ZEROS OF RANDOM REINHARDT POLYNOMIALS

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Abstract. For a Reinhardt domain Ω with the smooth boundary in $\mathbb{C}^{m+1}$ and a positive smooth measure μ on the boundary of Ω , we consider the ensemble P_N of polynomials of degree N with the Gaussian probability measure γ_N which is induced by $L^2(\partial\Omega, d\mu)$. Our aim is to compute scaling limit distribution function and scaling limit pair correlation function between zeros when $z \in \partial\Omega$. First of all we apply stationary phase method to the Boutet de Monvel-Sjöstrand theorem to get the asymptotic for the partial szegö kernel, $S_N(z,z)$, and then we compute the scaling limit partial szegö kernel in any direction in $\mathbb{C}^{m+1}$, then by using well-known Kac-Rice formula we compute scaling limit distribution function and scaling limit pair correlation function between zeros.

1. introduction

This paper is concerned with the scaling limit distribution of and pair correlation between zeros of random polynomials on $\mathbb{C}^{m+1}$, a famous result from Hammersley [Ha] which is the following work of Kac [Kac1], [Kac2] says that the zeros of random complex kac polynomials,

$$f(z) = \sum_{j \le N} a_j z^j, z \in \mathbb{C},$$

tend to concentrate on the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ as the degree of the polynomials goes to infinity when the coefficients a_j are independent complex Gaussian random variables of mean zero and variance one, later Shiffman and Bloom in the [BS1] proved a multi variable result, that the common zeros of $m+1$ random complex polynomials in $\mathbb{C}^{m+1}$,

$$f_k(z) = \sum_{|J| \le k} c_J^k z_0^{j_0} \ldots z_m^{j_m},$$

tend to concentrate on the product of unit circles $|z_j| = 1$, Shiffman in the joint work with Zelditch in [SZ] replaced S^1 with any closed analytic curve $\partial\Omega$ in $\mathbb{C}$ that bounds a simply connected domain Ω, so new inner product on the space of holomorphic polynomials P_N is

$$(f, g)_{\partial\Omega,\mu} = \int_{\partial\Omega} f\bar{g} d\mu(z).$$

where $d\mu(z)$ is a positive smooth volume measure on $\partial\Omega$. In their work they used Riemann mapping function Φ which maps the interior of Ω to interior of the unit disk, mapping $z_0 \in \partial\Omega$ to $1 \in S^1$ and they let $\hat{D}^N_{\mu,\partial\Omega} := D^N \circ \phi^{-1} |(\phi^{-1})'|^2$ be the expected zero density for the inner product with respect to the coordinate $w = \phi(z)$.

Then they proved that there is a scaling limit density function D^∞ such that,

$$\frac{1}{N^2}\hat{D}^N_{\partial\Omega,\mu}(1+\frac{u}{N}) \to D^\infty(u), \tag{1.1}$$

where $N \to \infty$. They also showed that there exists universal functions $\hat{K}^\infty : \mathbb{C}^2 \to \mathbb{R}$ independent of Ω, z_0, μ such that

$$\frac{1}{N^4}\hat{K}^N_{\partial\Omega,\mu}(1+\frac{u}{N}, 1+\frac{v}{N}) \to K^\infty(u,v), \tag{1.2}$$

as $N \to \infty$ which $\hat{K}^N_{\partial\Omega,\mu} = K^N_{\partial\Omega,\mu} \circ \Phi^{-1}$ is the pair correlation function written in terms of the complex coordinate $w = \phi(z)$. The first purpose of this paper is to generalize the scaling limit expected distribution result [SZ] to the boundary of any complete Rienhardt strictly pseudoconvex domain in $\mathbb{C}^{m+1}$. Our second purpose is to compute pair correlation between zeros. First of all we need to introduce our basic setting: We let Ω be a complete Reinhardt strictly pseudoconvex domain (see [BFG]) in $\mathbb{C}^{m+1}$ and let $X = \partial\Omega$ and μ to be a smooth positive volume measure on X which is invariant under torus action which means

$$d\mu(e^{i\theta_0}z_0, .., e^{i\theta_m}z_m) = d\mu(z_0, .., z_m),$$

where $z = (z_0, .., z_m) \in X, \theta_i \in [0, 2\pi]$. We give the space P_N of holomorphic polynomials of degree$\leq N$ on $\mathbb{C}^{m+1}$ the Gaussian probability measure γ_N induced by the Hermitian inner product

$$(f, g) = \int_X f\bar{g}d\mu(x). \tag{1.3}$$

In the next section in the theorem 2.1 we prove that all the homogeneouse polynomials of degree$\leq N$ make an Orthonormal basis for the P_N with respect to the inner product 1.3. The Gaussian measure γ_N which is induced from 1.3 can be described as follows: If we write

$$f = \sum_{|J|\leq N} a_J p_J, J = (j_0, ..., j_m),$$

where $\{p_J\}$ is the orthonormal basis of P_N that comes from 2.1 with respect to inner product 1.3 and $d(N) = dimP_N$. Identifying $f \in P_N$ with $a = (a_J)_{|J|\leq N} \in \mathbb{C}^{d(N)}$, we have

$$d\gamma_N(a) = \frac{1}{\pi^{d(N)}}e^{-|a|^2}da. \tag{1.4}$$

In other words, a random polynomial in the ensemble (P_N, γ_N) is a polynomial $f = \sum_J a_J p_J$ such that the a_J are independent complex Gaussian random variables with mean 0 and variance 1.

Our first result, theorem 1.1, gives asymptotic for the scaling partial szegö kernel respect to the inner product 1.3,

$$S_N(z, w) = \sum_{|J|\leq N} p_J(z)\bar{p}_J(w),$$

which gives the orthogonal projection onto the span of the all homogeneous polynomials of degree$\leq N$. We show that If $z = (z_0, .., z_m) \in X, u = (u_0, .., u_m), v = (v_0, ..., v_m) \in \mathbb{C}^{m+1}$,$z_i \neq 0$ then:

Theorem 1.1.

$$\lim_{N\to\infty}\frac{1}{N^{m+1}}S_N(z+\frac{u}{N},z+\frac{v}{N})=s_0(z,z)t_0^{m+1}F_m(t_0(d'\rho(z)\cdot u+d''\rho(z)\cdot\bar{v})), \tag{1.5}$$

which ρ is the defining function for Ω and s_0 is the first term of the amplitude that appears in the Boutet de Monvel-Sjöstrand theorem 1.6 and $t_0=\frac{1}{d'\rho(z)\cdot z}$ and $F_m(t)=\int_0^1 e^{ty}y^m dy$. Our method to compute asymptotic for the scaling partial szegö kernel is similar to the method that Zelditch had used in [Ze], in our proof we apply stationary phase method to,

$$\Pi_K(x,y)=\int_0^\infty\int_0^{2\pi}e^{-iK\theta}e^{it\psi(e^{i\theta x},y)}s(e^{i\theta}x,y,t)d\theta dt, \tag{1.6}$$

which, $s(x,y,t)\sim\sum_{k=0}^\infty t^{m-k}s_k(x,y)$ and the phase $\psi\in C^\infty(\mathbb{C}^{m+1}\times\mathbb{C}^{m+1})$ is determined by the following properties

