OBSERVING, REPORTING, AND DECIDING IN NETWORKS OF SENTENCES

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Abstract. In prior work [5] we considered networks of agents who prove facts from their knowledge bases and report them to their neighbors in their common languages in order to help a decider verify a single sentence. In report complete networks, the signatures of the agents and the links between agents are rich enough to verify any decider's sentence that can be proved from the combined knowledge base. This paper introduces a more general setting where new observations may be added to knowledge bases and the decider must choose a sentence from a set of alternatives. We consider the question of when it is possible to prepare in advance a finite plan to generate reports within the network. We obtain conditions under which such a plan exists and is guaranteed to produce the right choice under any new observations.

1. Introduction

This paper builds upon the paper [5]. In that paper, a signature network is a network of agents each labeled with a signature. Each agent has a knowledge base within its signature. A sentence is said to be report provable if it can be verified by agents who can only report sentences to their neighbors in their common languages. A signature network is said to be report complete if whatever the agent knowledge bases are, every consequence of the union of these knowledge bases is report provable. To obtain conditions for report completeness, the Craig Interpolation Theorem is applied at each edge in the network.

The present paper considers situations that may arise in applications where there is some systematic relationship among the possible knowledge bases and decision sentences. Specifically, we consider "observation networks" on a given signature network in which there are many possible observations that may be added to an underlying knowledge base (i.e., learning), and a finite set of alternative sentences one of which must be proved (i.e., making a decision). We exploit the fact that report completeness holds for every knowledge base on the signature

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network. We do this by introducing the notion of a "report plan" for an observation network — a finite scheme for finding a "report proof" for one of the alternative sentences once observations are added to the underlying knowledge bases. A report plan is decentralized; the agents only need to know their own observations and the reports they receive, not what happens elsewhere in the network.

Our main result, Theorem 5.6, shows that an observation network will always have a report plan for a given finite set of alternatives provided that: (1) the observation network is sufficient for selecting from the set of alternatives, and (2) the underlying signature network contains a signature tree in the sense of the paper [5]. We also prove two results that are complementary to our main theorem. Theorem 3.5 is a "finiteness theorem" that shows that if an observation network is sufficient for selecting from a given infinite set of alternatives, then some finite part of the observation network is sufficient for selecting from some finite subset of the alternatives. Theorem 5.3 shows that a report plan guarantees that under every possible observation by the agents, some alternative is report provable. In the last Section we briefly indicate a way that report plans might be applied to obtain approximate values for unknown quantities depending on observations.

2. Prerequisites

2.1. **Logic.** We assume familiarity with the notions of sentence, signature, and proof in first order logic with equality. For background, see [2] or [4]. The notation $\mathcal{K} \vdash B$ means that the sentence B is provable from the set of sentences $\mathcal{K}$, and $\vdash B$ means that B is provable. The set of all sentences in a signature L is denoted by $[L]$. First order logic is formulated so that the true sentence $\top$ and false sentence $\bot$ belong to $[L]$ for every signature L. A set of sentences $\mathcal{K}$ is **consistent** if it is not the case that $\mathcal{K} \vdash \bot$. Given a finite set $\mathcal{B}$ of sentences, $\bigvee \mathcal{B}$ is the disjunction of the sentences in $\mathcal{B}$, and $\bigwedge \mathcal{B}$ is the conjunction of the sentences in $\mathcal{B}$. The conjunction of the empty set of sentences is $\top$, and the disjunction of the empty set is $\bot$. We use $B \to D$ as an abbreviation for $\neg B \vee D$, and $B \leftrightarrow D$ for $(B \to D) \wedge (D \to B)$. Given a signature L and a set of sentences $\mathcal{K} \subseteq [L]$, a **complete**, or **maximal consistent**, extension of $\mathcal{K}$ (in L) is a set of sentences $\mathcal{M}$ such that $\mathcal{K} \subseteq \mathcal{M} \subseteq [L]$, $\mathcal{M}$ is consistent, and for each $B \in [L]$, either $B \in \mathcal{M}$ or $(\neg B) \in \mathcal{M}$. We will use the **Compactness Theorem** in the following form: *A set of sentences $\mathcal{K}$ is consistent if and only if every finite subset of $\mathcal{K}$ is consistent.*

2.2. **Graphs.** By a **(simple) directed graph** (V, E) we mean a non-empty finite set V of **vertices**, and a set $E \subseteq V \times V$ of **edges** (or arcs) (x, y) such that $x \neq y$. (We do not allow more than one edge from a vertex x to a vertex y, we do not allow edges from a vertex to itself, and we distinguish between the pair (x, y) and the pair (y, x).)

A **(directed) path** of length n from x to y is a sequence $(x_0, \ldots, x_n)$ of pairwise distinct vertices such that $x_0 = x, x_n = y$, and for each $i < n$, $(x_i, x_{i+1}) \in E$. (In particular, for each vertex x, (x) is a path of length 0 from x to itself.) A **source** is a vertex x such that there is no edge $(z, x) \in E$, and a **sink** is a vertex x such that there is no edge $(x, y) \in E$.

In a directed graph, by a **decider** we mean a vertex d such that for every other vertex x, there is at least one path from x to d. By a **pointed graph** we mean a directed graph (V, E) with at least one decider. Hereafter we will always assume that (V, E) is a pointed graph, and that d is a decider for (V, E).

A **directed cycle** of length n is a sequence $(x_0, \ldots, x_{n-1}, x_n)$ of vertices such that $(x_0, \ldots, x_{n-1})$ is a directed path, $x_n = x_0$, and $(x_{n-1}, x_n) \in E$.

By a **tree** we will mean a directed graph (V, E) with a decider d such that (V, E) has no directed cycles, and for every vertex $x \neq d$ there is a unique edge $(x, y) \in E$. Note that a tree has a unique decider, which is also the unique sink. Also, for every vertex x there is is a path from a source to x, and a unique path from x to the decider.

2.3. **Signature and Knowledge Base Networks.** We now review some notions that we will need from the paper [5]. We attach signatures and knowledge bases to the vertices of pointed graphs. From now on we will call the vertices **agents**.

A **signature network** on (V, E) is an object

$$\mathbb{S} = (V, E, L(\cdot))$$

where (V, E) is a pointed graph with a labeling $L(\cdot)$ that assigns a signature $L(x)$ to each agent $x \in V$. We let $L(V) = \bigcup_{x \in V} L(x)$, and call $L(V)$ the **combined signature**. We say that a symbol S **occurs** at a vertex x if $S \in L(x)$.

We say that a signature network $\mathbb{S}$ **contains** a signature network $\mathbb{T} = (V, F, L(\cdot))$ if $\mathbb{S}$ and $\mathbb{T}$ have the same agents and signatures, and (V, F) can be obtained from (V, E) by removing edges.

Let $\mathbb{T} = (V, F, L(\cdot))$ be a signature network. We say that $\mathbb{T}$ is a **signature tree** if:

- (V, F) is a tree;

- for every pair of agents $x, y \in V$ and symbol S that occurs at both x and y, there is a vertex $z \in V$ such that S occurs at every vertex on a path from x to z and at every vertex on a path from y to z.

Signature networks that contain signature trees will be of particular importance in this paper.

Given a signature network $\mathbb{S} = (V, E, L(\cdot))$, a **knowledge base** (or knowledge base network) over $\mathbb{S}$ is an object

$$\mathbb{K} = (V, E, L(\cdot), \mathcal{K}(\cdot))$$

where $\mathcal{K}(\cdot)$ is a labeling that assigns a knowledge base $\mathcal{K}(x) \subseteq [L(x)]$ to each agent $x \in V$. We write $\mathcal{K}(V) = \bigcup_{x \in V} \mathcal{K}(x)$, and we call the set $\mathcal{K}(V)$ the **combined knowledge base**.

Note that in a knowledge base $\mathbb{K}$ over $\mathbb{S}$, each symbol that occurs in a sentence in $\mathcal{K}(x)$ must belong to $L(x)$, but we allow the possibility that $L(x)$ also has additional symbols.

2.4. **Report Provability.** We summarize the concept of report provability and some related facts from [5].

Definition 2.1. *Let*

$$\mathbb{K} = (V, E, L(\cdot), \mathcal{K}(\cdot))$$

be a knowledge base over a signature network $\mathbb{S}$. A sentence C is 0***-reportable*** *in $\mathbb{K}$ along an edge (x, y) if*

$$C \in [L(x) \cap L(y)] \text{ and } \mathcal{K}(x) \vdash C.$$

*C is $(n+1)$**-reportable** in $\mathbb{K}$ along an edge (x, y) if*

$$C \in [L(x) \cap L(y)] \text{ and } \mathcal{K}(x) \cup \mathcal{R} \vdash C,$$

where $\mathcal{R}$ is a set of sentences each of which is n-reportable along some edge (z, x).

