N}, \mathcal{P})$ *of* WKL_0 *which is not an* ω*-model, there is a model* $\mathcal{M}_0 = (\mathcal{N}_0, \mathcal{P}_0)$ *such that* $\mathcal{M}_0 \cong \mathcal{M}$*,* $\mathcal{N}$ *is a proper end extension of* $\mathcal{N}_0$*, and* $\mathcal{P}_0 = \{X \cap \mathcal{N}_0 : X \in \mathcal{P}\}$*.*

THEOREM 5.7. *Every countable model* $\mathcal{M}$ *of* WKL_0 *which is not an* ω*-model can be expanded to a countable model* $(\mathcal{M}, {}^*\mathcal{N})$ *of* $^*\mathsf{WKL}_0$ *such that* $^*\mathcal{N} \cong \mathcal{N}$*.*

PROOF. By Theorem 5.6, there is a model $\mathcal{M}_1 = (\mathcal{N}_1, \mathcal{P}_1)$ such that $\mathcal{M}_1 \cong \mathcal{M}$, $\mathcal{N}_1$ is a proper end extension of $\mathcal{N}$, and $\mathcal{P} = \{X \cap \mathcal{N} : X \in \mathcal{P}_1\}$. Then $\mathcal{N}_1 \cong \mathcal{N}$. $(\mathcal{M}, \mathcal{N}_1)$ clearly satisfies all the axioms of $^*\mathsf{WKL}_0$ except possibly the Finiteness Axiom and STP.

Proof of STP: Let $x \in \mathcal{N}_1$ and let $X_1 = \{y \in \mathcal{N}_1 : (x)_y > 0\}$. Then $X_1 \in \mathcal{P}_1$ and $st(x) = X_1 \cap \mathcal{N} \in \mathcal{P}$. Now let $X \in \mathcal{P}$. Then $X = X_1 \cap \mathcal{N}$ for some $X_1 \in \mathcal{P}_1$. Pick $y \in \mathcal{N}_1 \setminus \mathcal{N}$. There exists $x \in \mathcal{N}_1$ such that $(\forall z < y)[(x)_z > 0 \leftrightarrow z \in X_1]$. Then x is a lifting of X.

Proof of the Finiteness Axiom: Let $z \in \mathcal{N}_1$. Suppose that $S((z)_0)$ and

$$\forall m[S((z)_m) \to S((z)_{m+1})].$$

The set $Y = \{(m, n) \in \mathcal{N}^2 : (z)_m = n\}$ belongs to $\mathcal{P}$, and using Σ^0_1 Induction with the formula $\exists n(m, n) \in Y$ we conclude that $\forall m S((z)_m)$. ⊣

COROLLARY 5.8. $^*\mathsf{WKL}_0$ *is conservative with respect to* WKL_0*.*

COROLLARY 5.9. *Every countable nonstandard model* $\mathcal{N}$ *of* $\Sigma\mathsf{PA}$ *can be expanded to a countable model* $(\mathcal{N}, {}^*\mathcal{N})$ *of* $^*\Sigma\mathsf{PA}$ *such that* $^*\mathcal{N} \cong \mathcal{N}$*.* $^*\Sigma\mathsf{PA}$ *is conservative with respect to* $\Sigma\mathsf{PA}$*.*

§6. The Theory $^*\mathsf{RCA}_0$**.** In this section we introduce the theory $^*\mathsf{RCA}_0$ in the language $L_2 \cup {}^*L_1$, and show that $^*\mathsf{RCA}_0$ implies and is conservative with respect to the base theory RCA_0 of reverse mathematics. $^*\mathsf{RCA}_0$ will contain the axioms of $^*\Sigma\mathsf{PA}$, the Upward STP, and a weakening of the

Downward STP which asserts that certain hyperintegers have standard sets.

We begin with a definability notion which is expressible in the language ${}^*L_1$. We say that x is **setlike** if $\forall m\,(x)_m < 2$.

DEFINITION 6.1. *Suppose that x and each y_i is setlike. x is **Δ^0_1-definable from** $\vec{y}$ if there are a Σ^0_1 formula $\varphi(k,m,\vec{p})$ and a Π^0_1 formula $\psi(k,m,\vec{p})$ of L_1 such that*

$$\forall m[\exists k\,\varphi(k,m,\vec{y}\restriction k) \leftrightarrow \forall k\,\psi(k,m,\vec{y}\restriction k)]$$

and

$$\forall m[(x)_m > 0 \leftrightarrow \exists k\,\varphi(k,m,\vec{y}\restriction k)].$$

DEFINITION 6.2. *The theory ${}^*\mathsf{RCA}_0$ has the following axioms:*

- *The axioms of ${}^*\Sigma\mathsf{PA}$,*
- *The Upward STP, $\forall X \exists x\,(X = st(x))$,*
- *Each constant function exists, $\forall n \exists f \forall k f(k) = n$.*
- *Δ^0_1-STP: If x and each y_i is setlike, x is Δ^0_1-definable from $\vec{y}$, and $\exists \vec{Y}(\vec{Y} = st(\vec{y}))$, then $\exists X(X = st(x))$.*

LEMMA 6.3. *(i)*

$${}^*\Delta\mathsf{PA} \vdash \forall v \exists x\,[x \text{ is setlike } \wedge \forall m\,[(x)_m > 0 \leftrightarrow (v)_m > 0]].$$

(ii) ${}^*\mathsf{RCA}_0 \vdash \forall X\,[X$ has a setlike lifting$]$.
(iii) ${}^*\mathsf{RCA}_0 \vdash \forall^\infty H \forall X\,[X$ has a setlike lifting $y < H]$.

PROOF. (i) By internal induction, there exists $x \le v$ such that $(\forall u < v)\,x_u = min((v)_u, 1)$.

(ii) By the Upward STP there is a lifting v of X. Then by (i), there is a setlike x such that $st(x) = st(v) = X$.

(iii) Take an infinite H and a set X. By (ii), X has a setlike lifting x. Then x is near-standard, and by Lemma 4.4 there exists $y \approx x$ such that $y < H$. y is a setlike lifting of X. ⊣

We need the following normal form theorem for Σ^0_1 formulas in ${}^*\Sigma\mathsf{PA}$.

