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On Dual Sets of Analytic Functions

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1. Introduction

Let $\boldsymbol{A}$ denote the set of analytic functions in the unit disc $\boldsymbol{D} = \{z: |z| < 1\}$, $\boldsymbol{A}_0$ the subset of functions $f \in \boldsymbol{A}$ with $f(0) = 1$, and $\boldsymbol{B}$ the subset of $f \in \boldsymbol{A}_0$ with $f(z) \neq 0$ in $\boldsymbol{D}$.

For $f, g \in \boldsymbol{A}$,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=0}^{\infty} b_k z^k$$

let

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k$$

denote the HADAMARD product (or convolution) of f and g. Obviously, $f * g \in \boldsymbol{A}$. For subsets $V \subset \boldsymbol{A}_0$ the notion of duality has been introduced in [3]: the dual set is given by

$$V^* = \{f \in \boldsymbol{A}_0 : f * g \in \boldsymbol{B} \quad \text{for all} \quad g \in V\}.$$

The set $(V^*)^*$ contains V and is called the second dual or *dual hull* du V of V. A set $W \subset \boldsymbol{A}_0$ is called "dual" if $W = V^*$ for some $V \subset \boldsymbol{A}_0$. It is easily seen that a set is dual if and only if it equals its dual hull.

We note that dual sets have the following general properties:

(i) Every dual set is closed in the topology of compact uniform convergence in $\boldsymbol{D}$,

(ii) Every dual set W is *complete*, i.e. with $f \in W$, $x \in \overline{\boldsymbol{D}}$ we have $f_x \in W$ where $f_x(z) = f(xz)$ in $\boldsymbol{D}$.

The concept of dual sets and dual hulls is somewhat similar to the concept of closed convex sets and closed convex hulls of sets in $\boldsymbol{A}_0$. To illustrate this let $\Lambda_{\boldsymbol{R}}$ and $\Lambda_{\boldsymbol{C}}$ denote the sets of real-valued respectively complex-valued continuous linear functionals on $\boldsymbol{A}$ (same topology as above). Then, for $\lambda \in \Lambda_{\boldsymbol{R}}$ and $V \subset \boldsymbol{A}_0$ compact and connected we have

$$(1) \qquad \lambda(V) = \lambda(\overline{\text{co}}\ V),$$

where $\overline{\text{co}}$ denotes the closed convex hull.

The duality principle [3] states: for $\lambda \in \Lambda_{\boldsymbol{C}}$, $V \subset \boldsymbol{A}_0$ compact and complete we have

$$(2) \qquad \lambda(V) = \lambda(du\ V).$$

The relation (2) is much stronger when compared with (1) since $\Lambda_{\boldsymbol{R}}$ is replaced by $\Lambda_{\boldsymbol{C}}$. In fact, (2) holds even for a class of non-linear continuous functionals λ. In particular, it implies the basic property

(3) $\overline{\text{co}}\,(du\ V) = \overline{\text{co}}\ V$

for compact and complete sets $V \subset \boldsymbol{A}_0$.

For a more detailed discussion of these facts we refer to [4].

On the other hand, it turned out that many of the (convex or not) sets of functions frequently studied in function theory are dual sets or closely related to dual sets. We mention a few examples.:

1) The convex set $\boldsymbol{P}$ of functions $f \in \boldsymbol{A}_0$ with $\operatorname{Re} f > 0$ in $\boldsymbol{D}$ is dual, but also the larger, non-convex set $\boldsymbol{H}$ of functions $f \in \boldsymbol{A}_0$ with $\operatorname{Re} e^{i\alpha} f > 0$ in $\boldsymbol{D}$, $\alpha = \alpha(f) \in \boldsymbol{R}$.

2) The set $\{f/z : f \in \boldsymbol{S}\}$ is dual where $\boldsymbol{S}$ is the usual class of normalized functions in $\boldsymbol{D}$. The same is true with $\boldsymbol{S}$ replaced by its subclasses of convex resp. starlike functions (but not for "close-to-convex").

3) The set of functions subordinate to a given univalent function $f \in \boldsymbol{A}_0$ is dual.

4) The set of non-vanishing univalent functions in $\boldsymbol{A}_0$ is dual.

5) The set of non-vanishing polynomials in $\boldsymbol{A}_0$ of a given degree is dual.

Although the study of dual sets has already been very successful in special situations (compare [4]), in particular in connection with convolution properties of certain subsets of $\boldsymbol{A}_0$, the general theory is not yet satisfactory. For example, the following important question is still open: is it true that

$$du\,(V_1 \cap V_2) = du\ V_1$$

if $V_j \subset \boldsymbol{A}_0$ are compact, complete and satisfy $du\ V_1 = du\ V_2$?