The word "reportable" means n-reportable for some n", and "reportable to x" means "reportable along (z, x) for some z".

*Given a decider d for $\mathbb{S}$, a sentence $D \in [L(d)]$ is **report provable** in $\mathbb{K}$ at d if D is provable from $\mathcal{K}(d)$ and a set $\mathcal{R}$ of sentences each of which is reportable to d in $\mathbb{K}$.*

This means that at each stage, for each edge (x, y), agent x can report to agent y a sentence C in their common language, where C is provable from the knowledge base $\mathcal{K}(x)$ and sentences reported to x at earlier stages. Finally, D is provable from the knowledge base $\mathcal{K}(d)$ and sentences reported to d. Thus the sentence D is established using only proofs within the languages $[L(x)]$ of single agents x, and communications along edges (x, y) in common language $[L(x) \cap L(y)]$.

The next fact shows that report provability implies provability.

Fact 2.2. *(Lemma 2.9 in* [5]*) Suppose d is a decider and a sentence $D \in [L(d)]$ is report provable in a knowledge base $\mathbb{K}$ over a signature network $\mathbb{S}$. Then D is provable from the combined knowledge base, $\mathcal{K}(V) \vdash D$.*

Definition 2.3. *A signature network $\mathbb{S}$ is* ***report complete*** *at a decider d if for every knowledge base $\mathbb{K}$ over $\mathbb{S}$, every sentence $D \in [L(d)]$ that is provable from the combined knowledge base $\mathcal{K}(V)$ is report provable in $\mathbb{K}$ at d. $\mathbb{S}$ is* ***report complete*** *if $\mathbb{S}$ is report complete at every decider d.*

Fact 2.4. *(Corollary 2.14 in* [5]*) A signature network is report complete at some decider if and only if it is report complete at every decider.*

Fact 2.5. *(*[1]*, Corollary 4.1 in* [5]*) Every signature network that contains a signature tree is report complete.*

In [5] we deal with the following two questions. Which signature networks are report complete? Which signature networks contain a signature tree?

3. Observation Networks

An observation network is a signature network where each agent has both a knowledge base and a set of sentences called potential observations.

The story: An organization has a finite set of agents arranged in a network indexed by a graph, with a decider d. Each agent x has a signature $L(x)$ and a knowledge base $\mathcal{K}(x) \subseteq [L(x)]$. Each agent x also has a set of potential observations $\mathcal{O}(x) \subseteq [L(x)]$, and the decider d has a set $\mathcal{A} \subseteq [L(d)]$ of alternatives. The organization is faced with a recurring situation where in every scenario, each agent x makes observations in $\mathcal{O}(x)$, and there is an alternative $A \in \mathcal{A}$ that is "correct" in the sense that it is provable from the combined knowledge base and the observations of the agents.

In Theorem 3.5 below we will use the Compactness Theorem to show that there must be a finite subset $\mathcal{A}_0 \subseteq \mathcal{A}$ and, for each agent $x \in V$, a finite set $\mathcal{O}_0(x) \subseteq \mathcal{O}(x)$, such that in every scenario, each agent x makes an observation in $\mathcal{O}_0(x)$, and some alternative $A \in \mathcal{A}_0$ is correct. We first formally state the definitions.

We say that a set $\mathcal{B}$ of sentences is **closed under finite conjunctions** if for each $B, C \in \mathcal{B}$, the conjunction $B \wedge C$ is logically equivalent to some sentence in $\mathcal{B}$.

Definition 3.1. *Given a knowledge base network*

$$\mathbb{K} = (V, E, L(\cdot), \mathcal{K}(\cdot)),$$

*an **observation network** (over $\mathbb{K}$) is an object*

$$\mathbb{O} = (V, E, L(\cdot), \mathcal{K}(\cdot), \mathcal{O}(\cdot))$$

consisting of $\mathbb{K}$ and a labeling $\mathcal{O}(\cdot)$ of (V, E) such that for each agent x in V, $\emptyset \neq \mathcal{O}(x) \subseteq [L(x)]$, and $\mathcal{O}(x)$ is closed under finite conjunctions.

The elements of $\mathcal{O}(x)$ are called **potential observations for** x. We write $L(V) = \bigcup_{x \in V} L(x)$ for the combined signature, $\mathcal{K}(V) = \bigcup_{x \in V} \mathcal{K}(x)$ for the combined knowledge base, and $\mathcal{O}(V) = \bigcup_{x \in V} \mathcal{O}(x)$ for the combined set of potential observations. For each decider d for $\mathbb{S}$, a non-empty set $\mathcal{A} \subseteq [L(d)]$ is called a set of **alternatives** at d.

Definition 3.2. *A **finite observation network** is an observation network*

$$\mathbb{O}_0 = (V, E, L(\cdot), \mathcal{K}(\cdot), \mathcal{O}_0(\cdot))$$

such that the combined set of potential observations $\mathcal{O}_0(V)$ is finite.

Given an observation network

$$\mathbb{O} = (V, E, L(\cdot), \mathcal{K}(\cdot), \mathcal{O}(\cdot)),$$

*a **finite part** of $\mathbb{O}$ is a finite observation network*

$$\mathbb{O}_0 = (V, E, L(\cdot), \mathcal{K}(\cdot), \mathcal{O}_0(\cdot))$$

over the same knowledge base network such that $\mathcal{O}_0(x) \subseteq \mathcal{O}(x)$ for every agent $x \in V$.

Definition 3.3. *An observation network*

$$\mathbb{O} = (V, E, L(\cdot), \mathcal{K}(\cdot), \mathcal{O}(\cdot))$$

*is **sufficient** for a set of alternatives $\mathcal{A}$ at a decider d if for every complete extension $\mathcal{M}$ of $\mathcal{K}(V)$, there exists an alternative $A^{\mathcal{M}} \in \mathcal{A}$ such that*

$$\mathcal{K}(V) \cup (\mathcal{M} \cap \mathcal{O}(V)) \vdash A^{\mathcal{M}}.$$

Definition 3.4. *By an **observation** in $\mathbb{O}$ we mean a function $O(\cdot)$ such that for each agent $x \in V$, $O(x)$ is either $\top$ or a finite conjunction of sentences in $\mathcal{O}(x)$, and $\{O(x) : x \in V\}$ is consistent with $\mathcal{K}(V)$. We will write $O(V)$ for the sentence $\bigwedge_{x \in V} O(x)$.*

For each observation $O(\cdot)$ in $\mathbb{O}$, let $\mathcal{K}^O(x) = \mathcal{K}(x) \cup \{O(x)\}$ and form the knowledge base network

$$\mathbb{K}^O = (V, E, L(\cdot), \mathcal{K}^O(\cdot))$$

by adding the sentence $O(x)$ to the knowledge base $\mathcal{K}(x)$ for each $x \in V$.

The following result is an application of the Compactness Theorem.

Theorem 3.5. *Let*

$$\mathbb{O} = (V, E, L(\cdot), \mathcal{K}(\cdot), \mathcal{O}(\cdot))$$

be an observation network, let d be a decider, and let $\mathcal{A} \subseteq [L(d)]$ be a set of alternatives at d. Then the following are equivalent:

(i) *$\mathbb{O}$ is sufficient for $\mathcal{A}$;*
(ii) *there exists a finite set $\mathcal{A}_0 \subseteq \mathcal{A}$ and a finite part $\mathbb{O}_0$ of $\mathbb{O}$ such that $\mathbb{O}_0$ is sufficient for $\mathcal{A}_0$.*
(iii) *there are finitely many observations $O_1(\cdot), \ldots, O_n(\cdot)$ in $\mathbb{O}$ and finitely many alternatives $A_1, \ldots, A_n$ in $\mathcal{A}$ such that:*

$$\mathcal{K}(V) \vdash O_1(V) \vee \cdots \vee O_n(V); \tag{1}$$

$$\textit{for each } k \le n, \mathcal{K}(V) \vdash O_k(V) \to A_k. \tag{2}$$

Discussion: In this result, all that matters is the combined knowledge base $\mathcal{K}(V)$ and observation set $\mathcal{O}(V)$. The knowledge bases and observations of the individual agents will play a role later on in this paper. Each complete extension $\mathcal{M}$ of the combined knowledge base $\mathcal{K}(V)$ corresponds to a possible scenario. Condition (i) says that in every possible scenario, some alternative $A \in \mathcal{A}$ can be proved from the combined knowledge bases and the observations of the agents. Condition (ii) says that there are predetermined finite sets of observations $\mathcal{O}_0(V)$ and of alternatives $\mathcal{A}_0$ such that in every possible scenario, some alternative in $\mathcal{A}_0$ can be proved from the combined knowledge bases and the observations of the agents in $\mathcal{O}_0(V)$.