LEMMA 6.4. *For each Σ^0_1 formula $\varphi(\vec{m}, \vec{Y})$ in L_2, there is a Δ^0_0 formula $\theta(k, \vec{m}, \vec{r})$ in L_1 such that*

$${}^*\Sigma\mathsf{PA} \vdash \overline{\varphi}(\vec{m}, \vec{y}) \leftrightarrow \exists k \theta(k, \vec{m}, \vec{y}\restriction k).$$

PROOF. We have $\overline{\varphi}(\vec{m}, \vec{y}) = \exists n \psi(n, \vec{m}, \vec{y})$ where ψ is Δ^S_0 and each y_i occurs only in subformulas of the form $(y_i)_t > 0$ where t is a term of sort N. Let $\vec{p}$ be a tuple containing all free and bound variables of sort N in ψ, and let T be the finite set of all terms $t(\vec{p})$ of sort N which occur in

ψ. Let $\vec{m} = \langle m_0, \ldots, m_j \rangle$ and let $|\vec{q}| = |\vec{p}|$. Then the result holds where $\theta(k, \vec{m}, \vec{y} \upharpoonright k)$ is

$$(\exists n < k)\,[\psi(n, \vec{m}, \vec{y} \upharpoonright k) \wedge (\forall \vec{q} < n + m_0 + \ldots + m_j) \bigwedge_{t \in T} t(\vec{q}) < k].$$

$\dashv$

THEOREM 6.5. $^*\mathsf{RCA_0} \vdash \mathsf{RCA_0}$.

PROOF. Work in $^*\mathsf{RCA_0}$. We have already shown in the proof of Theorem 5.4 that Σ^0_1-induction follows from the axioms of $^*\mathsf{RCA_0}$. We prove Δ^0_1-comprehension. The idea is to use Δ^S_1-comprehension in $^*\Delta\mathsf{PA}$ to get a hyperinteger x, and then use the Δ^0_1-STP axiom to get a set $X = st(x)$. Assume that

$$\forall m[\varphi(m, \vec{Y}) \leftrightarrow \psi(m, \vec{Y})]$$

where $\varphi \in \Sigma^0_1$ and $\psi \in \Pi^0_1$. We may suppose that there are no parameters of sort N, because such parameters can replaced by the corresponding constant functions and included in $\vec{Y}$. By Lemma 6.3, $\vec{Y}$ has a setlike lifting $\vec{y}$. By Lemma 4.6,

$$\varphi(m, \vec{Y}) \leftrightarrow \overline{\varphi}(m, \vec{y}), \quad \psi(m, \vec{Y}) \leftrightarrow \overline{\psi}(m, \vec{y}).$$

By Lemma 6.4, there are Δ^0_0 formulas $\alpha(k, m, \vec{n}), \beta(k, m, \vec{n})$ in L_1 such that

$$\overline{\varphi}(m, \vec{y}) \leftrightarrow \exists k\, \alpha(k, m, \vec{y} \upharpoonright k),$$
$$\overline{\psi}(m, \vec{y}) \leftrightarrow \forall k\, \beta(k, m, \vec{y} \upharpoonright k).$$

Then

$$\forall m\, [\exists k\, \alpha(k, m, \vec{y} \upharpoonright k) \leftrightarrow \forall k\, \beta(k, m, \vec{y} \upharpoonright k)].$$

Since $\vec{y}$ is setlike, $\forall k\, S(\vec{y} \upharpoonright k)$, and therefore $\exists k\, \alpha(k, m, \vec{y} \upharpoonright k)$ is Σ^S_1 and $\forall k\, \beta(k, m, \vec{y} \upharpoonright k)$ is Π^S_1. By Δ^S_1-comprehension (Lemma 3.5) there exists x such that

$$\forall m\, [(x)_m > 0 \leftrightarrow \exists k\, \alpha(k, m, \vec{y} \upharpoonright k)].$$

By Lemma 6.3 (i), we may take x to be setlike. This shows that x is Δ^0_1-definable from $\vec{y}$. By Δ^0_1-STP, there exists X with $X = st(x)$. Since

$$\varphi(m, \vec{Y}) \leftrightarrow \exists k\, \alpha(k, m, \vec{y} \upharpoonright k)$$

we have

$$\forall m\, [m \in X \leftrightarrow \varphi(m, \vec{Y})].$$

This proves Δ^0_1-CA. $\dashv$

THEOREM 6.6. *Every countable model* $\mathcal{M} = (\mathcal{N}, \mathcal{P})$ *of* $\mathsf{RCA_0}$ *which is not an* ω*-model can be expanded to a countable model* $(\mathcal{M}, {}^*\mathcal{N})$ *of* $^*\mathsf{RCA_0}$ *such that* $^*\mathcal{N} \cong \mathcal{N}$.

Proof. By Proposition 2.5, there is a countable model $\mathcal{M}' = (\mathcal{N}, \mathcal{P}')$ of WKL_0 with the same first order part $\mathcal{N}$ and with $\mathcal{P}' \supseteq \mathcal{P}$. By Theorem 5.7, $\mathcal{M}'$ can be expanded to a countable model $(\mathcal{M}', {}^*\mathcal{N})$ of ${}^*\mathsf{WKL_0}$ such that ${}^*\mathcal{N} \cong \mathcal{N}$.

We work in $(\mathcal{M}, {}^*\mathcal{N})$ and show that all the axioms of ${}^*\mathsf{RCA_0}$ hold. Since $\mathcal{P}' \supseteq \mathcal{P}$, the Upward STP holds. It remains to prove the Δ^0_1-STP. Suppose x and each y_i is setlike, $\vec{Y} = st(\vec{y})$, and x is Δ^0_1-definable from $\vec{y}$. This means that there are a Σ^0_1 formula $\varphi(k, m, \vec{n})$ and a Π^0_1 formula $\psi(k, m, \vec{n})$ of L_1 such that for each m the formulas

$$\exists k \varphi(k, m, \vec{y} \restriction k), \quad \forall k \psi(k, m, \vec{y} \restriction k), \quad \text{and} \quad (x)_m > 0 \tag{2}$$

are equivalent. There is a Δ^0_0 formula $q = Y \restriction k$ of L_2 which says that q is the integer which codes the set $\{j < k : j \in Y\}$. Then $\exists k \varphi(k, m, \vec{Y} \restriction k)$ is equivalent to a Σ^0_1 formula of L_2, and $\forall k \psi(k, m, \vec{Y} \restriction k)$ is equivalent to a Π^0_1 formula of L_2.

By Lemma 4.6, $\vec{Y} \restriction k = \vec{y} \restriction k$, and it follows that the formulas (2) are equivalent to $\exists k \varphi(k, m, \vec{Y} \restriction k)$ and to $\forall k \psi(k, m, \vec{Y} \restriction k)$. Since $\mathcal{M}$ satisfies Δ^0_1-comprehension, there is a set X such that

$$\forall m\, [m \in X \leftrightarrow \exists k \varphi(k, m, \vec{Y} \restriction k)],$$

and therefore $X = st(x)$. ⊣

Corollary 6.7. ${}^*\mathsf{RCA_0}$ *is conservative with respect to* RCA_0.

§7. The Theory ${}^*\mathsf{ACA_0}$.