1) $\psi(x,x)=\frac{\rho(x)}{i}$ where ρ is the defining function of X.
2) $\bar{\partial}_x\psi$ and $\partial_y\psi$ vanish to infinite order along diagonal.
3) $\psi(x,y)=-\bar{\psi}(y,x)$

In [BSZ1], [BSZ2] we see that the expected zero density and correlation function can be represented by given formulas involving only the Szegö kernel and its first and second derivatives, in the section 3 we show that the scaling limit for the expected zero density which is defined

$$E^N_{\mu,X}(|Z_f|\wedge\frac{\omega^m}{m!})=D^N_{\mu,X}(z)\frac{\omega^{m+1}}{(m+1)!} \tag{1.7}$$

Which $E^N_{\mu,X}$ is the expected zero current for the ensemble (P_N,γ_N) and $\omega=\frac{i}{2}\sum_{j=0}^m dz_j\wedge d\bar{z}_j$, can be given by the following theorem.

Theorem 1.2. *Let $D^N_{\mu,X}$ be the expected zero density for the probability space (P_N,γ_N) then*

$$\lim_{N\to\infty}\frac{1}{N^2}D^N_{\mu,X}(z+\frac{u}{N})=D^\infty_{z,X}(u),$$

where

$$D^\infty_{z,X}(u)=\frac{(t_0||P||)^2}{\pi}(\log F_m)''(\alpha),$$

and $P=(\frac{\partial\rho}{\partial\bar{z}_0},...,\frac{\partial\rho}{\partial\bar{z}_m})$, $\alpha=t_0(d'\rho(z)\cdot u+d''\rho(z)\cdot\bar{u})$.

Our main result, theorem 1.3, gives a formula for the scaling limit normalized pair correlation functions

$$\tilde{K}^N_{\mu,X}(z,w)=\frac{K^N_{\mu,X}(z,w)}{D^N_{\mu,X}(z)D^N_{\mu,X}(w)}, \tag{1.8}$$

which

$$E^N_{\mu,X}(|Z_f|^2\wedge\frac{\omega^{2m}}{(2m)!})=K^N_{\mu,X}(z,w)\frac{\omega^{2m+2}}{(2m+2)!}, \tag{1.9}$$

such that $\omega=\frac{i}{2}\sum_{i=0}^m dz_i\wedge d\bar{z}_i+dw_i\wedge d\bar{w}_i$.
Normalized pair correlation function $\tilde{K}^N_{\mu,X}(z,w)$ can be viewed as the probability

of finding simultaneous zeros at w, z. For example if $\tilde{K}^N_{\mu,X}(z,w)=1$ then it means having zero at z does not have any influence of probability having zero at w. If z, w are fixed and different then $\tilde{K}^N_{\mu,X}(z,w)\to 1$ as $N\to\infty$, but in the theorem 1.3 we show that we have nontrivial normalized pair correlations if the distance between points is $O(\frac{1}{N})$.

Theorem 1.3. *let $\tilde{K}^N_{\mu,X}(z,w)$ be the normalized pair correlation function for the probability space (P_N,γ_N) and choose $u\in\mathbb{C}^{m+1}$ such that $u\notin T^h_z X$ then,*

$$\lim_{N\to\infty}\frac{1}{N^4}K^N_{\mu,X}(z+\frac{u}{N},z)=K^\infty_{z,X}(u),$$
$$\lim_{N\to\infty}\tilde{K}^N_{\mu,X}(z+\frac{u}{N},z)=\tilde{K}^\infty_{z,X}(u),$$

where

$$K^\infty_{z,X}(u)=\frac{1}{\pi^2}\frac{perm(Q_m(\beta))}{det(G_m(\beta))}(||P||t_0)^2,$$
$$\tilde{K}^\infty_{z,X}(u)=\frac{1}{(\log F_m)''(\alpha)(\log F_m)''(0)}\frac{perm(Q_m(\beta))}{det(G_m(\beta))},$$

which $K^N_{\mu,X}(z,w)$, $\tilde{K}^N_{\mu,X}(z,w)$ are defined in 1.9, 1.8,

$$\beta=t_0 d'\rho(z).u,\alpha=\beta+\bar{\beta},$$

$$G_m(\beta)=\begin{pmatrix}F_m(\beta+\bar{\beta}) & F_m(\beta)\\ F_m(\bar{\beta}) & F_m(0)\end{pmatrix},$$

$$Q_m(\beta)=G_{m+2}(\beta)-G_{m+1}(\beta)G_m(\beta)^{-1}G_{m+1}(\beta).$$

One of the most interesting cases is to see how normalized pair correlation function behaves when we move in the normal direction, for example if we look at the sphere S^3 in the $\mathbb{C}^2$ and if we choose $z=(1,0)\in S^3\subset\mathbb{C}^2$ then normal vector at $(1,0)$ to S^3 would be $u^\perp=(1,0)$, then if we move along this vector from origin to infinity then in the normal direction, we obtain the scaling limit

$$k^\perp(\lambda):=\tilde{K}^\infty_{(1,0),S^3}(\lambda u^\perp)=\lim_{N\to\infty}\tilde{K}^N_{\mu,S^3}((1,0)+\lambda\frac{u^\perp}{N},(1,0)).$$

The graph of $k^\perp(\lambda)$ in Figure 1 converges to 1 when λ goes to infinity and as we expected form [SZ], in $\mathbb{C}^1$ it is not oscillatory and we have a zero repulsion when $\lambda\to 0$. In this example for the tangential scaling limit for the pair correlation function we move along the curve $\gamma(\theta)=e^{i\theta}(1,0)$, if we let u^θ to be the tangent vector to this curve at $\gamma(0)=(1,0)$, we see that $u^\theta=(i,0)$ and because $u^\theta\notin T^h_z X$ then

$$K^\infty_{(1,0),S^3}(u^\theta)=\lim_{N\to\infty}\frac{1}{N^4}K^N_{\mu,S^3}((1,0)+\frac{u^\theta}{N},(1,0)),$$

it means that the scaling limit pair correlation function grows as fast as N^4 along the curve $\gamma(\theta)$. After this step we compute normalized scaling limit pair correlation function between zeros along the curve $\gamma(\theta)$,

$$k^\theta(\lambda):=\tilde{K}^\infty_{(1,0),S^3}(\lambda u^\theta)=\lim_{N\to\infty}\tilde{K}^N_{\mu,S^3}((1,0)+\lambda\frac{u^\theta}{N},(1,0)),$$