Condition (iii) gives a characterization of sufficiency that does not mention complete extensions. (1) says that in every scenario, one of the observations $O_m(\cdot)$ will be made. (2) says that the alternative A_m can be proved from the combined knowledge base and the observations $O_m(x), x \in V$.

Proof of Theorem 3.5. (iii) $\Rightarrow$ (i): Assume (iii). Let $\mathcal{M}$ be a complete extension of $\mathcal{K}(V)$. By (1), $\mathcal{M} \vdash O_k(V)$ for some $k \le n$, so by (2) we have $\mathcal{M} \vdash A_k$. Since $A_k \in \mathcal{A}$, this proves (i).

(i) $\Rightarrow$ (ii): Assume (i). By (i), for each complete extension $\mathcal{M}$ of $\mathcal{K}(V)$ there is an alternative $A^{\mathcal{M}} \in \mathcal{A}$ such that

$$\mathcal{K}(V) \cup (\mathcal{M} \cap \mathcal{O}(V)) \vdash A^{\mathcal{M}}.$$

By the Compactness Theorem, for each complete extension $\mathcal{M}$ of $\mathcal{K}(V)$, there is a finite set $\mathcal{O}^{\mathcal{M}} \subseteq (\mathcal{M} \cap \mathcal{O}(V))$ such that

$$\mathcal{K}(V) \cup \mathcal{O}^{\mathcal{M}} \vdash A^{\mathcal{M}}.$$

By the Compactness Theorem again, every consistent set of sentences in $L(V)$ has at least one complete extension. The set of sentences

$$\mathcal{K}(V) \cup \{\neg \bigwedge \mathcal{O}^{\mathcal{M}} \ : \mathcal{M} \text{ is a complete extension of } \mathcal{K}(V)\}$$

does not have a complete extension, and hence is not consistent. By the Compactness Theorem yet again, this set has a finite subset that is inconsistent. Therefore there are complete extensions $\mathcal{M}_1, \ldots, \mathcal{M}_n$ of $\mathcal{K}(V)$ such that

$$\mathcal{K}(V) \cup \{\neg O_1, \ldots, \neg O_n\}$$

is inconsistent, where $O_i = \bigwedge \mathcal{O}^{\mathcal{M}_i}$. Then

$$\mathcal{K}(V) \vdash O_1 \vee \cdots \vee O_n.$$

Let

$$\mathcal{A}_0 = \{A^{\mathcal{M}_1}, \cdots, A^{\mathcal{M}_n}\}.$$

Then $\mathcal{A}_0$ is a finite subset of $\mathcal{A}$. For each agent $x \in V$ let $\mathcal{O}_0(x)$ be a finite set of sentences such that

$$\mathcal{O}(x) \supseteq \mathcal{O}_0(x) \supseteq \mathcal{O}(x) \cap (\mathcal{O}^{\mathcal{M}_1} \cup \cdots \cup \mathcal{O}^{\mathcal{M}_n}),$$

and $\mathcal{O}_0(x)$ is closed under finite conjunctions. Then

$$\mathbb{O}_0 = (V, E, L(\cdot), \mathcal{K}(\cdot), \mathcal{O}_0(\cdot))$$

is a finite part of $\mathbb{O}$. For each complete extension $\mathcal{N}$ of $\mathcal{K}(V)$ we have $\mathcal{N} \vdash O_k$ for some $k \leq n$, and therefore

$$(\mathcal{K}(V) \cup \mathcal{O}^{\mathcal{M}_k}) \subseteq (\mathcal{N} \cap \mathcal{O}_0(V)).$$

We have $\mathcal{K}(V) \vdash O_k \to A^{\mathcal{M}_i}$, so $\mathcal{K}(V) \cup \mathcal{O}^{\mathcal{M}_k} \vdash A^{\mathcal{M}_k}$. It follows that

$$(\mathcal{N} \cap \mathcal{O}_0(V)) \vdash A^{\mathcal{M}_k}$$

and $A^{\mathcal{M}_k} \in \mathcal{A}_0$. Then $\mathbb{O}_0$ is sufficient for $\mathcal{A}_0$, and (ii) is proved.

(ii) $\Rightarrow$ (iii): Assume (ii). Since $\mathcal{O}_0(V)$ is finite, there are finitely many subsets $\mathcal{P}_1, \ldots, \mathcal{P}_n$ of $\mathcal{O}_0(V)$ such that:

(a) For every complete extension $\mathcal{M}$ of $\mathcal{K}(V)$,

$$\mathcal{M} \cap \mathcal{O}_0(V) \in \{\mathcal{P}_1, \ldots, \mathcal{P}_n\};$$

(b) for each $k \leq n$, $\mathcal{K}(V) \cup \mathcal{P}_k$ is consistent.

For each $k \leq n$ and $x \in V$, let $O_k(x)$ be a sentence in $\mathcal{O}(x)$ that is logically equivalent to $\bigwedge(\mathcal{P}_k \cap \mathcal{O}_0(x))$. Condition (1) follows easily from (a). Note that each $O_k(\cdot)$ is an observation in $\mathbb{O}$. By (b), for each $k \leq n$ there is a complete extension $\mathcal{M}_k$ of $\mathcal{K}(V) \cup \mathcal{P}_k$, so there is an alternative $A_k \in \mathcal{A}_0$ such that $\mathcal{K}(V) \cup (\mathcal{M}_k \cap \mathcal{O}_0(V) \vdash A_k$. Hence $\mathcal{K}(V) \cup \mathcal{P}_k \vdash A_k$, and (2) follows. This proves (iii). □

4. Report Plans

In the rest of this paper, we will focus on what happens when information is passed between neighboring agents.

Suppose $\mathbb{S}$ is a signature network. Then for every observation network

$$\mathbb{O} = (V, E, L(\cdot), \mathcal{K}(\cdot), \mathcal{O}(\cdot))$$

over $\mathbb{S}$ and every observation $O(\cdot)$ in $\mathbb{O}$, $\mathbb{K}^O$ is a knowledge base network over $\mathbb{S}$. In this way, each observation network over $\mathbb{S}$ gives rise to a whole family of knowledge base networks over $\mathbb{S}$. So if $\mathbb{S}$ is report complete, then for every knowledge base network $\mathbb{K}^O$ in this family, every sentence that is provable from the combined knowledge base is report provable.

The following is an immediate consequence of Theorem 3.5

Corollary 4.1. *Let*

$$\mathbb{O} = (V, E, L(\cdot), \mathcal{K}(\cdot), \mathcal{O}(\cdot))$$

be an observation network over a report complete signature network $\mathbb{S}$. Let $\mathcal{A} \subseteq [L(d)]$ be a set of alternatives at a decider d. Suppose that $\mathbb{O}$ is sufficient for $\mathcal{A}$. Then there are finitely many observations $O_1(\cdot), \ldots, O_n(\cdot)$ in $\mathbb{O}$ and finitely many alternatives $A_1, \ldots, A_n$ in $\mathcal{A}$ such that:

(a) $\mathcal{K}(V) \vdash (O_1(V) \vee \cdots \vee O_n(V))$;
(b) *for each $k \leq n$, A_k is report provable in $\mathbb{K}^{O_k}$ at d.*

This corollary shows that there is a predetermined finite set of observations $O_k(\cdot)$ and alternatives A_k, $k \leq m$, such that in every scenario,

- one of the observations $O_k(\cdot)$ will be made;
- the alternative A_k will be correct in the sense that it is provable from $\mathcal{K}(V) \cup O_k(V)$;
- reports between neighboring agents will propagate through the network, and then the decider will be able to prove the alternative A_k from its knowledge base and the sentences that are reported to him.

An observation network $\mathbb{O}$ is called **Boolean closed** if for every agent $x \in V$, $\top \in \mathcal{O}(x)$, and for any $A, B \in \mathcal{O}(x)$ the conjunction $A \wedge B$ and negation $\neg A$ are logically equivalent to sentences in $\mathcal{O}(x)$.

In the case that the observation network is Boolean closed and its signature network contains a signature tree, one can improve upon Corollary 4.1. We will show that in that case there must be a finite plan that will always enable the decider to arrive at a correct alternative after a "single pass" of reports through the network. For each agent x and edge (x, y), this plan will provide x with a finite rule that gives a sentence C to report along (x, y), where C depends only on the observation by x and the reports received by x.