In this section we will find a nonstandard counterpart of the theory ACA_0 of arithmetical comprehension.

Given a class Γ of formulas in L_2, **Γ-comprehension** (Γ-CA) is the scheme

$$\exists Y \forall m\, [m \in Y \leftrightarrow \varphi(m, \dots)]$$

for all formulas $\varphi \in \Gamma$ in which Y does not occur.

Arithmetical Comprehension (ACA) is Γ-CA where Γ is the class of arithmetical formulas of L_2. The theory ACA_0 is defined by

$$\mathsf{ACA}_0 = \mathsf{WKL}_0 + \mathsf{ACA}.$$

It is well-known that the axioms of Peano arithmetic (PA) follow from ACA_0 (see [Si1999], page 7). The following result of Friedman and Harrington (see [Si1999] Theorem IX.1.5) is analogous to Proposition 2.5 (i) and shows that PA is the first order part of ACA_0.

Proposition 7.1. *For any model $\mathcal{N}$ of* PA *there is a model $\mathcal{M} = (\mathcal{N}, \mathcal{P})$ of* ACA_0 *with first order part $\mathcal{N}$.*

We now introduce a scheme corresponding to arithmetical comprehension in the language ${}^*L_1$ of nonstandard arithmetic.

DEFINITION 7.2. *Given a class* Γ *of formulas of* ${}^*L_1$, Γ***-comprehension*** (Γ-CA) *is the scheme*

$$\exists y \forall m\, [(y)_m > 0 \leftrightarrow \varphi(m, \dots)],$$

for all formulas $\varphi(m, \dots) \in \Gamma$ *in which* y *does not occur.*

Recall that in Lemma 3.4 we proved Δ_0^S comprehension in ${}^*\Delta\mathsf{PA}$.

DEFINITION 7.3. S***-arithmetical comprehension*** (S-ACA) *is* Γ-CA *where* Γ *is the set of* S*-arithmetical formulas. The theory* ${}^*\mathsf{ACA_0}$ *is defined by*

$${}^*\mathsf{ACA_0} = {}^*\mathsf{WKL_0} + S\text{-}\mathsf{ACA}.$$

The next result is the analogue of the fact that ACA is equivalent to Σ_1^0-CA in RCA_0 ([Si1999], page 105).

PROPOSITION 7.4. *In* ${}^*\Sigma\mathsf{PA}$, S-ACA *is equivalent to* Σ_1^S-CA*.*

PROOF. Work in ${}^*\Sigma\mathsf{PA}$ and assume Σ_1^S-CA. We prove Σ_k^S-CA by induction on k. This is trivial for $k = 1$. Suppose $k \geq 1$ and $\varphi(m, \vec{u}) \in \Sigma_{k+1}^S$, and assume Σ_k^S-CA. Then $\varphi(m, \vec{u}) = \exists n \psi(m, n, \vec{u})$ where $\psi \in \Pi_k^S$. By Σ_k^S-CA,

$$\exists y \forall m \forall n [(y)_{(m,n)} > 0 \leftrightarrow \neg\psi(m, n, \vec{u})].$$

Then by Σ_1^S-CA,

$$\exists x \forall m [(x)_m > 0 \leftrightarrow \exists n (y)_{(m,n)} = 0].$$

Therefore

$$\forall m [(x)_m > 0 \leftrightarrow \exists n \psi(m, n, \vec{u})].$$

$\dashv$

Here is a functional version of S-ACA.

PROPOSITION 7.5. *In* ${}^*\Sigma\mathsf{PA}$, S-ACA *is equivalent to the following scheme:*

$$\forall m \exists n\, \psi(m, n, \dots) \to \exists^{ns} z \forall m\, \psi(m, (z)_m, \dots) \tag{3}$$

for each S*-arithmetical formula* ψ*.*

PROOF. Work in ${}^*\Sigma\mathsf{PA}$. First assume S-ACA. Let $\psi(m, n, \vec{u})$ be S-arithmetical, and assume that $\forall m \exists n\, \psi(m, n, \vec{u})$. By S-ACA there exists y such that $\forall m \forall n\, [y_{(m,n)} > 0 \leftrightarrow \psi(m, n, \vec{u})]$. Pick an infinite H. By Lemma 3.6 there exists z such that $(\forall v < H)(z)_v = (\mu x < H) y_{(v,x)} > 0$. Then $\forall m\, \psi(m, (z)_m, \vec{u})$, and z is near-standard.

For the converse, assume (3). Let $\varphi(m, \vec{u})$ be S-arithmetical and let $\psi(m, \vec{u})$ be the S-arithmetical formula $n > 0 \leftrightarrow \varphi(m, \vec{u})$. By (3), we have

$$\exists x \forall m [(x)_m > 0 \leftrightarrow \varphi(m, \vec{u})].$$

$\dashv$

THEOREM 7.6. *In* $^*\mathsf{WKL}_0$*,* S-ACA *is equivalent to* ACA*.*

PROOF. Work in $^*\mathsf{WKL}_0$. We prove that Σ_1^S-CA is equivalent to Σ_1^0-CA. Assume Σ_1^S-CA and let $\varphi(m,n,\vec{Y})$ be Δ_0^0. By STP, $\vec{Y}$ has a lifting $\vec{y}$. By Σ_1^S-CA, $\exists x \forall m[(x)_m > 0 \leftrightarrow \exists n \varphi(m,n,\vec{y})]$. By STP, there exists X such that $X = st(x)$. Then by Lemma 4.6,

$$\forall m\,[m \in X \leftrightarrow \exists n \overline{\varphi}(m,n,\vec{Y})].$$

Now assume Σ_1^0-CA and let $\varphi(m,n,\vec{u})$ be Δ_0^S. By Δ_0^S-CA, (which holds by Lemma 3.4),

$$\exists z \forall m \forall n[(z)_{(m,n)} > 0 \leftrightarrow \varphi(m,n,\vec{u})].$$

By STP there exists Z with $Z = st(z)$. By Σ_1^0-CA,

$$\exists X \forall m[m \in X \leftrightarrow \exists n(m,n) \in Z].$$

By STP, X has a lifting x, and it follows that

$$\forall m[(x)_m > 0 \leftrightarrow \exists n \varphi(m,n,\vec{u})].$$

⊣

COROLLARY 7.7. $^*\mathsf{ACA}_0 \vdash \mathsf{ACA}_0$*.*

PROOF. By Theorems 5.4 and 7.6. ⊣

COROLLARY 7.8. $^*\Sigma\mathsf{PA} + S$-$\mathsf{ACA} \vdash \mathsf{PA}$*.*

DEFINITION 7.9. *Given a formula* φ *of* L_1*, a* ***star of*** φ *is a formula* $^*\varphi$ *of* $^*L_1$ *which is obtained from* φ *by replacing each bound variable in* φ *by a variable of sort* $^*N$ *in a one to one fashion.*

First Order Transfer (FOT) *is the scheme*

$$\varphi(\vec{n}) \rightarrow {}^*\varphi(\vec{n})$$

for each formula $\varphi(\vec{n})$ *of* L_1*.*

FOT says that $^*\mathcal{N}$ is an elementary extension of $\mathcal{N}$.