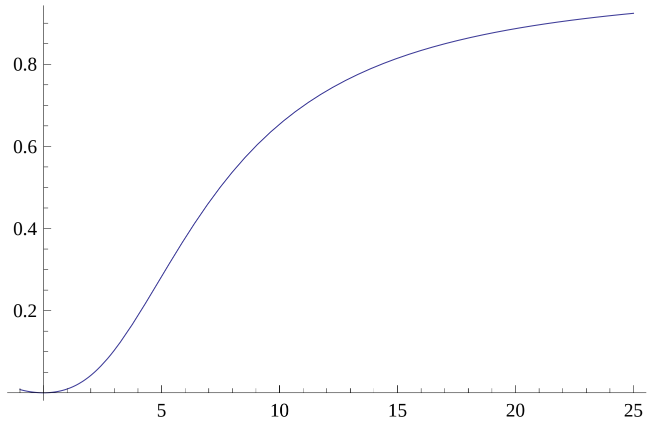

FIGURE 1. The normalized pair correlation function in the normal direction $k^{\perp}(\lambda)$

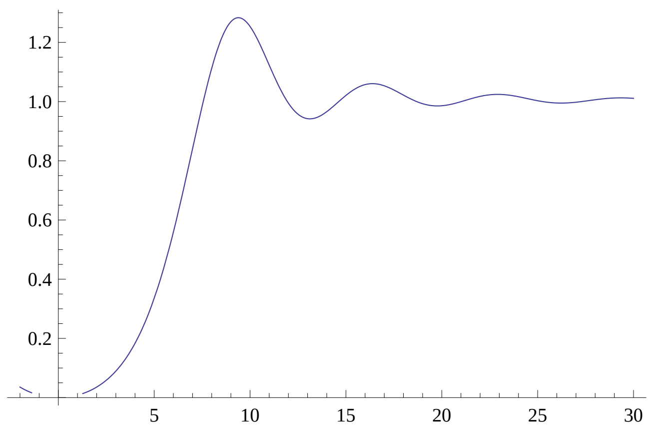

FIGURE 2. The tangential pair correlation function in the tangent direction $k^{\theta}(\lambda)$

which gives us the probability of finding a pair of zeros in small disks around two points on $\gamma(\theta)$ which the distance between them is $O(\frac{1}{N})$. As you see in the graph of k^{θ} in figure 2 the zeros repel when $\lambda \to 0$ and their correlations are oscillatory, same as the $\mathbb{C}^1$ case as shown in [SZ].

Now if we move along $h(t) = (\cos(t), i\sin(t)) \subset S^3$ then we see that

$$\lim_{N\to\infty} \frac{1}{N^5} K^N_{\mu,S^3}((1,0) + \frac{u^h}{N}, (1,0)) \to K^{\infty}_{(1,0),S^3}(u^h),$$

where $u^h = h^{'}(0) = (0,i)$, $u^h = (0,i) \in T^h_z X$. As we see the behavior of the scaling pair correlation function between zeros is totally different when we move in the u^h direction compare to $u^{\perp}$, u^{θ}. In this example we observe that if we move along u^h direction which belongs to $T^h_z S^3$ then $K^N_{\mu,S^3}((1,0) + \frac{u^h}{N}, (1,0))$ is asymptotic to N^5 but in other directions, $K^N_{\mu,S^3}((1,0) + \frac{u}{N}, (1,0))$ is asymptotic to N^4. Our result shows that $K^N_{\mu,X}(z + \frac{u}{N}, z)$ is asymptotic to N^4, in general case when $u \notin T^h_z X$, it would be interesting to see whether $K^N_{\mu,X}(z+\frac{u}{N},z)$ is asymptotic to N^5 in general case when $u\in T^h_z X$ or not.

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2. Asymptotics of orthogonal polynomials

Throughout this paper, we restrict ourself to a smooth boundary complete Reinhardt strictly pseudoconvex domain in $\mathbb{C}^{m+1}$, this is by far one of the most interesting case to study, and it includes many interesting examples. we recall the elementary definitions:

A domain Ω is strictly pseudoconvex if its levi form is strictly positive definite at every boundary point. the levi form of $\Omega=\{z\in\mathbb{C}^{m+1}:\rho(z)<0\}$ with ρ is a real valued C^∞ function on $\mathbb{C}^{m+1}$, $\rho'\neq 0$ on $\partial\Omega$ is defined as the restriction of quadratic form

$$(v_0,...,v_m)\to\sum_{j,k}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}(z)v_j\bar{v}_k,$$

to the subspace $\{(v_0,...,v_m)\in\mathbb{C}^{m+1}:\sum\frac{\partial\rho}{\partial z_j}(z)z_j=0\}$. it is defined independently of ρ up to constants.

Complete Reinhardt means that if $z\in\Omega$ implies that $(\mu_0z_0,...,\mu_mz_m)\in\Omega$ for all $\mu_j\in\mathbb{C}$ with $|\mu_j|\leq 1, j=0,...,m$ if we let $X=\partial\Omega=\{z\in\mathbb{C}^{m+1}:\rho(z)=0\}$ then we see that X is a complete circular subset of $\mathbb{C}^{m+1}$ which means if $z\in X$

$$(e^{i\theta_0}z_0,...,e^{i\theta_m}z_m)\in X, j=0,..,m,$$

for all $\theta_j\in(0,2\pi)$, it is because Ω is a complete Reinhardt domain so the boundary of Ω is a complete circular subset of $\mathbb{C}^{m+1}$, form now we use X for the boundary of Ω, the proofs of Theorems 3.1 and 4.1 are based on asymptotic properties of orthogonal polynomials associated to X, for that purpose we need to have a measure on X, so during this article we assume that $d\mu$ is a volume measure on X which is invariant under torus actions.

2.1. **Szegö kernel and orthogonal polynomials.** The Hardy space $H^2(X,d\mu)$ is the space of boundary values of holomorphic functions on Ω which are in $L^2(X,d\mu)$, or equivalently $H^2(X,d\mu)=(ker\bar{\partial}_b)\bigcap L^2(X,d\mu)$. The S^1 action commutes with $\bar{\partial}_b$, hence

$$H^2(X,d\mu)=\bigoplus_{n=0}^{\infty}H^2_n(X,d\mu),$$

where

$$H^2_n=\{f\in H^2(X,d\mu):f(r_\theta x)=e^{in\theta}f(x)\}.$$

The Szegö kernel of X with respect to the measure $d\mu$ on X is the orthogonal projection

$$S:L^2(X,d\mu)\to H^2(X,d\mu).$$

Onto the Hardy space of boundary values of holomorphic functions in Ω that belongs to $L^2(X,d\mu)$. The Schwartz kernel of S is denoted by $S(z,w)$, $S(z,w)$ admits an analytic continuation (holomorphic in z and anti holomorphic in w) to $\bar{\Omega}\times\bar{\Omega}$. We will refer to this as the *regularity* theorem.