Questions of this type lead immediately to the problem how to decide whether a given set is dual or not and whether there exists a concept similar to the extreme point theory in the convex case, in particular a theorem of Krein-Milman type. Almost nothing is known in this direction.

In the present paper we discuss the apparently simplest case of these questions, namely to determine those dual sets which consists of only one function plus its rotations f_x, $x \in \overline{\boldsymbol{D}}$. Even this problem cannot be solved completely and its investigation leads deep into extremely difficult questions concerning the value distribution of gap power series.

We shall call a set $V \subset \boldsymbol{A}_0$ *semi-dual* if its completion

$$\{f_x : f \in V,\ x \in \overline{\boldsymbol{D}}\}$$

is dual. Similarly, a function $f \in \boldsymbol{A}_0$ is called semi-dual if $\{f\}$ is semi-dual. The only previous results concerning semi-dual functions are as follows:

Theorem A ([4], p. 41). *Let $f \in \boldsymbol{A}_0$ such that a formal solution $f^{(-1)}$ of the equation $f * f^{(-1)} = 1/(1-z)$ exists and is in $\boldsymbol{A}_0$. Then f is semi-dual. In particular, the function*

(4) $e(z) = 1/(1-z)$

is semi-dual.

Theorem B ([4], p. 13). *The functions $f_n(z) = (1+z)^n$, $n \in \boldsymbol{N} \setminus \{1\}$, are not semi-dual.*

Note that $f_1(z) = 1 + z$ is trivially semi-dual.

For $T \subset N \cup \{0\}$, $0 \in T$, let A_T be the set of functions $f \in A_0$ with power series expansion

$$f(z) = \sum_{k \in T} a_k z^k, \qquad a_k \neq 0 \text{ for } k \in T,$$

and

$$e_T(z) = \sum_{k \in T} z^k.$$

Hence $e_{T_0} = e$ for $T_0 = N \cup \{0\}$ and e defined as in (4).

Our first result shows that semi-duality of a function f depends only on the class A_T in which f is located rather than of f itself.

Theorem 1. *Let $f \in A_T$. Then f is semi-dual if and only if e_T is semi-dual. Furthermore,*

$$du\,\{f\} = f * du\,\{e_T\}. \tag{5}$$

We remark that the set $du\,\{e_T\}$ is called "shadow of the identity" and has — as a consequence of the duality principle — many interesting properties, compare [4], p. 143. Note that Theorem 1 extends Theorem A once it is known that e is semi-dual. This latter fact, however, seems not to be so elementary as it looks like. A really simple proof is not yet known.

Theorem 2. $0 = t_0 < t_1 < \ldots$ *be the elements of $T \neq \{0, 1\}$ and assume*

$$\sum_{k=1}^{\infty} 1/t_k < \infty. \tag{6}$$

Then e_T is not semi-dual.

This result disproves an earlier — tentative — conjecture that e_T is semi-dual iff T is infinite.

In the opposite direction we show:

Theorem 3. *Let $n \in N$, $T_n = \{0, n, 2n, 3n, \ldots\}$. Then e_T is semi-dual.*

We are not able to fill the gap between these Theorems. There are, however, some observations and connections with other problems which we should like to point out. Let $\tilde{A}_T$ be the union of all $A_{T'}$ with $T' \subset T$.

1) If $W \subset \tilde{A}_T$ is a dual set, then we say that W has the *neighborhood property* if and only if there exist numbers $\varrho_k \in (0, 1)$, $k \in T \setminus \{0\}$, such that W contains the set

$$\left\{f = \sum_{k \in T} a_k z^k \colon |a_k| \leqq \varrho_k \text{ for } k \in T \setminus \{0\}\right\}$$

We conjecture that du $\{e_T\}$ has the neighborhood property if and only if e_T is not semi-dual (except for the trivial case $e_T = 1 + z$). This would imply that $e_{T'}$ is not semi-dual if e_T is not semi-dual and $T' \subset T$.

2) We have $\{e_T\}^* \cap \tilde{A}_T = B_T$, B_T being the set of all functions in $\tilde{A}_T$ which are non-vanishing in D.

It seems not unlikely that e_T is semi-dual if and only if the sets

$$\{f_\varrho \colon f \in B_T\} \tag{7}$$

are not compact for every $0 < \varrho \leqq 1$.

This is true for all cases covered by Theorems 2, 3. Note that the sets (7) are indeed non-compact whenever $\tilde{A}_T$ contains an entire function ($\neq 1$) which does not vanish in C.