The following conservation result for $^*\mathsf{ACA}_0 + \mathsf{FOT}$ is a consequence of Theorem B in the paper Enayat [En2005].

THEOREM 7.10. *Every countable model of* ACA_0 *can be expanded to a model of* $^*\mathsf{ACA}_0 + \mathsf{FOT}$*.*

COROLLARY 7.11. *The theory* $^*\mathsf{ACA}_0 + \mathsf{FOT}$ *is conservative with respect to* ACA_0*.*

This follows in the usual way from Theorem 7.10. It also follows from earlier results in [HKK1984] (Theorem 4.1 and Lemma 4.12), which show that every countable model of ACA_0 has an elementary extension which can be expanded to a model of $^*\mathsf{ACA}_0 + \mathsf{FOT}$.

COROLLARY 7.12. *The theory* $^*\Sigma\mathsf{PA} + S\text{-}\mathsf{ACA} + \mathsf{FOT}$ *is conservative with respect to* PA.

PROOF. By Proposition 7.1 and Theorem 7.10. $\dashv$

It is worth noting that Theorem B in [En2005] is actually stronger than Theorem 7.10 above. Let j be a new unary function symbol and let AUT be the sentence in the language $^*L_1 \cup \{j\}$ which says that j is an automorphism of $^*\mathcal{N}$ with fixed point set $\mathcal{N}$. In our setting, the result can be stated as follows.

THEOREM 7.13. *(Enayat [En2005], Theorem B). Every countable model of* ACA_0 *can be expanded to a model of* $^*\mathsf{ACA}_0 + \mathsf{AUT} + \mathsf{FOT}$.

Hence $^*\mathsf{ACA}_0 + \mathsf{AUT} + \mathsf{FOT}$ *is conservative with respect to* ACA_0*, and* $^*\Sigma\mathsf{PA} + S\text{-}\mathsf{ACA} + \mathsf{AUT} + \mathsf{FOT}$ *is conservative with respect to* PA.

§8. The Theory $^*\mathsf{ATR}_0$**.** In this section we find two nonstandard counterparts of the theory ATR_0 of arithmetic transfinite recursion.

In the language L_2, a Σ^1_1 formula is a formula of the form $\exists Y \psi(Y, \dots)$ where ψ is arithmetical, and a Π^1_1 formula is a formula of the form $\forall Y \psi(Y, \dots)$ where ψ is arithmetical.

Σ^1_1**-separation** (Σ^1_1-SEP) is the scheme which says that any two disjoint Σ^1_1 properties can be separated by a set. That is, for any two Σ^1_1 formulas $\psi(n, \dots), \theta(n, \dots)$ in which X does not occur,

$$\neg\exists n[\psi(n, \dots) \wedge \theta(n, \dots)] \rightarrow$$

$$\exists X \forall n\, [(\psi(n, \dots) \rightarrow n \in X) \wedge (\theta(n, \dots) \rightarrow n \notin X)].$$

It is known (see [Si1999], Theorem V.5.1) that in RCA_0, Σ^1_1-SEP is equivalent to the scheme of Arithmetical Transfinite Recursion. Thus the theory ATR_0 can be defined as

$$\mathsf{ATR}_0 = \mathsf{WKL}_0 + \Sigma^1_1\text{-}\mathsf{SEP}.$$

In the language $^*L_1$, we define Γ**-separation** (Γ-SEP) as the scheme that for all formulas $\psi(n, \dots), \theta(n, \dots) \in \Gamma$ in which x does not occur,

$$\neg\exists n[\psi(n, \dots) \wedge \theta(n, \dots)] \rightarrow$$

$$\exists x \forall n\, [(\psi(n, \dots) \rightarrow (x)_n > 0) \wedge (\theta(n, \dots) \rightarrow (x)_n = 0)].$$

We will consider two classes of formulas in $^*L_1$ analogous to the class Σ^1_1, which we call Σ^b_1 and Σ^*_1. The Σ^b_1 formulas are formed by putting a bounded existential quantifier in front of an S-arithmetical formula, and the Σ^*_1 formulas are formed by putting an unbounded existential quantifier in front of an S-arithmetical formula. We will then compare Σ^1_1-SEP with Σ^b_1-SEP and Σ^*_1-SEP.

Definition 8.1. *A* Σ_1^b ***formula*** *is a formula of the form* $(\exists x < H)\varphi(x, \vec{u})$*, and a* Π_1^b ***formula*** *is a formula of the form* $(\forall x < H)\varphi(x, \vec{u})$*, where* φ *is* S*-arithmetical.*

The theory ATR_0^b *is defined by*

$$\mathsf{ATR}_0^b = {}^*\mathsf{WKL}_0 + \Sigma_1^b\text{-}\mathsf{SEP}.$$

Definition 8.2. *A* Σ_1^* ***formula*** *is a formula of the form* $\exists x\, \varphi(x, \vec{u})$*, and a* Π_1^* ***formula*** *is a formula of the form* $\forall x, \varphi(x, \vec{u})$*, where* φ *is* S*-arithmetical.*

The theory ${}^*\mathsf{ATR}_0$ *is defined by*

$${}^*\mathsf{ATR}_0 = {}^*\mathsf{WKL}_0 + \Sigma_1^*\text{-}\mathsf{SEP}.$$

Since $\Sigma_1^b \subseteq \Sigma_1^*$, we see at once that

Proposition 8.3. ${}^*\mathsf{ATR}_0 \vdash \mathsf{ATR}_0^b$.

We do not know whether or not $\mathsf{ATR}_0^b \vdash {}^*\mathsf{ATR}_0$. To make a connection between the theories ATR_0 and ATR_0^b, we will use an analogue of the Kleene normal form theorem which applies to Σ_1^b formulas. From Section 3, $q = f \upharpoonright m$ is a Δ_0^0 formula which says that q is the integer which codes the tuple $\langle f(n) : n < m\rangle$. The Kleene Normal Form Theorem for Σ_1^1 formulas (see [Si1999], page 169) shows that for each Σ_1^1 formula $\theta(\vec{p}, \vec{U})$ there is a Π_1^0 formula $\psi(m, n, \vec{p}, \vec{U})$ such that

$$\mathsf{ACA}_0 \vdash \theta(\vec{p}, \vec{U}) \leftrightarrow \exists f \forall m\, \psi(m, f \upharpoonright m, \vec{p}, \vec{U}).$$

Here is the analogue for Σ_1^b formulas.