Theorem 2.1. *The monomials in $\mathbb{C}^{m+1}$ make an orthogonal basis for $H^2(X, d\mu)$.*

Proof. Let $J = (j_0, .., j_m)$ and $K = (k_0, .., k_m)$ and assume $j_0 \neq k_0$ and let z^J, z^K be the associated polynomials then induced inner product from measure μ is going to be

$$(z^J, z^K) = \int_X z^J \bar{z}^K d\mu(x),$$

we know that X is a complete circular subset of $\mathbb{C}^{m+1}$ it means we can define this change of variable

$$(z_0, .., z_m) \to (e^{i\theta} z_0, .., z_m),$$

then for every $\theta \in (0, 2\pi)$ we have this equation

$$\int_X z^J \bar{z}^K d\mu(x) = e^{i\theta(j_0 - k_0)} \int_X z^J \bar{z}^K d\mu(x),$$

it means

$$(z^J, z^K) = e^{i\theta(j_0 - k_0)} (z^J, z^K).$$

So if $J \neq K$ then $(z^J, z^K) = 0$ and if $J = K$ then

$$(z^J, z^J) = \int_X |z_0|^{2j_0} ..|z_m|^{2j_m} d\mu(x) > 0.$$

So if we let $c_J = (\frac{1}{(z^J, z^J)})^{\frac{1}{2}}$ then the $\{p_J = c_J z^J : J = (j_0, ..j_m), j_i \geq 0\}$ make an orthonormal subset of $H^2(\partial\Omega, \mu)$, and proof for completeness is same as the proof that comes in [SK] for the unit ball. now we show that in fact this forms a basis. It suffices to show that if f is smooth on X , $\bar{\partial}_b f = 0$ and if $(f, z^J) = 0$ then $f = 0$. By knowing if $f \in H^2(X, d\mu)$ then f has fourier series which converges to f in $L^2(X, d\mu)$ it means f can be written like,

$$f(z) = \sum_{n=0}^{\infty} f_n(z)$$

which

$$f_n(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta} z) e^{-in\theta} d\theta, \tag{2.1}$$

f_n is a smooth function on X such that

$$f_n(e^{i\theta} z) = e^{in\theta} f_n(z). \tag{2.2}$$

Now we want to extend f_n to a smooth function on the $\mathbb{C}^{m+1}$, for that purpose we use the fact that any $w \in \mathbb{C}^{m+1}$ can be written like $w = \lambda z$ such that $\lambda \in \mathbb{C}$ and $z \in X$, now we define

$$F_n(w) = F_n(\lambda z) = \lambda^n f_n(z),$$

bye using 2.1, 2.2 we see that F_n is a well-define and smooth function on $\mathbb{C}^{m+1}$. Next step is to see that F_n is a holomorphic function, in other word we want to show that $\frac{\partial F_n}{\partial \bar{w}_j} = 0$ for $j = 0, ..., m$, we assume that $v_1, ..., v_m$ is a basis for $T^h_z X$ then

$$\frac{\partial}{\partial \bar{w}_j} = a_0 \frac{\partial}{\partial \bar{\lambda}} + \sum_{k=1}^{m} a_k \bar{v}_k.$$

we assume that $f \in H^2_n(X, du)$ and smooth on X, it concludes that $\bar{\partial}_b f = 0$ so it means

$$\begin{aligned} \bar{v}_k(F_n) &= \lambda^n \bar{v}_k(f_n) \\ &= \lambda^n \frac{1}{2\pi} \int_0^{2\pi} \bar{v}_k(f(e^{i\theta} z)) e^{-in\theta} d\theta = 0 \end{aligned} \tag{2.3}$$

also

$$\frac{\partial}{\partial \bar{\lambda}} F_n = \frac{\partial \lambda^n}{\partial \bar{\lambda}} f_n(z) = 0, \tag{2.4}$$

so from 2.3, 2.4 we conclude $\frac{\partial}{\partial \bar{w}_j} F_n = 0$ so it means F_n is holomorphic on $\mathbb{C}^{m+1}$ and also bye using 2.2 we see that $F_n(\lambda w) = \lambda^n F_n(w)$ for all $\lambda \in \mathbb{C}, w \in \mathbb{C}^{m+1}$ it means F_n is a homogeneous polynomial of degree n. So we showed that every smooth function in $H^2(X, du)$ can be written in sum of homogeneous polynomials and we know that smooth functions on X are dense in $H^2(X, du)$, it means monomials are a basis for $H^2(X, du)$, so we showed that monomials make an orthogonal basis for $H^2(X, du)$. □

Since S is an orthogonal projection, we may express it in terms of any orthonormal basis. Hence when $(z, w) \in \bar{\Omega} \times \bar{\Omega}$ then

$$\begin{aligned} S(z, w) &= \sum_J p_J(z) \bar{p_J}(w), \\ S(z, z) &< \infty, z \in \Omega. \end{aligned} \tag{2.5}$$

By using theorem 2.1 we see that $S_N(z, z) = \sum_{|J| \leq N} p_J(z) \bar{p}_J(w)$ and also by using Regularity theorem we conclude that these partial szegö kernels converges uniformly on compact subsets of Ω to the $S(z, z)$.

From now we are using X instead of $\partial\Omega$, we already talked about $H^2_N(X, d\mu)$ but we want to talk about this spaces more,

Let's look at this orthogonal projection

$$\Pi_K : L^2(X, d\mu) \to H^2_K(X, d\mu). \tag{2.6}$$

We know that the set of all the homogeneous polynomials of degree K is an orthogonal basis for $H^2_K(X, d\mu)$ and also Π_K is an orthogonal projection so we have

$$\Pi_K(z, w) = \sum_{|J| = K} p_J(z) \bar{p}_J(w),$$

So $S_N(z, w) = \sum_{K=0}^N \Pi_k(z, w)$ it means if we find a nice formula for Π_K then we can compute S_N.

2.2. **Boutet de Monvel-Sjöstrand Theorem and Partial Szegö kernels.**

Theorem 2.2. *Let $\Pi(x, y)$ be the Szegö kernel of the boundary X of a bounded strictly pseudo-convex domain Ω in a complex manifold. Then there exists a symbol $s \in S^n(X \times X \times \Re^+)$ of the type*
$s(x, y, t) \sim \sum_{k=0}^\infty t^{m-k} s_k(x, y)$,
So that

$$\Pi(x, y) = \int_0^\infty e^{it\psi(x,y)} s(x, y, t) dt,$$

where the phase $\psi \in C^\infty(X \times X)$ is determined by the following properties:

1) $\psi(x,x) = \frac{\rho(x)}{i}$ *where ρ is the defining function of X.*
2) $\bar{\partial}_x\psi$ *and* $\partial_y\psi$ *vanish to infinite order along diagonal.*
3) $\psi(x,y) = -\bar{\psi}(y,x)$.