3) The "heuristic principle" of E. HILLE [2], p. 250, states that a family of analytic functions in a domain $\Omega \subset C$, defined by some property is normal iff there exists no nonconstant entire function which has the same property in D. This principle has been established for quite a number of situations by ZALCMAN [5]. The following statement reflects a situation described by HILLE's principle but is not covered by ZALCMAN's theorem:

For every T the set B_T is normal in D if and only if there is no entire function in $\tilde{A}_T$ which is non-vanishing in C.

We are not able to prove this. Its truth, however, would imply the following

Conjecture. *e_T is semi-dual if and only if there exists a non-constant entire function in $\tilde{A}_T$ which is non-vanishing in C.*

A small step in this direction is contained in our next theorem:

Theorem 4. *Let T be such that there exists an entire function $f \in A_T$, which is non-vanishing in C. Then du $\{e_T\}$ contains no non-constant polynomial. In particular, du $\{e_T\}$ does not have the neighborhood property.*

Regarding the question whether there are semi-dual sets containing more than one but finitely many elements we give just two (possibly typical) examples:

Theorem 5. 1) *Let $a, b \in C \setminus \{1\}$, $a \neq b$. Then $V = \{(1-az)/(1-z), (1-bz)/(1-z)\}$ is not semi-dual.*

2) *The set $V = \{1/(1-z), 1/(1-z^2)\}$ is not semi-dual.*

Note that in both cases the elements of V are semi-dual functions.

We agree that the results in this paper are rather rudimentary. However, it is hoped that it stimulates further research in these questions.

2. Proofs

To prove Theorem 1 we make use of the following result ([4], p. 107):

Lemma 1. *Let $Q: A \to A$ be a continuous linear operator with $(Qf)(0) = f(0)$. Then, for every compact and complete set $V \subset A_0$ we have*

$$Q(du\ V) \subset du\ (QV).$$

Proof of Theorem 1. 1) Let $Q: A \to A$ be defined by $Q: g \mapsto f * g$. Then Q fulfills the assumptions of Lemma 1 and we conclude $f * du\ \{e_T\} \subset du\ \{f\}$.

2) Let $T = \{t_0, t_1, \ldots\}$, $0 = t_0 < t_1 < \cdots$, and $T_n = \{t_0, t_1, \ldots, t_n\}$, $n \in N$.

For f we have

$$f(z) = \sum_{k \in T} a_k z^k, \qquad a_k \neq 0 \text{ for } k \in T.$$

Hence, the operators $Q_n: A \to A$. with

$$Q_n: g \mapsto g * \sum_{k \in T_n} (1/a_k)\, z^k$$

fulfil the conditions of Lemma 1 and we obtain $Q_n(\{f\}) = \{e_{T_n}\}$ and

(8) $$Q_n(du\,\{f\}) \subset du\,\{e_{T_n}\}.$$

Let $g \in du\,\{f\}$. Then, by the duality principle

$$g(z) = \sum_{k \in T} b_k z^k, \qquad |b_k| \leqq |a_k| \text{ for } k \in T,$$

and thus

$$h(z) = \sum_{k \in T} (b_k/a_k)\, z^k \in \tilde{A}_T.$$

Let $p \in B_{T_n}$. Then (8) and the duality principle applied to $\{e_{T_n}\}$ show that

(9) $$h * p = (Q_n g) * p \neq 0 \text{ for } z \in D.$$

Now let q be an arbitrary element of $\{e_T\}^*$, i.e. $q * e_T \neq 0$ in D. There exists a compact convergent (in D) sequence $p_n \in B_{T_n}$ with $p_n \to q * e_T$.

Since (9) holds for every p_n we deduce from Hurwitz's Theorem that $h * (q * e_T) \neq 0$ in D and — equivalently — $h * q \neq 0$ in D. This implies $h \in$ du $\{e_T\}$. Since $g = f * h$ we see that indeed $g \in f * du\,\{e_T\}$ and thus $du\,\{f\} \subset f * du\,\{e_T\}$. This together with 1) completes the proof.

For the proof of Theorem 2 we need the following result of Fekete [1], p. 305:

Lemma 2. *Let* $T = \{t_0, t_1, \ldots\}$, $0 = t_0 < t_1 < \cdots$, *with* $\tau = \sum_{k=1}^{\infty} (1/t_k) < \infty$ *and* $f(z) = \sum_{k \in T} a_k z^k$ *an entire function. Then f has a zero in the circle* $|z| \leqq 4e^\tau \inf_T |a_k|^{-1/k}$.