Theorem 8.4. *Let* $\theta(\vec{u})$ *be a* Σ_1^b *formula. There is a* Π_1^S *formula* $\psi(k, v, \vec{u})$ *such that*

$${}^*\Sigma\mathsf{PA} + S\text{-}\mathsf{ACA} \vdash \theta(\vec{u}) \leftrightarrow \exists^{ns} w \forall k\, \psi(k, w \upharpoonright k, \vec{u}).$$

Proof. Work in ${}^*\Sigma\mathsf{PA}+S\text{-}\mathsf{ACA}$. Suppose first that $\theta(\vec{u})$ is S-arithmetical. Then there is a formula $\psi \in \Pi_j^S$ such that j is minimal and

$$\theta(\vec{u}) \leftrightarrow \exists^{ns} w \forall k\, \psi(k, w \upharpoonright k, \vec{u}).$$

We wish to show that $j \le 1$. Suppose $j > 1$, so that

$$\psi(k, v, \vec{u}) \leftrightarrow \forall m \exists n\, \varphi(m, n, k, v, \vec{u})$$

where $\varphi \in \Pi_{j-2}^S$. Then

$$\forall k \psi(k, v, \vec{u}) \leftrightarrow \forall k \forall m \exists n\, \varphi(m, n, k, v, \vec{u}).$$

By Proposition 7.5,

$$\forall k\, \psi(k, v, \vec{u}) \leftrightarrow \exists^{ns} z \forall k \forall m\, \varphi(m, (z)_{(m,k)}, k, v, \vec{u}),$$

and therefore

$$\theta(\vec{u}) \leftrightarrow \exists^{ns} w \exists^{ns} z \forall k \forall m\, \varphi(m, (z)_{(m,k)}, k, w \upharpoonright k, \vec{u}).$$

By combining quantifiers and simplifying, we get a formula $\psi' \in \Pi^S_{j-2}$ such that
$$\theta(\vec{u}) \leftrightarrow \exists^{ns} w \forall k\, \psi'(k, w \restriction k, \vec{u}),$$
contradicting the assumption that j is minimal. We conclude that $j \leq 1$, so $\psi \in \Pi^S_1$, and the result is proved in the case that $\theta(\vec{u})$ is S-arithmetical.

For the general case, suppose $\theta(\vec{u})$ is of the form $(\exists x < H)\delta(x, \vec{u})$ where δ is S-arithmetical. Then
$$\delta(x, \vec{u}) \leftrightarrow \exists^{ns} w \forall k \forall n\, \psi(x, k, n, w \restriction k, \vec{u})$$
for some Δ^S_0 formula ψ. It follows that
$$\theta(\vec{u}) \leftrightarrow (\exists x < H)\exists^{ns} w \forall k \forall n\, \psi(x, k, n, w \restriction k, \vec{u}),$$
so
$$\theta(\vec{u}) \leftrightarrow \exists^{ns} w (\exists x < H)\forall k \forall n\, \psi(x, k, n, w \restriction k, \vec{u}).$$
By Overspill,
$$(\exists x < H)\forall k \forall n\, \psi(x, k, n, w \restriction k, \vec{u}) \leftrightarrow$$
$$\forall p (\exists x < H)(\forall k < p)(\forall n < p)\, \psi(x, k, n, w \restriction k, \vec{u}).$$
Therefore
$$\theta(\vec{u}) \leftrightarrow \exists^{ns} w \forall p\, \psi'(p, w \restriction p, \vec{u})$$
where $\psi'(p, w \restriction p, \vec{u})$ is the Δ^S_0 formula
$$(\exists x < H)(\forall k < p)(\forall n < p)\, \psi(x, k, n, w \restriction k, \vec{u}).$$
⊣

THEOREM 8.5. *In* $^*\mathsf{WKL_0}$, Σ^1_1-SEP *is equivalent to* Σ^b_1-SEP.

PROOF. Work in $^*\mathsf{WKL_0}$. First assume Σ^b_1-SEP. Let $\psi(m, Y, \vec{U})$, $\theta(m, Y, \vec{U})$ be arithmetical formulas such that
$$\neg\exists m\, [\exists Y \psi(m, Y, \vec{U}) \wedge \exists Y \theta(m, Y, \vec{U})].$$
By STP there is a lifting $\vec{u}$ of $\vec{U}$. By Lemma 4.6 we have
$$\neg\exists m\, [\exists y\, \overline{\psi}(m, y, \vec{u}) \wedge \exists y\, \overline{\theta}(m, y, \vec{u})].$$
Let H be infinite. Then
$$\neg\exists m\, [(\exists y < H)\, \overline{\psi}(m, y, \vec{u}) \wedge (\exists y < H)\, \overline{\theta}(m, y, \vec{u})].$$
The formulas $(\exists y < H)\, \overline{\psi}(m, y, \vec{u})$ and $(\exists y < H)\, \overline{\theta}(m, y, \vec{u})$ are Σ^b_1. By Σ^b_1-SEP there is an x such that
$$\forall m\, [((\exists y < H)\, \overline{\psi}(m, y, \vec{u}) \to (x)_m > 0) \wedge (\exists y < H)\, \overline{\theta}(m, y, \vec{u}) \to (x)_m = 0)].$$
By STP there exists a set $X = st(x)$. By Lemma 6.3, each set Y has a lifting $y < H$. Thus by Lemma 4.6 we have
$$\forall m\, [(\exists Y\, \psi(m, Y, \vec{U}) \to m \in X) \wedge (\exists Y\, \theta(m, Y, \vec{U}) \to m \notin X)].$$
This proves Σ^1_1-SEP.