The integral is defined as a complex oscillatory integral and is regularized by taking the principal value.

So our goal is to find a nice formula for $\Pi_K(z,z)$ by using above theorem.

Homogeneous polynomials of degree K make an orthogonal basis for $H^2_K(X, d\mu)$ so

$$\Pi_K(x,y) = \int_0^\infty \int_0^{2\pi} e^{-iK\theta} e^{it\psi(e^{i\theta x},y)} s(e^{i\theta}x, y, t) d\theta dt. \tag{2.7}$$

Lemma 2.3. *if $z = (z_0, ..z_m) \in X = \partial\Omega$, $z_i \neq 0$ then $d'\rho(z) \cdot z > 0$*

Proof. let $z = (z_0, ..z_m) \in X$ and $z_i = r_i e^{i\theta_i}$ then

$$\frac{\partial \rho}{\partial z_i} = \frac{\partial \rho}{\partial r_i} \cdot \frac{\partial r_i}{\partial z_i} + \frac{\partial \rho}{\partial \theta_i} \cdot \frac{\partial \theta_i}{\partial z_i} = \frac{\partial \rho}{\partial r_i} \cdot \frac{\bar{z}_i}{2r_i},$$

So

$$d'\rho(z) \cdot z = \sum_{i=0}^{m} \frac{\partial \rho}{\partial z_i} z_i = \frac{1}{2} \sum_{i=0}^{n} \frac{\partial \rho}{\partial r_i} \frac{\bar{z}_i}{r_i} z_i = \frac{1}{2} \sum_{i=0}^{m} \frac{\partial \rho}{\partial r_i} r_i.$$

We want to show that $\frac{\partial \rho}{\partial r_i} \geq 0$ when $i = 0, .., m$, for that purpose we define $f(t) = \rho(tz_0, .., z_m)$ it is clear that $f(1) = 0$ and also because Ω is complete Reinhardt domain then $(tz_0, .., z_m) \in \Omega^c$ for $|t| \geq 1$ it means $f(t) \geq 0$ when $|t| \geq 1$ so

$$\frac{\partial \rho}{\partial r_0} r_0 = f'_+(1) = \lim_{t \to 1} \frac{\rho(tz_0, .., z_m) - \rho(z_0, .., z_m)}{t-1} \geq 0.$$

Also we know that $d'\rho(z) \neq 0$ it means there is $i \in \{0, .., m\}$ such that $\frac{\partial \rho}{\partial r_i} > 0$ so it means

$$d'\rho(z).z > 0. \tag{2.8}$$

□

By using Boutet de Monvel-Sjöstrand theorem,

$$\Pi_K(z,z) = \int_0^\infty \int_0^{2\pi} e^{-iK\theta} e^{it\psi(r_\theta z, z)} s(r_\theta z, z, t) d\theta dt. \tag{2.9}$$

If we let $t \to Kt$, $\phi(t, \theta; z, z) = \theta - t\psi(r_\theta z, z)$ then

$$\Pi_K(z,z) = K \int_0^\infty \int_0^{2\pi} e^{-iK\phi(\theta,t;z,z)} s(r_\theta z, z, Kt) d\theta dt. \tag{2.10}$$

Also we have

$$Im\psi(x,y) \geq c(d(x,X) + d(y,X) + |x-y|^2) + O(|x-y|^3), \tag{2.11}$$

which c is a positive constant so it means that $Im\psi(x,y) \geq 0$ so we can use Stationary phase method, for this purpose we need to consider phase function, first thing to do is to find critical points of phase function,

Lemma 2.4. *$(0, \frac{1}{d'\rho(z)\cdot z})$ is the only critical point of the phase function $\phi(\theta, t; z, z) = \theta - t\psi(r_\theta z, z)$.*

Proof. $\frac{\partial\phi}{\partial t} = 0$ then $\psi(r_\theta z, z) = 0$,now by using 2.11 $\psi(r_\theta z, z) = 0$ if and only if $r_\theta z = z$ if and only if $\theta = 0$
and also if we take derivative respect to t then

$$\frac{\partial\phi}{\partial\theta} = 1 - te^{i\theta}\sum_{i=0}^{i=m}\frac{\partial_x\psi(r_\theta z, z)}{\partial x_i}z_i,$$

and we plug in $\theta = 0$ we get this formula

$$\frac{\partial\phi}{\partial\theta} = 1 - td'\rho(z)\cdot z = 0 \to t_0 = \frac{1}{d'\rho(z)\cdot z}, \tag{2.12}$$

and it is well defined because of lemma 2.3 also $(0, t_0)$ is a nondegenerate critical point because,

$$|\phi''(0, t_0)| = |\begin{pmatrix} 0 & \frac{1}{t_0} \\ \frac{1}{t_0} & \frac{\partial^2\phi}{\partial\theta^2}\end{pmatrix}| = -\frac{1}{t_0^2} < 0. \tag{2.13}$$

□

Theorem 2.5. *For $z = (z_0, ..., z_m) \in X$ which $z_i \neq 0$ for each z_i we have*

$$\Pi_K(z, z) = s_0(z, z)t_0(Kt_0)^m + R_{K,0}, \tag{2.14}$$

such that $|R_{K,0}| \leq C_0K^{m-1}$ and C_0 just depends on X, ψ, z.

Proof. By using inequality 2.11 we see that Imaginary part of $-\phi(t, \theta)$ which is equal to Imaginary part of $t\psi(r_\theta z, z)$ is non negative and also our critical point is nondegenerate so we are ready to use stationary phase method, by using theorem (7.7.5) from [Hö],

$$\begin{gathered}\Pi_K(z, z) \sim \frac{K}{\sqrt{\frac{K\phi''(0,t_0)}{2\pi i}}}\sum_{j,k=0}^{\infty}K^{m-j-k}L_j(t^{m-k}s_k(r_\theta z, z)),\\ \textit{which}\\ L_j(a) = \sum_{\nu-\mu=j}\sum_{2\nu\geq 3\mu}\frac{i^{-j}2^{-\nu}}{\mu!\nu!}\langle\phi''(0, t_0)^{-1}D, D\rangle(g^{\mu}_{(0,t_0)}(t, \theta)a).\end{gathered} \tag{2.15}$$

In this equation $g_{(0,t_0)}$ is equal to the third order reminder of $\phi(\theta, t)$ at $(0, t_0)$ in the left hand side you can see that if $j = 0, k = 0$ then we will get the highest degree of K, by looking at the definition of L_j we have $L_0(t^m s_0(r_\theta z, z)) = t_0^m s_0(z, z)$, by using the stationary phase theorem from [Hö]:

(2.16)

$$|\Pi_K(z, z) - t_0K^mL_0(t^m s_0(r_\theta z, z))| = |\Pi_K(z, z) - K^m t_0^m s_0(z, z)t_0| \leq K^{m-1}CM,$$

which

$$M = \sum_{|\alpha|\leq 2}||D^\alpha s||_\infty.$$

□

For the next step we need to find a good estimate for the derivatives of $\Pi_K(z,z)$ by using 2.10, for that purpose we need to introduce some notations which help us to understand derivatives of $\Pi_K(z,z)$ much better and also we know that $s(x,y,t)$ is a smooth function on X and we don't know about behavior of $s(x,y,t)$ on the neighborhood of X so we can only use (2.10) for computing derivatives of $\Pi_k(z,z)$ in real tangential directions. Now let's talk more about real tangent plane on X at point $z=(z_0,..,z_m)\in X$.