A finite set of sentences $\{C_1, \ldots, C_n\}$ is a **partition** if the sentences are mutually exclusive and their disjunction is logically valid, that is,

$$\vdash \neg(C_i \wedge C_j) \text{ whenever } i < j, \text{ and } \vdash C_1 \vee \cdots \vee C_n.$$

Definition 4.2. *Let $\mathbb{O}$ be an observation network and $\mathcal{A}$ be a set of alternatives at a decider d. A* ***report plan*** *for $(\mathbb{O}, \mathcal{A}, d)$ is an object*

$$\mathbb{P} = (\mathcal{O}_0(\cdot), \mathcal{C}_0(\cdot), \mathcal{A}_0)$$

such that:

- *for each agent $x \in V$, $\mathcal{O}_0(x) \subseteq \mathcal{O}(x)$ and $\mathcal{O}_0(x)$ is a partition;*
- *for each edge $(x, y) \in E$, $\mathcal{C}_0(x, y)$ is a finite set of sentences in the common language $[L(x) \cap L(y)]$,*
- *$\mathcal{A}_0$ is a finite subset of $\mathcal{A}$;*
- *by a* ***potential report*** *(to x) we mean a set $\mathcal{R}(x)$ of sentences consisting of one element of $\mathcal{C}_0(z, x)$ for each edge (z, x) ($\mathcal{R}(x)$ is empty if x is a source);*
- *for each edge $(x, y) \in E$, observation $O(x) \in \mathcal{O}_0(x)$, and potential report $\mathcal{R}(x)$, there is a sentence $C \in \mathcal{C}_0(x, y)$ such that $\mathcal{K}(x) \cup \{O(x)\} \cup \mathcal{R}(x) \vdash C$;*
- *for each observation $O(d) \in \mathcal{O}_0(d)$ and potential report $\mathcal{R}(d)$, there is a sentence $A \in \mathcal{A}_0$ such that $\mathcal{K}(d) \cup \{O(d)\} \cup \mathcal{R}(d) \vdash A$;*
- *$\mathbb{P}$* ***avoids cycles****, that is, every directed cycle contains an edge e such that $\mathcal{C}_0(e) = \{\top\}$.*

Remark 4.3. *Let $\mathbb{P}$ be a report plan for $(\mathbb{O}, \mathcal{A}, d)$. Then for each complete extension $\mathcal{M}$ of $\mathcal{K}(V)$ there is a unique observation $O^{\mathbb{P},\mathcal{M}}(\cdot)$ in $\mathbb{O}$ such that $O^{\mathbb{P},\mathcal{M}}(x) \in \mathcal{O}_0(x)$ for each agent $x \in V$ and $\mathcal{M} \vdash O^{\mathbb{P},\mathcal{M}}(V)$. This follows from the fact that each set $\mathcal{O}_0(x)$ in a report plan is a partition.*

The story: When $(x, y) \in E$, we will say that x is a **child** of y and that y is a **parent** of x. Intuitively, a report plan provides each agent

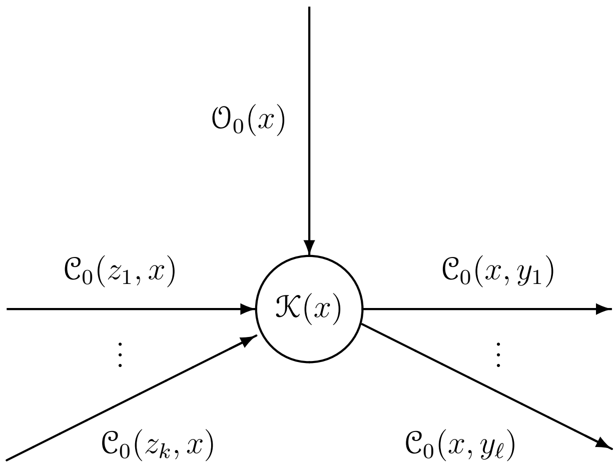

FIGURE 1. Checklist for agent x.

x with a checklist consisting of a finite partition $\mathcal{O}_0(x)$ of potential observations, and finite sets of $\mathcal{C}_0(z, x)$ and $\mathcal{C}_0(x, y)$ for each child z of x and each parent y of x (see Figure 1). From the point of view of agent x, $\mathcal{C}_0(z, x)$ is a finite set of "question sentences" to watch for, and $\mathcal{C}_0(x, y)$ is a finite set of "answer sentences" to try to prove. (Thus for each edge (x, y), $\mathcal{C}_0(x, y)$ is both a set of questions asked by y and a set of possible answers by x). The decider d is also provided with a finite set of alternatives $\mathcal{A}_0$. Meaningful reports can only be passed along edges (x, y) such that $\mathcal{C}_0(x, y) \neq \{\top\}$.

Remark 4.3 says that in each scenario $\mathcal{M}$, every agent x makes exactly one observation $O^{\mathbb{P},\mathcal{M}}(x) \in \mathcal{O}_0(x)$. Each agent x asks each of its children z the finite set of questions $\mathcal{C}_0(z, x)$ and receives an answer in this set. These answers together form a potential report $\mathcal{R}(x)$ to x. Then for each parent y of x, agent x proves one of the sentences in $\mathcal{C}_0(x, y)$ from its knowledge base $\mathcal{K}(x)$, its observation $O^{\mathbb{P},\mathcal{M}}(x)$, and $\mathcal{R}(x)$, and reports this sentence as its answer to y. Finally, the decider d receives a potential report $\mathcal{R}(d)$ from its children and proves an alternative in $\mathcal{A}_0$ from its knowledge base $\mathcal{K}(x)$, its observation $O^{\mathcal{M}}(x)$, and $\mathcal{R}(x)$. We will call this process a report proof in $\mathbb{P}$ (defined formally in Definition 5.1).

In a report proof, each agent x selects an answer sentence $C(x, y) \in \mathcal{C}_0(x, y)$ for each edge (x, y), and the decider d selects an alternative

$A \in \mathcal{A}$. To do its part in a report proof, each agent x only needs its own knowledge base $\mathcal{K}(x)$ and observation $O^{\mathbb{P},\mathcal{M}}(x)$, and the sets sentences $\mathcal{C}_0(z, x)$ for each child z and $\mathcal{C}_0(x, y)$ for each parent y. The agents do not need to know the observations or question sentences of the other agents. A report proof is a particular realization of the report plan where each agent makes an observation, receives a single potential report from its children, and proves one of finitely many possible answers to report to each of its parents.

Why do we require that a report plan avoids cycles? A key requirement for a report plan is that every report proof in it produces a report provable sentence (Lemma 5.2). The following small example shows that if $\mathbb{P}$ does not avoid cycles, there can be an object that looks like a report proof but produces a sentence that is not report provable. We also note that avoiding cycles guarantees that the report proofs have only be a single pass through the network.

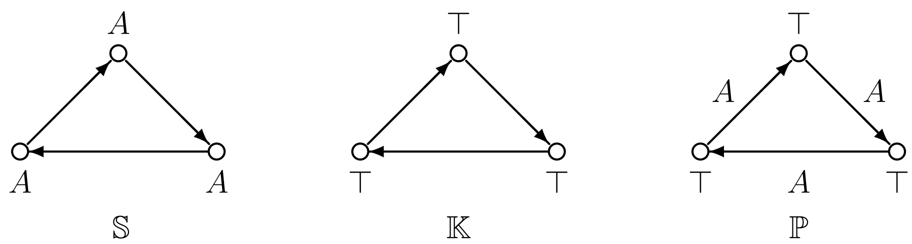

FIGURE 2. Example 4.4.

Example 4.4. Let $\mathbb{K}$ be the knowledge base network shown in Figure 2. It is a triangle where each agent has the signature $\{A\}$ and the knowledge base $\{\top\}$. The underlying signature network $\mathbb{S}$ is report complete. Let d be any of the three agents. Then d is a decider. Let $\mathbb{O}$ be the Boolean closed observation network over $\mathbb{K}$ where $\mathcal{O}(x) = \{\top, \bot\}$ for each agent x. The only observation is the function $O(x) = \top$, and the combined knowledge base for $\mathbb{K}^O$ is $\mathcal{K}^O(V) = \{\top\}$. Let $\mathcal{A}_0 = \{A\}$, and let $\mathbb{P} = (\mathcal{O}_0(\cdot), \mathcal{C}_0(\cdot), \mathcal{A}_0)$ assign $\mathcal{O}_0(x) = \{\top\}$ to every agent x, and $\mathcal{C}_0(x, y) = \{A\}$ to every edge (x, y) (so the sentence A appears out of thin air and is reported along every edge). $\mathbb{P}$ satisfies all the requirements for being a report plan for $(\mathbb{O}, \mathcal{A}_0, d)$ except for avoiding cycles. But the sentence A is not provable from $\mathcal{K}^O(V)$, and hence not report provable at d in the knowledge base network $\mathbb{K}^O$.