Now assume Σ^1_1-SEP. Let $\psi(m,\dots),\theta(m,\dots)$ be Σ^b_1 formulas such that

$$\neg\exists m\,[\psi(m,\dots)\wedge\theta(m,\dots)].$$

By Theorem 8.4, there are Π^S_0 formulas $\psi'(m,k,n,\dots)$ and $\theta'(m,k,n,\dots)$ such that

$$\psi(m,\dots)\leftrightarrow\exists^{ns}w\forall k\,\psi'(m,k,w\restriction k,\dots),$$

$$\theta(m,\dots)\leftrightarrow\exists^{ns}w\forall k\,\theta'(m,k,w\restriction k,\dots).$$

It is clear that Σ^1_1-SEP implies ACA. By Theorem 7.6, S-ACA holds, so there exist y,z such that

$$\forall m\forall k\forall n\,[(y)_{(m,k,n)}>0\leftrightarrow\psi'(m,k,n,\dots)],$$

$$\forall m\forall k\forall n\,[(z)_{(m,k,n)}>0\leftrightarrow\theta'(m,k,n,\dots)].$$

By STP there are sets $Y=st(y)$ and $Z=st(z)$. By Lemma 4.6,

$$\exists^{ns}w\forall k\,(y)_{(m,k,w\restriction k)}>0\leftrightarrow\exists f\forall k\,(m,k,f\restriction k)\in Y$$

and

$$\exists^{ns}w\forall k\,(z)_{(m,k,w\restriction k)}>0\leftrightarrow\exists f\forall k\,(m,k,f\restriction k)\in Z.$$

By Σ^1_1-SEP,

$$\exists X\forall m\,[(m\in X\rightarrow\exists f\forall k\,(m,k,f\restriction k)\in Y)\wedge$$

$$(m\notin X\rightarrow\exists f\forall k\,(m,k,f\restriction k)\in Z)].$$

By STP, X has a lifting x. By Lemma 4.6,

$$\forall m\,[((x)_m>0\rightarrow\exists^{ns}w\forall k\,(y)_{(m,k,w\restriction k)}>0)\wedge$$

$$((x)_m=0\rightarrow\exists^{ns}w\forall k\,(z)_{(m,k,w\restriction k)}>0)].$$

Then

$$\forall m\,[(x)_m>0\rightarrow\psi(m,\dots)\wedge((x)_m=0\rightarrow\theta(m,\dots)],$$

and x is the required witness for Σ^b_1-SEP. ⊣

Corollary 8.6. $\mathsf{ATR}^b_0\vdash\mathsf{ATR}_0$ *and* $\mathsf{ATR}^*_0\vdash\mathsf{ATR}_0$.

Theorem 8.7. *Every countable model* $\mathcal{M}=(\mathcal{N},\mathcal{P})$ *of* ATR_0 *can be expanded to a model* $(\mathcal{M},{}^*\mathcal{N})$ *of* $\mathsf{ATR}^b_0+\mathsf{FOT}$.

Proof. By Theorem 7.10, every countable model $\mathcal{M}$ of ATR_0 can be expanded to a model $(\mathcal{M},{}^*\mathcal{N})$ of ${}^*\mathsf{ACA}_0{+}\mathsf{FOT}$. By Theorem 8.5, $(\mathcal{M},{}^*\mathcal{N})$ also satisfies Σ^b_1-SEP, and hence is a model of ATR^b_0. ⊣

This shows that $\mathsf{ATR}^b_0+\mathsf{FOT}$ is conservative with respect to ATR_0. We now improve this by showing that the larger theory ${}^*\mathsf{ATR}_0+\mathsf{FOT}$ is still conservative with respect to ATR_0.

LEMMA 8.8. *Let* $(\mathcal{N}, {}^*\mathcal{N})$ *be a model of* ${}^*\Sigma\mathsf{PA}$ *such that* ${}^*\mathcal{N}$ *is* ω_1*-like, that is,* ${}^*\mathcal{N}$ *has cardinality* ω_1 *and every initial segment is countable. Then for every* S*-arithmetical formula* $\varphi(x, m, \vec{u})$,

$$(\mathcal{N}, {}^*\mathcal{N}) \models \exists H \forall m\, [\exists x\, \varphi(x, m, \vec{u}) \to (\exists x < H)\varphi(x, m, \vec{u})].$$

PROOF. Fix a tuple $\vec{u}$ in ${}^*\mathcal{N}$. For each $m \in \mathcal{N}$ such that $(\mathcal{N}, {}^*\mathcal{N}) \models \exists x\, \varphi(x, m, \vec{u})$, pick an x_m such that $(\mathcal{N}, {}^*\mathcal{N}) \models \varphi(x_m, \vec{u})$. Since $\mathcal{N}$ is countable, and ${}^*\mathcal{N}$ is ω_1-like, there is an $H \in {}^*\mathcal{N}$ such that $x_m < H$ for each $m \in \mathcal{N}$. ⊣

THEOREM 8.9. *Every countable model* $\mathcal{M} = (\mathcal{N}, \mathcal{P})$ *of* ATR_0 *can be expanded to a model* $(\mathcal{M}, {}^*\mathcal{N})$ *of* ${}^*\mathsf{ATR}_0 + \mathsf{FOT}$.

PROOF. By Theorem 8.7 and the Löwenheim-Skolem theorem, $\mathcal{M} = (\mathcal{N}, \mathcal{P})$ can be expanded to a countable model $(\mathcal{M}, \mathcal{N}')$ of $\mathsf{ATR}_0^b + \mathsf{FOT}$. Since $\mathsf{ATR}_0^b \vdash \mathsf{PA}$, $\mathcal{N}' \models \mathsf{PA}$. By the MacDowell-Specker theorem (MS1961] and the Löwenheim- Skolem theorem, each countable model of PA has a countable end elementary extension. Applying this ω_1 times, $\mathcal{N}'$ has an ω_1-like end elementary extension ${}^*\mathcal{N}$. We show that $(\mathcal{M}, {}^*\mathcal{N}) \models {}^*\mathsf{ATR}_0$. By Proposition 5.3, $(\mathcal{M}, {}^*\mathcal{N}) \models {}^*\mathsf{WKL}_0$. By hypothesis, Σ^1_1-SEP holds in $\mathcal{M}$. By Theorem 8.5, Σ^b_1-SEP holds in $(\mathcal{M}, {}^*\mathcal{N})$. FOT holds in $(\mathcal{M}, {}^*\mathcal{N})$ because it holds in $(\mathcal{M}, \mathcal{N}')$ and ${}^*\mathcal{N}$ is an elementary extension of $\mathcal{N}'$. We work in $(\mathcal{M}, {}^*\mathcal{N})$ and show that Σ^*_1-SEP holds.

Let $\psi(m, \vec{u}), \theta(m, \vec{u})$ be Σ^*_1 formulas in which x does not occur, and suppose that

$$\neg\exists m[\psi(m, \vec{u}) \wedge \theta(m, \vec{u})].$$

We have

$$\psi(y, \vec{u}) = \exists z\, \psi'(z, y, \vec{u}), \quad \theta(y, \vec{u}) = \exists z\, \theta'(z, y, \vec{u})$$

where ψ', θ' are S-arithmetical. By Lemma 8.8, there exists H such that

$$\forall m[\exists z\, \psi'(z, m, \vec{u}) \to (\exists z < H)\psi'(z, m, \vec{u})],$$
$$\forall m[\exists z\, \theta'(z, m, \vec{u}) \to (\exists z < H)\theta'(z, m, \vec{u})].$$

Then

$$\forall m[\psi(m, \vec{u}) \leftrightarrow (\exists z < H)\psi'(z, m, \vec{u})],$$
$$\forall m[\theta(m, \vec{u}) \leftrightarrow (\exists z < H)\theta'(z, m, \vec{u})].$$

By Σ^b_1-SEP,

$$\exists x \forall m[((\exists z < H)\psi'(z, m, \vec{u}) \to (x)_m > 0) \wedge ((\exists z < H)\theta'(z, m, \vec{u}) \to (x)_m = 0)].$$

Therefore

$$\exists x \forall m[(\psi(m, \vec{u}) \to (x)_m > 0) \wedge (\theta(m, \vec{u}) \to (x)_m = 0)].$$

⊣

COROLLARY 8.10. ${}^*\mathsf{ATR}_0 + \mathsf{FOT}$ *is conservative with respect to* ATR_0.