Proof of Theorem 2. Let $f \in \{e_T\}^*$. Then

$$f * e_T = \sum_{k \in T} a_k z^k \in B_T$$

and there exists a sequence $\sigma_n \in (0, 1)$, $\sigma_n \to 1$, such that

$$p_n(z) = \sum_{\substack{k \in T \\ k \leqq n}} a_k z^k$$

is different from 0 in $|z| \leqq \sigma_n$. Hence, by Lemma 2,

$$\sigma_n \leqq 4e^\tau\, |a_k|^{-1/k}, \qquad k \in T, \qquad k \leqq n.$$

Hence, for $k \in T$ fixed and $n \to \infty$ we obtain

$$|a_k| \leqq (4e^\tau)^k, \qquad k \in T.$$

Now, if $c_k \in C$, $k \in T$, with $c_0 = 1$ and

$$|c_k| \leqq (8e^\tau)^{-k} := \varrho_k$$

otherwise, we find for $g(z) = \sum_{k \in T} c_k z^k \in \tilde{A}_T$:

$$|(f * g)\,(z)| > 1 - \sum_{\substack{k \in T \\ k \neq 0}} |a_k c_k| \geqq 1 - \sum_{\substack{k \in T \\ k \neq 0}} 2^{-k} > 0.$$

Since this is true for every $f \in \{e_T\}^*$ we deduce $g \in du\,\{e_T\}$. Hence $du\,\{e_T\}$ has the neighborhood property which shows that e_T is not semi-dual.

Proof of Theorem 3. We define $Q: \boldsymbol{A} \to \boldsymbol{A}$ with

$$Q: \sum_{k=0}^{\infty} a_k z^k \mapsto \sum_{k=0}^{\infty} a_{nk} z^k.$$

Now let $V = \{e_T(xz): x \in \bar{\boldsymbol{D}}\}$. Then, by Lemma 1, we see that

$$\begin{aligned} Q(du\ V) &\subset du\, Q(V) \\ &= du\ \{e_{T_0}(x^n z): x \in \bar{\boldsymbol{D}}\} \\ &= \{e_{T_0}(x^n z): x \in \bar{\boldsymbol{D}}\} \end{aligned}$$

because e_{T_0} us semi-dual by Theorem A. But the duality principle shows that du $V \subset \boldsymbol{A}_{T_n}$ and the operator Q is invertible on $\boldsymbol{A}_{T_n}$. This implies that every function f in $du\ V$ is of the form

$$f(z) = \sum_{k=0}^{\infty} x^{nk} z^{nk} = e_{T_n}(xz)$$

for a certain $x \in \bar{\boldsymbol{D}}$. Hence e_{T_n} is semi-dual.

Proof of Theorem 4. For every $\varrho > 0$ the function f_ϱ belongs to $\{e_T\}^*$. Let $p \in du\ \{e_T\}$ be a non-constant polynomial. Then, by the duality principle, $p \in \tilde{\boldsymbol{A}}_T$ and $g_\varrho = p * f_\varrho$ is non-constant in $\boldsymbol{B}_T$, $\varrho > 0$. But $g_\varrho = g_1(\varrho z)$ is a polynomial which has a zero in $\boldsymbol{D}$ for ϱ large enough. This contradicts $g_\varrho \in \boldsymbol{B}_T$ and the proof is complete.

Proof of Theorem 5. 1) Let V as in the Theorem and $f \in V^*$. Then, for $c \in \{a, b\}$ we get

$$f * (1 - cz)/(1 - z) = c + (1 - c) f(z) \neq 0 \text{ in } \boldsymbol{D}.$$

Hence $f(z) \neq a/(a-1),\ b/(b-1)$.

From SCHOTTKY's theorem we deduce that V^* is compact. By a standard argument this implies that there are numbers $\sigma_k > 0$, $k \in \boldsymbol{N}$, such that for

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \in V^*$$

the inequalities $|a_k| \leqq \sigma_k$ hold.

Now let $\varrho_k = 1/(2\sigma_k)$. A similar estimate as in the proof of Theorem 2 shows that du V has the neighborhood property (generated by the ϱ_k).

2) It is easy to see that the set

$$\{f(-z)/f(z): f \in V^*\}$$

is a compact family.

Thus there exist numbers $\sigma_k > 0$ such that for $f \in V^*$ and

$$f(-z)/f(z) = \sum_{k=0}^{\infty} c_k z^k$$

we have $|c_k| \leqq \sigma_k$.

Now for

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \in V^*$$

we compute

$$f(-z)/f(z) = 1 - 2a_1 z + \cdots$$

Hence $|a_1| \leqq \sigma_1/2$ for all $f \in V^*$.

It follows that du V contains elements of the form $p(z) = 1 + \varepsilon z$ for ε sufficiently small. Thus du V is not semi-dual.

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