Example 4.5. Here is an example of a report plan. Figure 3 shows a signature network $\mathbb{S}$ with decider d, a knowledge base network $\mathbb{K}$ over

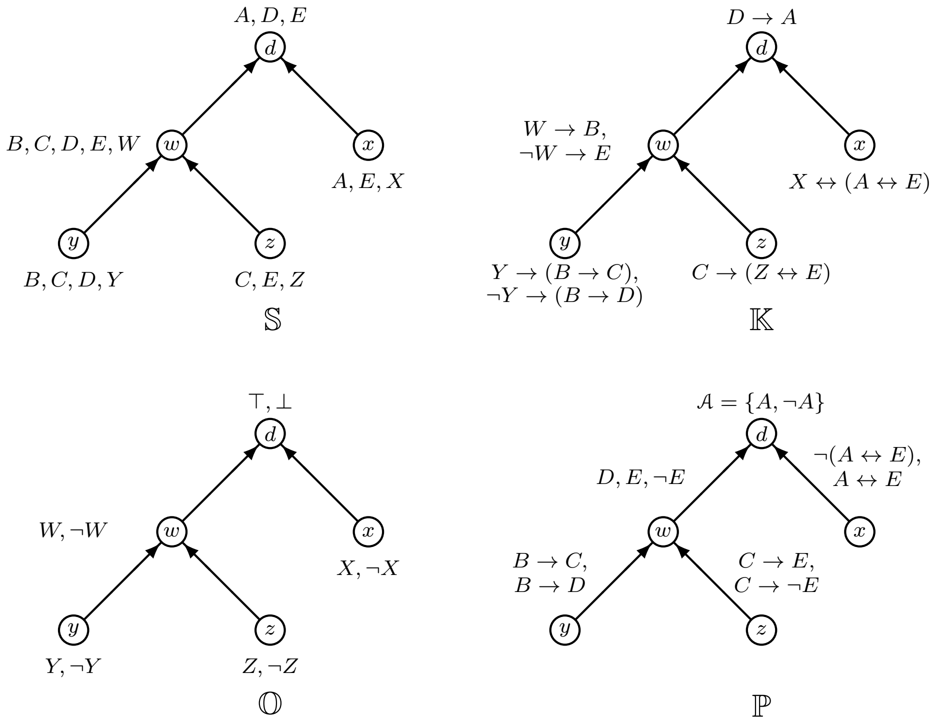

FIGURE 3. Example 4.5.

$\mathbb{S}$, a Boolean closed observation network $\mathbb{O}$ over $\mathbb{K}$, a set of alternatives $\mathcal{A} = \{A, \neg A\}$, and a report plan $\mathbb{P}$ for $(\mathbb{O}, \mathcal{A}, d)$.

For instance, agent w has signature $L(w) = \{W, B, C, D, E\}$, knowledge base $\mathcal{K}(w) = \{W \to B, \neg W \to E\}$, and potential observations $\mathcal{O}(w) = \{W, \neg W\}$. In the report plan $\mathbb{P}$, w's checklist consists of the sets of potential observations $\mathcal{O}(w)$, question sentences $\mathcal{C}_0(y, w) = \{B \to C, B \to D\}$ and $\mathcal{C}_0(z, w) = \{C \to E, C \to \neg E\}$, and possible answers $\mathcal{C}_0(w, d) = \{D, E, \neg E\}$. By checking through each of the 16 possible observation functions $O(\cdot)$, one can show that $\mathbb{P}$ actually is a report plan for $(\mathbb{O}, \mathcal{A}, d)$.

The remark below says that being a report plan is preserved under adding new sentences to the knowledge bases, new potential observations, or new alternatives.

Remark 4.6. *Suppose* $\mathbb{P}$ *is a report plan for* $(\mathbb{O}, \mathcal{A}, d)$. *Let*

$$\mathbb{O}' = (V, E, L(\cdot), \mathcal{K}'(\cdot), \mathcal{O}'(\cdot))$$

be an observation network over the same $\mathbb{S}$ *such that* $\mathcal{K}(x) \subseteq \mathcal{K}'(x)$ *and* $\mathcal{O}(x) \subseteq \mathcal{O}'(x)$ *for each agent* $x \in V$. *Also suppose that* $\mathcal{A} \subseteq \mathcal{A}' \subseteq [L(d)]$. *Then* $\mathbb{P}$ *is still a report plan for* $(\mathbb{O}', \mathcal{A}', d)$.

5. Main Results

In this section we prove two theorems about report plans. Theorem 5.3 will show that a report plan guarantees that for every observation, some alternative is report provable from the observation and the knowledge bases. Theorem 5.6 will show that when the sets of potential observations are Boolean closed, the signature network contains a signature tree, and the observation network is sufficient, then there will be a report plan.

The proof of Theorem 5.3 will describe how a report plan $\mathbb{P}$ can be executed. This will be done by means of a report proof in $\mathbb{P}$, which is a finite process that can be carried out whenever each agent x makes an observation $O(x) \in \mathcal{O}_0(x)$. For each such observation and for every edge $(x, y) \in E$, a report proof will eventually produce a sentence $C \in \mathcal{C}_0(x, y)$ that is reportable along (x, y). The decider d will then be able to prove some sentence $A \in \mathcal{A}_0$ from its knowledge base, its observation, and the sentences reported to d, and it will follow that A is report provable.

We first give a formal definition of report proof, and show that report proofs give report provability.

Definition 5.1. *Let $\mathbb{P} = (\mathcal{O}_0(\cdot), \mathcal{C}_0(\cdot), \mathcal{A}_0)$ be a report plan for $(\mathcal{O}, \mathcal{A}, d)$, $O(\cdot)$ be an observation in $\mathbb{P}$, and $A \in \mathcal{A}_0$. A* ***report proof*** *of A from $O(\cdot)$ in $\mathbb{P}$ is a function $C(\cdot)$ such that:*

- *for each $(x, y) \in E$, $C(x, y) \in \mathcal{C}_0(x, y)$ and*
$$\mathcal{K}(x) \cup \{O(x)\} \cup \{C(z, x) : (z, x) \in E\} \vdash C(x, y);$$
- $$\mathcal{K}(d) \cup \{O(d)\} \cup \{C(z, d) : (z, d) \in E\} \vdash A.$$

Lemma 5.2. *Suppose that $C(\cdot)$ is a report proof of A from an observation $O(\cdot)$ in a report plan $\mathbb{P}$. Then in the knowledge base network $\mathbb{K}^O$, $C(x, y)$ is reportable along (x, y) for each edge $(x, y) \in E$, and A is report provable at d.*

Proof. Let F be the set of all edges $(x, y) \in E$ such that $\mathcal{C}_0(x, y) \neq \{\top\}$. For each agent $X \in V$, define the **height** of x to be the length of the longest path in (V, F) that ends in x. Since the report plan $\mathbb{P}$ avoids cycles, (V, F) contains no directed cycles, so each agent has finite height. An easy induction on the height of x shows that $C(x, y)$ is reportable along (x, y) for each edge $(x, y) \in E$. It then follows that A is report provable at d. $\square$

Theorem 5.3. *Let $\mathbb{O}$ be an observation network with decider d, and let $\mathbb{P} = (\mathcal{O}_0(\cdot), \mathcal{C}_0(\cdot), \mathcal{A}_0)$ be a report plan for $(\mathbb{O}, \mathcal{A}, d)$. Let $O(\cdot)$ be an* *observation such that $O(x) \in \mathcal{O}_0(x)$ for every agent x. Then for some alternative $A \in \mathcal{A}_0$, there exists a report proof of A from $O(\cdot)$ in $\mathbb{P}$, and hence A is report provable at d in the knowledge base network $\mathbb{K}^O$.*

Proof. To construct a report proof, we describe a process in which each agent x will act at most once. so there will be a single pass through the network.

At stage 0, no agents will act, each edge (x, y) such that $\mathcal{C}_0(x, y) = \{\top\}$ will receive the report $\top$, and the other edges will not yet have a report. At stage $n+1$, an agent x will act if it has not previously acted, and every edge (z, x) has a report. These reports together will form a potential report $\mathcal{R}(x)$ to x. In this action, x will prove a sentence $C(x, y) \in \mathcal{C}_0(x, y)$ from x's knowledge base $\mathcal{K}(x)$, observation $O(x)$, and the incoming report $\mathcal{R}(x)$. x will then report the sentence $C(x, y)$ along the edge (x, y). The following Claim shows that this process can be carried out.

Claim.

- Every agent acts exactly once.
- Whenever an agent x acts, the set of sentences reported along the edges (z, x) form a potential report to x.
- Whenever a sentence C is reported along an edge (x, y) in the report proof, C is reportable in $\mathbb{K}^O$ along (x, y).

Proof of Claim: As in the proof of Lemma 5.2, we let F be the set of all edges $(x, y) \in E$ such that $\mathcal{C}_0(x, y) \neq \{\top\}$, and we define the **height** of an agent x to be the length of the longest path in (V, F) that ends in x. Each agent x of height 0 acts at stage 1, because there can be no edge $(z, x) \in F$, so every edge $(z, x) \in E$ (if any) has the report $\top$. Thus from the definition of a report plan, the claim holds when x has height 0.