Lemma 2.6. *If $z=(z_0,..,z_m)\in X$ and $z_j\neq 0$ for $0\le j\le m$ then,*

$$T_0=(iz_0,..,0),..,T_j=(0,..,iz_j,..,0),..,T_m=(0,..,iz_m)\in T_zX.$$

Proof. If we look at T_0 in the $\mathbb{R}^{2m+2}$ then T_0 can be written like $(-y_0,x_0,0,0...,0,0)$ and normal vector at z on X is

$$\begin{aligned}\vec{N}&=(\frac{\partial\rho}{\partial x_0},\frac{\partial\rho}{\partial y_0},......,\frac{\partial\rho}{\partial x_m},\frac{\partial\rho}{\partial y_m})\\&=(\frac{\partial\rho}{\partial r_0}\frac{\partial r_0}{\partial x_0},\frac{\partial\rho}{\partial r_0}\frac{\partial r_0}{\partial y_0},...,\frac{\partial\rho}{\partial r_m}\frac{\partial r_m}{\partial x_m},\frac{\partial\rho}{\partial r_0}\frac{\partial r_m}{\partial y_m})\\&=(\frac{\partial\rho}{\partial r_0}\frac{x_0}{r_0},\frac{\partial\rho}{\partial r_0}\frac{y_0}{r_0},...,\frac{\partial\rho}{\partial r_m}\frac{x_m}{y_m},\frac{\partial\rho}{\partial r_m}\frac{y_m}{r_m}).\end{aligned}\tag{2.17}$$

So:

$$\vec{N}\cdot T_0=-\frac{\partial\rho}{\partial r_0}\frac{x_0}{r_0}y_0+\frac{\partial\rho}{\partial r_0}\frac{y_0}{r_0}x_0=0.\tag{2.18}$$

It means $T_0\in T_zX$ and similarly we can show that each T_j belongs to the real tangent space at z on X. □

Lemma 2.7. *If $f:\mathbb{C}^{m+1}\to\mathbb{C}$ is an anti holomorphic function then*

$$D_{T_j}f(z)=-i\bar{z}_j\frac{\partial f}{\partial\bar{z}_j},$$

for each T_j

Notations:

$$\begin{aligned}&\alpha=(\alpha_0,..,\alpha_m),\beta=(\beta_0,..,\beta_m),\gamma_i=(\gamma_{i,0},..,\gamma_{i,m}),\{k_i\},\\&\textit{which}\\&\alpha_i,\beta_i,\gamma_{i,j},k_i\in\{0,1,2,..\},\\&I_\alpha=\{l=(\beta,\{\gamma_i\},\{k_i\}):\sum k_i\gamma_i+\beta=\alpha\},\\&T=T_0+...+T_m.\end{aligned}\tag{2.19}$$

For any multi indices like $\alpha=(\alpha_0,...,\alpha_m)$ we define:

$$D_T^\alpha=D_{T_m}^{\alpha_m}....D_{T_0}^{\alpha_0},$$

if $l\in I_\alpha$ we define

$$Z_l(f,g)=\Pi(D_T^{\gamma_i}f)^{k_i}(D_T^\beta g),\tag{2.20}$$

for example if we let, $l_0 = (\beta, \{\gamma_i\}, \{k_i\})$ such that $\beta = (0,..,0), \gamma_0 = (1,0..,0), .\gamma_m = (0,..,1), k_0 = \alpha_0, ..k_m = \alpha_m$ then

$$\begin{aligned} Z_{l_0}(i\psi(r_\theta z,z), s_0(r_\theta z,z))|_{\theta=0} &= \Pi(iD_{y,T}\psi(r_\theta z,z))^{\alpha_i} s_0(r_\theta z,z)|_{\theta=0} \\ &= \Pi(i\frac{\partial_y \psi(r_\theta z,z)}{\partial \bar{z}_i}(-i\bar{z}_i))^{\alpha_i} s_0(r_\theta z,z)|_{\theta=0} \\ &= (\frac{\partial\rho}{\partial\bar{z}_0}...\frac{\partial\rho}{\partial\bar{z}_m})^{\alpha}(-i\bar{z})^{\alpha} s_0(z,z). \end{aligned} \tag{2.21}$$

Lemma 2.8. $D_T^\alpha(e^f g) = e^f \sum_{c_l \in I_\alpha} c_l Z_l(f,g)$.

Now by using lemma 2.8 we have this result:

$$\begin{aligned} D_{y,T}^\alpha(e^{-iK\phi}s(r_\theta z,z,Kt)) &= \sum_{c_l\in I_\alpha} c_l e^{-iK\phi} Z_l(-iK\phi, s(r_\theta z,z,Kt)) \\ &= \sum_{k=0}^{\infty}\sum_{l\in I_\alpha} c_l e^{-iK\phi}(Kt)^{\sum k_i} Z_l(i\psi, s_k)(Kt)^{m-k} \\ &= \sum_{k=0}^{\infty}\sum_{l\in I_\alpha} c_l e^{-iK\phi}(Kt)^{m+\sum k_i - k} Z_l(i\psi, s_k). \end{aligned} \tag{2.22}$$

Theorem 2.9. *If $z = (z_0, ..., z_m) \in X$ and $z_i \neq 0$ for each z_i then there is a constant C_α which it just depends on z, α, ψ, X such that:*

$$D_{y,T}^\alpha \Pi_K(z,z) = s_0(z,z)t_0(Kt_0)^{m+|\alpha|}(\frac{\partial\rho}{\partial\bar{z}_0}...\frac{\partial\rho}{\partial\bar{z}_m})^{\alpha}(-i\bar{z})^{\alpha} + R_{K,\alpha}, \tag{2.23}$$