§9. The Theory $^*\Pi^1_1$-CA_0. We give two nonstandard counterparts of the theory Π^1_1-CA_0 of Π^1_1 comprehension, a theory with a bounded outer quantifier and a larger theory with an unbounded outer quantifier.

In the language L_2, Π^1_1-CA_0 is defined as the theory

$$\Pi^1_1\text{-}\mathsf{CA}_0 = \mathsf{WKL}_0 + \Pi^1_1\text{-}\mathsf{CA}.$$

DEFINITION 9.1. *In the language $L_2 \cup {}^*L_1$, the theories Π^b_1-CA_0 and $^*\Pi^1_1$-CA_0 are defined by*

$$\Pi^b_1\text{-}\mathsf{CA}_0 = {}^*\mathsf{WKL_0} + \Pi^b_1\text{-}\mathsf{CA},$$
$${}^*\Pi^1_1\text{-}\mathsf{CA}_0 = {}^*\mathsf{WKL_0} + \Pi^*_1\text{-}\mathsf{CA}.$$

Since $\Pi^b_1 \subseteq \Pi^*_1$, we have:

PROPOSITION 9.2. $^*\Pi^1_1$-$\mathsf{CA}_0 \vdash \Pi^b_1$-$\mathsf{CA}_0$.

As in the preceding section, we do not know whether or not Π^b_1-$\mathsf{CA}_0 \vdash {}^*\Pi^1_1$-$\mathsf{CA}_0$.

THEOREM 9.3. *In $^*\mathsf{WKL_0}$, Π^b_1-CA is equivalent to Π^1_1-CA.*

PROOF. Work in $^*\mathsf{WKL_0}$. Assume Π^b_1-CA and let $\varphi(m, Z, \vec{U})$ be arithmetical. By STP, $\vec{U}$ has a lifting $\vec{u}$. Take an infinite H. By Lemmas 4.6 and 6.3 (iii),

$$\forall m\,[\forall Z \varphi(m, Z, \vec{U}) \leftrightarrow (\forall z < H)\overline{\varphi}(m, z, \vec{u})].$$

The formula $(\forall z < H)\overline{\varphi}(m, z, \vec{u})$ is Π^b_1. By Π^b_1-CA, there exists x such that

$$\forall m\,[(x)_m > 0 \leftrightarrow (\forall z < H)\overline{\varphi}(m, z, \vec{u})].$$

By STP there exists $X = st(x)$. Then

$$\forall m\,[m \in X \leftrightarrow \forall Z \varphi(m, Z, \vec{U})],$$

and Π^1_1-CA is proved.

For the converse, assume Π^1_1-CA, and let $\theta(v, \vec{u})$ be a Π^b_1 formula. By Theorem 7.6, S-ACA holds, and therefore by Theorem 8.4 there is a Π^S_1 formula $\psi(v, k, z, \vec{u})$ such that

$$\theta(v, \vec{u}) \leftrightarrow \forall^{ns} w \exists k\, \psi(v, k, w \restriction k, \vec{u}).$$

By ACA there exists a set Y such that

$$\forall m \forall k \forall n[(m, k, n) \in Y \leftrightarrow \psi(m, k, n, \vec{u})].$$

Then by Π^1_1-CA there is a set X such that

$$\forall m[m \in X \leftrightarrow \forall f \exists k\, (m, k, f \restriction k) \in Y].$$

Using Lemma 4.6 again,

$$\forall f \exists k\, (m, k, f \restriction k) \in Y \leftrightarrow \forall^{ns} w \exists k\, \psi(v, k, w \restriction k, \vec{u}).$$

Therefore

$$\forall m\,[m \in X \leftrightarrow \theta(m, \vec{u})].$$

By STP, X has a lifting x, and it follows that

$$\forall m[(x)_m > 0 \leftrightarrow \theta(m, \vec{u})].$$

⊣

Corollary 9.4. $\Pi^b_1\text{-}\mathsf{CA}_0 \vdash \Pi^1_1\text{-}\mathsf{CA}_0$ *and* $\Pi^*_1\text{-}\mathsf{CA}_0 \vdash \Pi^1_1\text{-}\mathsf{CA}_0$.

Theorem 9.5. *Every countable model* $\mathcal{M} = (\mathcal{N}, \mathcal{P})$ *of* $\Pi^1_1\text{-}\mathsf{CA}_0$ *can be expanded to a model* $(\mathcal{M}, {}^*\mathcal{N})$ *of* $\Pi^b_1\text{-}\mathsf{CA}_0 + \mathsf{FOT}$.

Proof. By Theorems 9.3 and 8.7. ⊣

Theorem 9.6. *Every countable model* $\mathcal{M} = (\mathcal{N}, \mathcal{P})$ *of* $\Pi^1_1\text{-}\mathsf{CA}_0$ *can be expanded to a model* $(\mathcal{M}, {}^*\mathcal{N})$ *of* ${}^*\Pi^1_1\text{-}\mathsf{CA}_0 + \mathsf{FOT}$.

Proof. As in the proof of Theorem 8.9, but using Theorems 9.3 and 9.5 instead of Theorems 8.5 and 8.7, $\mathcal{M}$ can be expanded to a model $(\mathcal{M}, {}^*\mathcal{N})$ of $\Pi^b_1\text{-}\mathsf{CA}_0 + \mathsf{FOT}$ such that ${}^*\mathcal{N}$ is ω_1-like. We work in $(\mathcal{M}, {}^*\mathcal{N})$ and show that $\Pi^*_1\text{-}\mathsf{CA}$ holds.