Assume that the Claim holds for all agents x of height at most n. Note that for each edge $(z, x) \in F$, the height of z is $\leq n$. By the Claim for agents of height $\leq n$, there is a sentence reported on each edge $(z, x) \in E$, and this sentence is reportable in $\mathbb{K}^O$ along (z, x). Therefore x acts at some stage $\leq n + 1$. Since $\mathbb{P}$ is a report plan, for each edge $(x, y) \in$ there is a sentence $C(x, y) \in \mathcal{C}_0(x, y)$ that is provable from $\mathcal{K}(x) \cup \{O(x)\} \cup \mathcal{R}(x)$. It follows that the Claim holds for x.

One can also show that an agent of height n acts at stage $n + 1$ and not earlier, but this will not be needed.

By the Claim, there is a potential report $\mathcal{R}(d)$ to d such that for each edge (z, d) ending in d, $\mathcal{R}(d)$ contains a sentence reportable along (z, d). Since $\mathbb{P}$ is a report plan, there is a sentence $A \in \mathcal{A}_0$ such that

$\mathcal{K}(d) \cup \{O(d)\} \cup \mathcal{R}(d) \vdash A$. This shows that $C(\cdot)$ is a report proof of A from $O(\cdot)$ in $\mathbb{P}$. Finally, by Lemma 5.2, A is report provable in $\mathbb{K}^O$ at d. □

Consider a Boolean closed observation network $\mathbb{O}$ over a signature network $\mathbb{S}$. If $\mathbb{O}$ is sufficient for a set of alternatives $\mathcal{A}$, must there exist a report plan for $(\mathbb{O}, \mathcal{A}, d)$? Not necessarily. For one thing, $\mathbb{S}$ must be report complete. But even that is not enough. We will also need to assume that $\mathbb{S}$ contains a signature tree, in order to avoid examples such as the following.

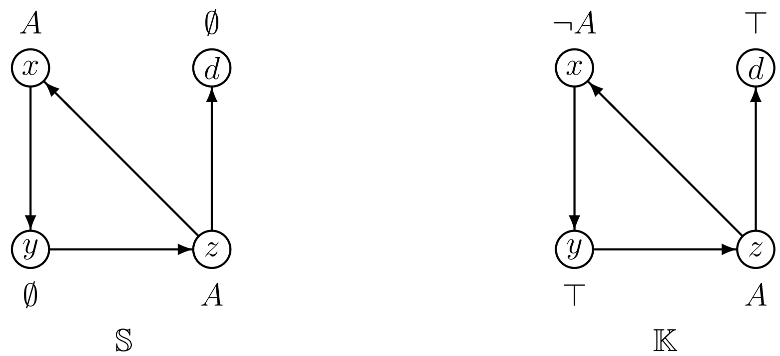

FIGURE 4. Example 5.4

Example 5.4. We give a report complete signature network $\mathbb{S}$, an observation network $\mathbb{O}$ over $\mathbb{S}$, and a set of alternatives $\mathcal{A}$ at a decider d such that $\mathbb{O}$ is sufficient for $\mathcal{A}$, but there is no report plan $\mathbb{P}$ for $(\mathbb{O}, \mathcal{A}, d)$. $\mathbb{S}$ will not contain a signature tree.

Figure 4 shows a signature network $\mathbb{S}$ and knowledge base network $\mathbb{K}$ over $\mathbb{S}$. The upper right agent d is the only decider, and $\mathbb{S}$ is report complete at d. Let $\mathbb{O}$ be the observation network over $\mathbb{K}$ such that $\mathcal{O}(u) = \{\top, \bot\}$ for each agent u. $\mathbb{O}$ is Boolean closed. We have $\mathcal{K}^O(V) \vdash \bot$, so $\mathbb{O}$ is sufficient for $\{\bot\}$ at d. We show that there does not exist a report plan for $(\mathbb{O}, \{\bot\}, d)$.

Proof. To prove $\bot$ at d, agent z reports A (or a sentence equivalent to A) to x, x reports $\bot$ to y, y reports $\bot$ to z, z reports $\bot$ to d, and then d proves $\bot$. Note that z acts twice. If there were a report plan for $(\mathbb{O}, \{\bot\}, d)$, then (by cycle avoidance) some edge e on the triangle must have $\mathcal{C}_0(e) = \{\top\}$, so only $\top$ could be reported along the edges (y, z) and (z, d), and d could not prove $\bot$ from the report it receives. □

Notice that we have only defined report proofs in the context of a report plan. In cases like Example 5.4, where the signature network is report complete but there is no report plan, an alternative can be report provable but have no report proof in the sense of Definition 5.1. In more complicated cases, several passes through the network may be needed to establish report provability, while a report proof in the sense of Definition 5.1 has only a single pass.

Definition 5.5. *We say that an observation network $\mathbb{O}$ is* ***plan complete*** *at a decider d if for every set of sentences $\mathcal{A} \subseteq [L(d)]$ for which $\mathbb{O}$ is sufficient, there exists a report plan for $(\mathbb{O}, \{A\}, d)$.*

In Example 5.4, the signature network $\mathbb{S}$ is report complete but the observation network $\mathbb{O}$ is not plan complete at d.

Theorem 5.6. *Let $\mathbb{S} = (V, E, L(\cdot))$ be a signature network that contains a signature tree $\mathbb{T} = (V, F, L(\cdot))$ with decider d. Then every Boolean closed observation network $\mathbb{O}$ over $\mathbb{S}$ is plan complete at d.*

The proof will show a bit more: For each $\mathcal{A} \subseteq [L(d)]$ there is report plan $\mathbb{P}$ for $(\mathbb{O}, \mathcal{A}, d)$ such that $\mathcal{C}_0(x, y) = \{\top\}$ whenever $(x, y) \in E \setminus F$. Intuitively, this means that each agent reports only to its unique parent in the tree (V, F).

Proof of Theorem 5.6. Let $\mathcal{A} \subseteq [L(d)]$. Note that if we are able to get a report plan for $(\mathbb{O}', \mathcal{A}, d)$ where $\mathbb{O}'$ is the observation network

$$\mathbb{O}' = (V, F, L(\cdot), \mathcal{K}(\cdot), \mathcal{O}(\cdot))$$

formed by replacing E by F and leaving everything else unchanged, then we at once get a report plan for $(\mathbb{O}, \mathcal{A}, d)$ by putting $\mathcal{C}_0(x, y) = \{\top\}$ for each edge $(x, y) \in E \setminus F$. We may therefore assume without loss of generality that $\mathbb{S}$ is already a signature tree, so that $E = F$.

Since (V, E) is a tree, each agent $x \neq d$ has a unique parent $y = p(x)$, and d has no parents. Moreover, (V, E) has no directed cycles. By Fact 2.5, $\mathbb{S}$ is report complete. Then by Corollary 4.1, there are finitely many observations $O_1(\cdot), \ldots, O_n(\cdot)$ in $\mathbb{O}$ and finitely many alternatives $A_1, \ldots, A_n$ in $\mathcal{A}$ such that conditions 4.1 (a) and (b) hold. Let $\mathcal{A}_0 = \{A_1, \ldots, A_n\}$.

For each agent $x \in V$, let $\mathcal{O}_0(x)$ be the set of all sentences of the form $P_1(x) \wedge \cdots \wedge P_n(x)$ where for each $k \leq n$, $P_k(x) \in \{O_k(x), \neg O_k(x)\}$. It is clear that each $\mathcal{O}_0(x)$ is a finite partition. Since $\mathbb{O}$ is Boolean closed, we have $\mathcal{O}_0(x) \subseteq \mathcal{O}(x)$ for each x.

We now build the sets of question sentences $\mathcal{C}_0(x, y)$ for each $(x, y) \in E$. Consider an observation $O(\cdot)$ in $\mathcal{O}_0(\cdot)$. For each x, $O(x)$ has the form $O(x) = P_1(x) \wedge \cdots \wedge P_n(x)$ where $P_k(x) \in \{O_k(x), \neg O_k(x)\}$ for each $k \leq n$. Therefore the sentence $\bigwedge_{x\in V} O_k(x)$ is either provable from $\bigwedge_{x\in V} O(x)$ or inconsistent with $\bigwedge_{x\in V} O(x)$. If $\bigwedge_{x\in V} O(x)$ is not consistent with $\mathcal{K}(V)$, then every sentence is provable from $\mathcal{K}(V)$ and $\bigwedge_{x\in V} O(x)$. By 4.1 (a), we have

$$\mathcal{K}(V) \vdash (O_1(V) \vee \cdots \vee O_n(V)).$$

Hence if $\bigwedge_{x\in V} O(x)$ is consistent with $\mathcal{K}(V)$, there exists $k \leq n$ such that $\bigwedge_{x\in V} O(x)$ is consistent with $\bigwedge_{x\in V} O_k(x)$, and hence $\bigwedge_{x\in V} O_k(x)$ is provable from $\bigwedge_{x\in V} O(x)$. So in every case there exists $k \leq n$ such that

$$\mathcal{K}(V) \vdash \bigwedge_{x\in V} O(x) \to \bigwedge_{x\in V} O_k(x),$$

and hence

$$\mathcal{K}^O(V) \vdash \bigwedge_{x\in V} O_k(x).$$

By 4.1 (b), A_k is report provable in $\mathbb{K}^{O_k}$ at d, so A_k is provable from $\mathcal{K}^O(V)$. Since $\mathbb{S}$ is report complete, A_k is report provable in $\mathbb{K}^O$ at d. We let $A^O = A_k$ and note that $A^O \in \mathcal{A}_0$.