$|R_{K,\alpha}| \leq C_\alpha K^{m+|\alpha|-1}$

Proof. If we use equation 2.10 and lemma 2.8 then

(2.24)

$$\begin{aligned} D_{y,T}^\alpha \Pi_K(z,z) &= D_{y,T}^\alpha K\int_0^\infty\int_0^{2\pi} e^{-iK\phi(\theta,t;z,z)} s(r_\theta z,z,Kt)d\theta dt \\ &= K\int_0^\infty\int_0^{2\pi} D_{y,T}^\alpha(e^{-iK\phi(\theta,t;z,z)} s(r_\theta z,z,Kt))d\theta dt \\ &= K\sum_{l\in I_\alpha} c_l \int_0^\infty\int_0^{2\pi} e^{-iK\phi} Z_l(-iK\phi, s(r_\theta z,z,Kt))d\theta dt \\ &= K\sum_{l\in I_\alpha} c_l \int_0^\infty\int_0^{2\pi} e^{-iK\phi} Z_l(iKt\psi, s(r_\theta z,z,Kt))d\theta dt \\ &= K\sum_{l\in I_\alpha} c_l \int_0^\infty\int_0^{2\pi} e^{-iK\phi}(Kt)^{\sum k_i} Z_l(i\psi, s(r_\theta z,z,Kt))d\theta dt \\ &\sim \sum_{l\in I_\alpha} c_l \frac{K}{\sqrt{|K\phi''(0,t_0)/2\pi i|}} \sum_{k,j=0}^{\infty} K^{-j} L_j((Kt)^{m+\sum k_i - k} Z_l(i\psi, s_k)), \end{aligned}$$

if we look at in the series, the highest degree of K happens whenever $l = l_0, k = j = 0$ in this case $k_i = \alpha_i, c_{l_0} = 1$ and by using equation 2.20 and using theorem(7.7.5) from [Hö] we will get this result,

(2.25)

$$|D^{\alpha}_{y,T}\Pi_K(z,z)-(Kt_0)^{m+|\alpha|}(\frac{\partial\rho}{\partial\bar z_0}...\frac{\partial\rho}{\partial\bar z_m})^{\alpha}(-i\bar z)^{\alpha}s_0(z,z)t_0| \le K^{m+|\alpha|-1}C\sum_{l\in I_\alpha}\sum_{|\beta|\le 2}||D^{\beta}Z_l(i\psi,s)||_{\infty},$$

If we let $M = C\sum_{l\in I_\alpha}\sum_{|\beta|\le 2}||D^{\beta}Z_l(-i\psi,s)||_{\infty}$ then

$$|D^{\alpha}_{y,T}\Pi_K(z,z)-(Kt_0)^{m+|\alpha|}(\frac{\partial\rho}{\partial\bar z_0}...\frac{\partial\rho}{\partial\bar z_m})^{\alpha}(-i\bar z)^{\alpha}s_0(z,z)t_0| \le MK^{m+|\alpha|-1}, \tag{2.26}$$

which M is a constant that just depends on ψ,ρ and their partial derivatives and $m+1$ which is the dimension of our space, so i can tell,

$$D^{\alpha}_{y,T}\Pi_K(z,z) = (Kt_0)^{m+|\alpha|}(\frac{\partial\rho}{\partial\bar z_0}...\frac{\partial\rho}{\partial\bar z_m})^{\alpha}(-i\bar z)^{\alpha}s_0(z,z)t_0 + R_{K,\alpha}, \tag{2.27}$$

which $|R_{K,\alpha}| \le MK^{m+|\alpha|-1}$. □

We can also get an upper bound for $D^{\alpha}_{y,T}\Pi_K(z,z)$ like,

$$\begin{aligned} D^{\alpha}_{y,T}\Pi_K(z,z) &= (Kt_0)^{m+|\alpha|}(\frac{\partial\rho}{\partial\bar z_0}...\frac{\partial\rho}{\partial\bar z_m})^{\alpha}(-i\bar z)^{\alpha}s_0(z,z)t_0 + R_{K,\alpha} \\ &\le (Kt_0)^{m+|\alpha|}(\frac{\partial\rho}{\partial\bar z_0}...\frac{\partial\rho}{\partial\bar z_m})^{\alpha}(-i\bar z)^{\alpha}s_0(z,z)t_0 + MK^{m+|\alpha|-1} \\ &\le K^{m+|\alpha|}((\frac{\partial\rho}{\partial\bar z_0}...\frac{\partial\rho}{\partial\bar z_m})^{\alpha}(-i\bar z)^{\alpha}s_0(z,z)t_0 + M) \\ &\le K^{m+|\alpha|}M'_{\alpha}, \end{aligned} \tag{2.28}$$

which M'_α just depends on M,ρ,z,α.

Lemma 2.10. *If f is an anti holomorphic function on $\mathbb{C}^{m+1}$ then,*

$$\frac{\partial^{\alpha} f}{\partial\bar z^{\alpha}} = \frac{1}{(-i\bar z)^{\alpha}}D^{\alpha}_T f + \sum_{|\beta|<|\alpha|} e_\beta D^{\beta}_T f,$$

which e_β just depends on α,β,z.

Theorem 2.11. *If $z=(z_0,...,z_m)\in X$ and $z_i\neq 0$ for each z_i then there is a constant C'_α which it just depends on z,α,ψ,X such that,*

$$\frac{\partial^{\alpha}}{\partial\bar z^{\alpha}}\Pi_K(z,z) = s_0(z,z)t_0(Kt_0)^{m+|\alpha|}(\frac{\partial\rho}{\partial\bar z_0}...\frac{\partial\rho}{\partial\bar z_m})^{\alpha} + R'_{K,\alpha}, \tag{2.29}$$

and $|R'_{K,\alpha}| \le C'_\alpha K^{m+|\alpha|-1}$.

Proof. By using lemma 2.10 and theorem 2.9 we have,

$$\begin{aligned}\frac{\partial^\alpha}{\partial\bar{z}^\alpha}\Pi_K(z,z) &= \frac{1}{(\bar{z})^\alpha}D_T^\alpha\Pi_K(z,z)+\sum_{|\beta|<|\alpha|}e_\beta D_T^\beta\Pi_K(z,z)\\ &= \frac{1}{(\bar{z})^\alpha}(s_0(z,z)t_0(Kt_0)^{m+|\alpha|}(\frac{\partial\rho}{\partial\bar{z}_0}...\frac{\partial\rho}{\partial\bar{z}_m})^\alpha(-i\bar{z})^\alpha+R_{K,\alpha})+\sum_{|\beta|<|\alpha|}e_\beta D_T^\beta\Pi_K(z,z)\\ &= s_0(z,z)t_0(Kt_0)^{m+|\alpha|}(\frac{\partial\rho}{\partial\bar{z}_0}...\frac{\partial\rho}{\partial\bar{z}_m})^\alpha+\frac{1}{(-i\bar{z})^\alpha}R_{K,\alpha}+\sum_{|\beta|<|\alpha|}e_\beta D_T^\beta\Pi_K(z,z)\\ &= s_0(z,z)t_0(Kt_0)^{m+|\alpha|}(\frac{\partial\rho}{\partial\bar{z}_0}...\frac{\partial\rho}{\partial\bar{z}_m})^\alpha+R'_{K,\alpha},\end{aligned} \tag{2.30}$$

which

$$R'_{K,\alpha}=\frac{1}{(\bar{z})^\alpha}R_{K,\alpha}+\sum_{|\beta|<|\alpha|}e_\beta D_T^\beta\Pi_K(z,z).$$