Let $\varphi(m, \vec{u})$ be a Π^*_1 formula in which x does not occur. Then $\varphi(m, \vec{u}) = \forall z \varphi'(z, m, \vec{u})$ where φ' is S-arithmetical. By Lemma 8.8, there exists H such that

$$\forall m[\exists z \neg\varphi'(z, m, \vec{u}) \to (\exists z < H)\neg\varphi'(z, m, \vec{u})],$$

so

$$\forall m[\varphi(m, \vec{u}) \leftrightarrow (\forall z < H)\varphi'(z, m, \vec{u}).$$

By $\Pi^b_1\text{-}\mathsf{CA}$,

$$\exists x \forall m[(x)_m > 0 \leftrightarrow (\forall z < H)\varphi'(z, m, \vec{u})],$$

so

$$\exists x \forall m[(x)_m > 0 \leftrightarrow \varphi(m, \vec{u})].$$

⊣

Corollary 9.7. ${}^*\Pi^1_1\text{-}\mathsf{CA}_0 + \mathsf{FOT}$ *is conservative with respect to* $\Pi^1_1\text{-}\mathsf{CA}_0$.

This proves a conjecture stated in [HKK1984], page 1054.

In [HKK1984] and [En2005], conservation results were obtained for induction, choice, and dependent choice schemes in L_2 which are stronger than $\Pi^1_1\text{-}\mathsf{CA}_0$. To clarify the connection of those results with this paper, we restate the results for dependent choice in our present setting.

In L_2, Σ^1_k**-dependent choice** ($\Sigma^1_k\text{-}\mathsf{DC}$) (where $k > 0$) is the scheme

$$\forall X \exists Y \varphi(X, Y, \dots) \to \forall X \exists Y[Y^{(0)} = X \wedge \forall m \varphi(Y^{(m)}, Y^{(m+1)}, \dots)]$$

where $\varphi(X, Y, \dots) \in \Sigma^1_k$ and $Y^{(m)} = \{n : (m,n) \in Y\}$. The theory Σ^1_k-DC_0 is defined by

$$\Sigma^1_k\text{-}\mathsf{DC}_0 = \mathsf{RCA}_0 + \Sigma^1_k\text{-}\mathsf{DC}.$$

We also write Σ^1_∞-$\mathsf{DC} = \bigcup_{k\in\mathbb{N}} \Sigma^1_k$-$\mathsf{DC}$.

The theory Σ^1_1-DC_0 implies ACA_0 and is incomparable with ATR_0 (H. Friedman; see [Si1999], Section VIII.5). Σ^1_2-DC_0 is strictly stronger than Π^1_1-CA_0. The very strong theory Σ^1_∞-DC_0 is called **second order arithmetic with dependent choice**.

Definition 9.8. *In* $^*L_1$, *we define*

$$\Sigma^*_{k+1} = \{\exists x\varphi(x, \dots) : \varphi \in \Pi^*_k\}, \quad \Pi^*_{k+1} = \{\forall x\varphi(x, \dots) : \varphi \in \Sigma^*_k\}.$$

Σ^*_k***-dependent saturation*** (Σ^*_k-DSAT) *is the scheme*

$$\forall x\exists y\varphi(x, y, \dots) \rightarrow \forall x\exists y[(y)_0 = x \wedge \forall m\varphi((y)_m, (y)_{m+1}, \dots)]$$

where $\varphi \in \Sigma^*_k$. *We also write* Σ^*_∞-$\mathsf{DSAT} = \bigcup_{k\in\mathbb{N}} \Sigma^*_k$-$\mathsf{DSAT}$.

We omit the case $k = 0$ because Σ^1_0-DC is equivalent to Σ^1_1-DC, and Σ^*_0-DSAT is equivalent to Σ^*_1-DSAT. We now restate two theorems from [HKK1984] in our present setting.

Theorem 9.9. *([HKK1984], Theorem 3.4) Whenever* $0 < k \leq \infty$, $^*\mathsf{WKL}_0 + \Sigma^*_k$-$\mathsf{DSAT} \vdash \Sigma^1_k$-$\mathsf{DC}_0$.

Theorem 9.10. *([HKK1984], Theorem 4.1). Whenever* $1 < k \leq \infty$, *every countable model of* Σ^1_k-DC_0 *can be expanded to a model of* $^*\mathsf{WKL}_0 + \Sigma^*_k$-$\mathsf{DSAT} + \mathsf{FOT}$.

Whenever $0 < k \leq \infty$, $^*\mathsf{WKL}_0 + \Sigma^*_k$-$\mathsf{DSAT} + \mathsf{FOT}$ *is conservative with respect to* Σ^1_k-DC_0.

In the case $k = \infty$ this is a conservation result for second order arithmetic with dependent choice. Theorem C in [En2005] can also be stated as a conservation result for second order arithmetic with dependent choice. Recall from the discussion after Theorem 7.10 that AUT is the sentence in the language $^*L_1 \cup \{j\}$ which says that j is an automorphism of $^*\mathcal{N}$ with fixed point set $\mathcal{N}$.

Theorem 9.11. *([En2005], Theorem C). Let* Γ *be the set of all formulas of the language* $^*L_1 \cup \{j\}$. *Every countable model of* Σ^1_∞-DC_0 *can be expanded to a model of* $^*\mathsf{WKL}_0 + \mathsf{AUT} + \Gamma$-$\mathsf{CA} + \mathsf{FOT}$.

Hence the theory $^*\mathsf{WKL}_0 + \mathsf{AUT} + \Gamma$-$\mathsf{CA} + \mathsf{FOT}$ *is conservative with respect to* Σ^1_∞-DC_0.

References:

[CK1990] C.C. Chang and H.J. Keisler. Model Theory, Third Edition. Elsevier 1990.

[En2005] A. Enayat. From bounded arithmetic to second order arithmetic via automorphisms. To appear, Logic in Tehran: Proceedings of the Conference in Logic, Algebra, and Arithmetic held in Tehran, 2003. Available online at http://academic2.american.edu/∼enayat.

[HKK1984] C.W. Henson, M. Kaufmann, and H.J. Keisler. The strength of nonstandard methods in arithmetic. Journal of Symbolic Logic 49 (1984), pp. 1039-1058.

[HK1986] C.W. Henson and H.J. Keisler. On the strength of nonstandard analysis. Journal of Symbolic Logic 51 (1986), pp. 377-386.

[Ke2005] H.J. Keisler. The strength of nonstandard analysis. To appear. Available online at http://www.math.wisc.edu/∼keisler.

[MS1961] R. MacDowell and E. Specker. Modelle der arithmetic. Pp. 257-263 in Infinitistic Methods, Pergamon Press 1961.

[Si1999] S.G. Simpson. Subsystems of second order arithmetic. Perspectives in Mathematical Logic. Springer-Verlag 1999.

[Ta1997] K. Tanaka. The self-embedding theorem of WKL_0 and a non-standard method. Annals of Pure and Applied Logic 84 (1997), pp. 41-49.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
480 LINCOLN DRIVE
MADISON WI 53706
E-mail: keisler@math.wisc.edu