It follows that there are sentences $B^O(x,y) \in [L(x) \cap L(y)]$ for each edge $(x,y) \in E$ such that

$$\mathcal{K}(x) \cup \{O(x)\} \cup \{B^O(z,x) \,:\, (z,x) \in E\} \vdash B^O(x,y)$$

and

$$\mathcal{K}(d) \cup \{O(d)\} \cup \{B^O(z,d) \,:\, (z,d) \in E\} \vdash A^O.$$

For each edge $(x,y) \in E$, let

$$\mathcal{D}_0(x,y) = \{B^O(x,y) : O(\cdot) \text{ is an observation in } \mathcal{O}_0(\cdot)\}.$$

We write $x \leq y$ if there is a path from x to y in (V,E), and write $x < y$ if $x \leq y$ and $x \neq y$. As before, the **height** of y is the length of the longest path ending in y. Note that if $x < y$ then the height of x is less than the height of y.

Now, for each observation $O(\cdot)$ in $\mathcal{O}_0(\cdot)$, define the sentences $D^O(x,y)$, $(x,y) \in E$, inductively on the height of x in the tree (V,E) as follows: $D^O(x,y)$ is the conjunction of all sentences in $\mathcal{D}_0(x,y)$ that are provable from

$$\mathcal{K}(x) \cup \{O(x)\} \cup \{D^O(z,x) : (z,x) \in E\}.$$

Finally, for each edge $(x,y) \in E$ define

$$\mathcal{C}_0(x,y) = \{D^O(x,y) : O(\cdot) \text{ is an observation in } \mathcal{O}_0(\cdot)\}.$$

Since there are finitely many observations in $\mathcal{O}_0(\cdot)$, $\mathcal{D}_0(x,y)$ and $\mathcal{C}_0(x,y)$ are finite subsets of $[L(x) \cap L(y)]$ for each edge $(x,y) \in E$. We let $\mathbb{P} = (\mathcal{O}_0(\cdot), \mathcal{C}_0(\cdot), \mathcal{A}_0)$, and will show that $\mathbb{P}$ is a report plan for $(\mathbb{O}, \mathcal{A}, d)$.

Consider an arbitrary agent $x \in V$ and let $B \in \mathcal{O}_0(x)$. An argument by induction on the height of z in (V, E) shows that for each edge $(z, y) \in E$, the sentences $D^O(z, y)$ depend only on the observations $O(u)$ such that $u \leq z$. Let $\{z_1, \ldots, z_m\}$ be the set of all children of x in (V, E), and let $\mathcal{R}(x)$ be a potential report to x in $\mathbb{P}$. Then $\mathcal{R}(x)$ has the form

$$\mathcal{R}(x) = \{D_1, \ldots, D_m\}$$

where each D_j is the conjunction of one or more sentences in $\mathcal{C}_0(z_j, x)$. Then for each $j \leq m$ there is an observation $O_j(\cdot)$ in $\mathcal{O}_0(\cdot)$ such that the sentence $D^{O_j}(z_j, x)$ occurs in the conjunction D_j.

Since (V, E) is a tree, for any distinct $i, j \leq m$, the sets $\{u : u \leq z_i\}$ and $\{u : u \leq z_j\}$ are disjoint. Therefore there is an observation $O(\cdot)$ in $\mathcal{O}_0(\cdot)$ such that $O(u) = O_j(u)$ for each $j \leq m$ and agent $u \leq z_j$, and $O(x) = B$. Then for each $j \leq m$ we have $D^O(z_j, x) = D^{O_j}(z_j, x)$, so $\mathcal{R}(x) \vdash D^O(z, x)$ whenever $(z, x) \in E$.

Whenever $(x, y) \in E$, $D^O(x, y)$ is a conjunction of sentences that are provable from $\mathcal{K}(x) \cup \{O(x)\} \cup \mathcal{R}(x)$, so

$$\mathcal{K}(x) \cup \{O(x)\} \cup \mathcal{R}(x) \vdash D^O(x, y).$$

By the definition of $\mathcal{C}_0(x, y)$ we have $D^O(x, y) \in \mathcal{C}_0(x, y)$.

In the case $x = d$, we have $\mathcal{R}(d) \vdash D^O(z, d)$ whenever $(z, d) \in E$. By induction on height, for each observation $O(\cdot)$ in $\mathcal{O}_0(\cdot)$ and edge $(z, y) \in E$, we have $D^O(z, y) \vdash B^O(z, y)$. Then $\mathcal{R}(d) \vdash B^O(z, d)$ whenever $(z, d) \in E$. We recall that

$$\mathcal{K}(d) \cup \{O(d)\} \cup \{B^O(z, d) : (z, d) \in E\} \vdash A^O.$$

Therefore

$$\mathcal{K}(d) \cup \{O(d)\} \cup \mathcal{R}(d) \vdash A^O.$$

This completes the proof that $\mathbb{P}$ is a report plan for $(\mathbb{O}, \mathcal{A}, d)$, and hence that $\mathbb{O}$ is plan complete at d. □

We pose an open question, which is a possible converse to Theorem 5.6.

Question 5.7. *Let $\mathbb{S}$ be a signature network and d be a decider for $\mathbb{S}$. Suppose that every Boolean closed observation network $\mathbb{O}$ over $\mathbb{S}$ is plan complete at d. Must $\mathbb{S}$ contain a signature tree with decider d?*

6. Special Cases

6.1. **Knowledge Base Networks.** What do the theorems in Section 5 tell us if we just have a knowledge base network instead of an observation network, and a single sentence to be proved instead of a set of alternatives (as in [5])? In other words, what do they tell us when the agents do not have options and there is just one alternative? To answer this, we note that every knowledge base network

$$\mathbb{K} = (V, E, L(\cdot), \mathcal{K}(\cdot))$$

can be made into a finite Boolean closed observation network

$$\mathbb{O} = (V, E, L(\cdot), \mathcal{K}(\cdot), \mathcal{O}(\cdot))$$

by putting $\mathcal{O}(x) = \{\top, \bot\}$ for each agent x. We call $\mathbb{O}$ the **minimal observation network** over $\mathbb{K}$. In the minimal observation network over $\mathbb{K}$, in any scenario every agent will observe the true sentence $\top$, because $\mathcal{O}(V) = \{\top, \bot\}$, and for every complete extension $\mathcal{M}$ of $\mathcal{K}(V)$, we have $\mathcal{M} \cap \mathcal{O}(V) = \{\top\}$.

Corollary 6.1. *Suppose that $\mathbb{O}$ is a minimal observation network, and there exists a report plan for $\mathbb{P} = (\mathcal{O}_0(\cdot), \mathcal{C}_0(\cdot), \mathcal{A}_0)$ for $(\mathbb{O}, \mathcal{A}, d)$. Then there is a report plan $\mathbb{P}' = (\mathcal{O}'_0(\cdot), \mathcal{C}'_0(\cdot), \mathcal{A}'_0)$ for $(\mathbb{O}, \mathcal{A}, d)$ such that:*

(i) *$\mathcal{O}'_0(x) = \{\top\}$ for every agent $x \in V$,*
(ii) *$\mathcal{C}'_0(x, y) = \{C(x, y)\}$ where $C(x, y) \in \mathcal{C}_0(x, y)$ for each edge (x, y),*
(iii) *$\mathcal{A}'_0 = \{A\}$ where $A \in \mathcal{A}_0$,*
(iv) *the function $C(\cdot)$ is a report proof of A from the observation $O(\cdot) = \top$ in $\mathbb{P}'$.*

Proof. We first note that the only partitions contained in $\mathcal{O}(V)$ are $\{\top\}$ and $\{\top, \bot\}$, so the sentence $\top$ belongs to $\mathcal{O}_0(x)$ for every agent x. Therefore the function $O(x) = \top$ is an observation in $\mathbb{O}$ with $O(x) \in \mathcal{O}_0(x)$ for each agent x. For each agent x, let $\mathcal{O}'_0(x) = \{\top\}$, so (i) will hold. The proof of Theorem 5.3 shows that there is a report proof of a sentence $A \in \mathcal{A}_0$ in $\mathbb{P}$ from $O(\cdot)$ such that every agent x acts only once, and reports a sentence $C(x, y) \in \mathcal{C}_0(x, y)$ for each edge (x, y). It follows that $\mathbb{P}'$ is a report plan for $(\mathbb{O}, \mathcal{A}, d)$, where $\mathcal{C}'_0(x, y) = \{C(x, y)\}$ for each edge $(x, y) \in E$, and $\mathcal{A}'_0 = \{A\}$. $\mathbb{P}'$ has the required properties (i)–(iv). □

Here is another open question.

Question 6.2. *Let $\mathbb{S}$ be a signature network and d a decider. Suppose that every minimal observation network $\mathbb{O}$ over $\mathbb{S}$ is plan complete. Must every Boolean closed observation network over $\mathbb{S}$ also be plan complete?*

Remark 6.3. *The following are equivalent:*

(a) *The answers to Questions 5.7 and 6.2 are both "yes";*

(b) *for every signature network $\mathbb{S}$ that does not contain a signature tree with decider d, there is a minimal observation network $\mathbb{O}$ over $\mathbb{S}$ that is not plan complete at d.*

Proof. Let $\mathbb{S}$ be a signature network with decider d.

Assume (a). Suppose that $\mathbb{S}$ does not contain a signature tree with decider d. By Question 5.7 there is a Boolean closed observation network $\mathbb{O}$ over $\mathbb{S}$ that is not plan complete at d. By Question 6.2, there is a minimal observation network over $\mathbb{S}$ that is not plan complete at d. Thus (b) holds.

Now assume (b). Suppose every minimal observation network $\mathbb{O}$ over $\mathbb{S}$ is plan complete. Then by (b), $\mathbb{S}$ contains a signature tree at d. Therefore (b) implies Question 5.7. By Theorem 5.6, every Boolean closed observation network over $\mathbb{S}$ is plan complete. Thus (b) also implies Question 6.2. □

6.2. **Fragments of First Order Logic.** It is pointed out in [5] that all the results in that paper hold for propositional logic. Therefore Theorem 5.6 holds for propositional logic.

Corollary 6.4. *Assume the hypotheses of Theorem 5.6. If each sentence in the combined knowledge base $\mathcal{K}(V)$, the combined set of potential observations $\mathcal{O}(V)$, and the set of alternatives $\mathcal{A}$ is a sentence in propositional logic, then there is a report plan $\mathbb{P}$ composed entirely of sentences in propositional logic.*

Similarly, all the results of [5] and hence Theorem 5.6 hold for the fragment of first order logic without quantifiers. A first order sentence is said to be **quantifier-free** if it has no quantifiers.

Corollary 6.5. *Assume the hypotheses of Theorem 5.6. If each sentence in the combined knowledge base $\mathcal{K}(V)$, the combined set of potential observations $\mathcal{O}(V)$, and the set of alternatives $\mathcal{A}$ is quantifier-free, then there is a report plan $\mathbb{P}$ composed entirely of quantifier-free sentences.*

6.3. **Approximate Values.** In many situations, there is a need to determine an approximate value of one or more unknown quantities. We briefly indicate how report plans might be used to approximate one unknown quantity. The idea can easily be generalized to the case of finitely many unknown quantities.

Suppose $\mathbb{O}$ is an observation network in which the signature $L(d)$ of the decider has at least the symbols $+, -, \leq$, a constant symbol for each rational number, and one extra constant symbol c for an "unknown quantity". Suppose the knowledge base $\mathcal{K}(d)$ has at least the axioms for an ordered abelian group and all true equations and inequalities involving rational numbers. We allow the possibility that $L(d)$ also has other symbols and $\mathcal{K}(d)$ has additional sentences, and make no restrictions about the other agents. We let $\mathbb{Q}$ denote the set of rational numbers. For each positive rational number r, let $\mathcal{A}(r)$ be the set of sentences

$$\{q \le c \wedge c \le q + r \,:\, q \in \mathbb{Q}\}.$$

Each sentence in $\mathcal{A}(r)$ says that c belongs to a closed interval of length r with rational endpoints.

Theorem 5.6 tells us that if $\mathbb{O}$ is Boolean closed, contains a signature tree with decider d, and is sufficient for $\mathcal{A}(r)$, then there exists a report plan $\mathbb{P}$ for $(\mathbb{O}, \mathcal{A}(r), d)$.

If $\mathbb{P} = (\mathcal{O}_0(\cdot), \mathcal{C}_0(\cdot), \mathcal{A}_0)$ is a report plan for $(\mathbb{O}, \mathcal{A}(r), d)$, and $O(x) \in \mathcal{O}_0(x)$ for each agent x, then Theorem 5.3 tells us that some sentence $A \in \mathcal{A}_0$ is report provable in the knowledge base network $\mathbb{K}^O$. This sentence A belongs to $\mathcal{A}(r)$, so it approximates the unknown value c within r.

Now let $r_0, r_1, \ldots$ be a sequence of positive rational numbers that converges to 0, and consider a sequence of report plans $\mathbb{P}^n$ for $(\mathbb{O}, \mathcal{A}(r_n), d)$. In each scenario, the sequence of report plans will produce a sequence of better and better approximations of the unknown quantity c. The following corollary to Theorem 5.3 shows that in every scenario, the sequence of report plans produces a unique real value for the unknown constant c.

Corollary 6.6. *Suppose that $r_0, r_1, \ldots$ is a sequence of positive rational numbers that converges to 0, and that for each n, $\mathbb{P}^n = (\mathcal{O}_0^n(\cdot), \mathcal{C}_0^n(\cdot), \mathcal{A}_0^n)$ is a report plan for $(\mathbb{O}, \mathcal{A}(r_n), d)$. Let $\mathcal{M}$ be a complete extension of $\mathcal{K}(V)$ and let $O^n(\cdot) = O^{\mathbb{P}^n, \mathcal{M}}(\cdot)$ be the unique observation given in Remark 4.3. Then the real number*

$$s = \sup\{q \in \mathbb{Q} \,:\, \mathcal{M} \vdash q \le c\}$$

exists and for each n, s belongs to an interval $[q_n, q_n + r_n]$ where the sentence

$$q_n \le c \wedge c \le q_n + r_n$$

belongs to $\mathcal{A}_0^n$ and is report provable in $\mathbb{K}^{O^n}$.

Proof. By Theorem 5.3, for each n there is a rational q_n such that the sentence $A_n = (q_n \le c \wedge c \le q_n + r_n)$ belongs to $\mathcal{A}_0^n$ and is report provable in $\mathbb{K}^{O^n}$. By Fact 2.2, A_n is provable from the combined knowledge base $\mathbb{K}^{O^n}(V)$. But $\mathcal{K}(V) \subseteq \mathcal{M}$ and $\mathcal{M} \vdash O^n(x)$ for each n and x. Therefore $\mathcal{M} \vdash A^n$ for each n.

Since $\mathcal{M}$ is consistent and contains the axioms for ordered abelian groups with constants for each rational, the set

$$\{q \in \mathbb{Q} : \mathcal{M} \vdash q \leq c\}$$

contains q_1 and is bounded above by q_1+r_1+1. Therefore the supremum of this set exists and is a real number s. Moreover, s belongs to the interval $[q_n, q_n + r_n]$ for each n, as required. □

7. Conclusion

This paper and the paper [5] provide a "report theory" for analyzing situations of the following kind: Agents make inferences based on the information they have, and report them other agents in order to make decisions. Such reporting is natural when decentralized information is to be incorporated by a central agent. We provide conditions under which reporting can lead to correct decisions. These conditions involve signatures, knowledge bases, and possible observations of the agents, the way the agents are connected in a network, and the alternatives to be decided. Future work might involve knowledge bases and networks tailored to specific applications.

References

[1] Amir, E. and S. McIlraith. Partition-Based Logical Reasoning for First-Order and Propositional Theories. Artificial Intelligence Journal, 162 49-88 (2005).

[2] Chang, C. C. and H. J. Keisler. Model Theory, Dover Publications (2012).

[3] Craig, W. Linear Reasoning. Journal of Symbolic Logic 22 250-268. Three Uses of the Herbrand-Gentzen Theorem in Relating Model Theory and Proof Theory. Ibid. 269-285 (1957).

[4] Enderton, H. A Mathematical Introduction to Logic. Harcourt Academic Press (2001).

[5] Keisler, H. J. and J. M. Keisler. Craig Interpolation for Networks of Sentences. Annals of Pure and Applied Logic 163, 1322–1344 (2012).

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