Now by using inequality 2.28

$$\begin{aligned}R'_{K,\alpha} &= \frac{1}{(-i\bar{z})^\alpha}R_{K,\alpha}+\sum_{|\beta|<|\alpha|}e_\beta D_T^\beta\Pi_K(z,z)\\ &\le \frac{1}{(-i\bar{z})^\alpha}C_\alpha K^{M+|\alpha|-1}+\sum_{|\beta|<|\alpha|}e_\beta M'_\beta K^{M+|\beta|}\\ &\le K^{M+|\alpha|-1}(\frac{1}{(-i\bar{z})^\alpha}C_\alpha+\sum_{|\beta|<|\alpha|}e_\beta M'_\beta)\\ &= K^{M+|\alpha|-1}C'_\alpha,\end{aligned} \tag{2.31}$$

which

$$C'_\alpha=\frac{1}{(-i\bar{z})^\alpha}C_\alpha+\sum_{|\beta|<|\alpha|}e_\beta M'_\beta.$$

□

Theorem 2.12. *For any $z=(z_0,...,z_m)\in X$ which $z_i\neq 0$ for each z_i and $\alpha=(\alpha_0,..,\alpha_m)$ which $\alpha_i\in\{0,1,..\}$ we have,*

$$\lim_{N\to\infty}\frac{1}{N^{m+|\alpha|+1}}\frac{\partial^\alpha}{\partial\bar{z}^\alpha}S_N(z,z)=s_0(z,z)(t_0)^{m+1}\int_0^1 y^m(yt_0\frac{\partial\rho}{\partial\bar{z}_0}...\frac{\partial\rho}{\partial\bar{z}_m})^\alpha dy. \tag{2.32}$$

Proof.

(2.33)
$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N^{m+|\alpha|+1}} \frac{\partial^\alpha}{\partial \bar{z}^\alpha} S_N(z,z) &= \lim_{N\to\infty} \frac{1}{N^{m+|\alpha|+1}} \sum_{K=0}^{N} \frac{\partial^\alpha}{\partial \bar{z}^\alpha} \Pi_K(z,z) \\
&= \lim_{N\to\infty} \frac{1}{N^{m+|\alpha|+1}} (\sum_{K=0}^{N} (Kt_0)^{m+|\alpha|} (\frac{\partial \rho}{\partial \bar{z}_0} ... \frac{\partial \rho}{\partial \bar{z}_m})^\alpha s_0(z,z) t_0 + \sum_{K=0}^{N} R_{K,\alpha}) \\
&= (\frac{\partial \rho}{\partial \bar{z}_0} ... \frac{\partial \rho}{\partial \bar{z}_m})^\alpha s_0(z,z) t_0 \lim_{N\to\infty} (\sum_{K=0}^{N} (\frac{Kt_0}{N})^{m+|\alpha|} \frac{1}{N} + \sum_{K=0}^{N} \frac{R_{K,\alpha}}{N^{m+|\alpha|}} \frac{1}{N}) \\
&= (\frac{\partial \rho}{\partial \bar{z}_0} ... \frac{\partial \rho}{\partial \bar{z}_m})^\alpha s_0(z,z) (t_0)^{m+|\alpha|+1} \int_0^1 y^{m+|\alpha|} dy + 0 \\
&= s_0(z,z)(t_0)^{m+1} (t_0 \frac{\partial \rho}{\partial \bar{z}_0} ... \frac{\partial \rho}{\partial \bar{z}_m})^\alpha \int_0^1 y^{m+|\alpha|} dy \\
&= s_0(z,z)(t_0)^{m+1} \int_0^1 y^m (y t_0 \frac{\partial \rho}{\partial \bar{z}_0} ... \frac{\partial \rho}{\partial \bar{z}_m})^\alpha dy.
\end{aligned}$$

□

For the next step we consider behavior of scaling Szegö kernel when N goes to infinity, for this purpose we pick a point on the X which we call it $z = (z_0, .., z_m)$ and we move in the direction of $u = (u_0, .., u_m) \in \mathbb{C}^{m+1}$, for the simplicity we define,

$$G_N(u) = \{\frac{S_N(z + \frac{u}{N}, z)}{N^{m+1}}\},$$

we want to use Arzela Ascoli theorem to show that $G_N(u)$ uniformly converges on any compact set in $\mathbb{C}^{m+1}$. I should mention that we fix point $z \in X$.

Lemma 2.13. *$G_N(u)$ is uniformly bounded on $\bar{B}(0,1) \subset \mathbb{C}^{m+1}$.*

Proof.

$$\begin{aligned}
|G_N(u)| = |\frac{1}{N^{m+1}} S_N(z + \frac{u}{N}, z)| &= |\frac{1}{N^{m+1}} \sum_{|J|\le N} c_J (1 + \frac{u}{Nz})^J z^J \bar{z}^J| \\
&= |\frac{1}{N^{m+1}} \sum_{|J|\le N} ((1 + \frac{u_0}{Nz_0})^{J_0} .. (1 + \frac{u_m}{Nz_m})^{J_m}) c_J z^J \bar{z}^J| \\
&\le \frac{1}{N^{m+1}} \sum_{|J|\le N} (|1 + \frac{u_0}{Nz_0}|^{J_0} .. |1 + \frac{u_m}{Nz_m}|^{J_m}) c_J z^J \bar{z}^J \qquad (2.34) \\
&\le e^{\sum_{i=0}^m |\frac{u_i}{z_i}|} \frac{1}{N^{m+1}} \sum_{|J|\le N} c_J z^J \bar{z}^J \\
&= e^{\sum_{i=0}^m |\frac{u_i}{z_i}|} \frac{1}{N^{m+1}} S_N(z,z).
\end{aligned}$$

At the end we have,

$$|G_N(u)| \le e^{\sum_{i=0}^m |\frac{u_i}{z_i}|} \frac{1}{N^{m+1}} S_N(z,z). \tag{2.35}$$

Bye using theorem 2.12, we see that $\frac{1}{N^{m+1}}S_N(z,z)$ converges, so there is a positive constant M such that $|\frac{1}{N^{m+1}}S_N(z,z)| \leq M$ so

$$|G_N(u)| \leq Me^{\sum_{i=0}^{m}|\frac{u_i}{z_i}|}$$

□

Lemma 2.14. $\frac{\partial}{\partial u_i}G_N(u)$ *is uniformly bounded on* $\bar{B}(0,1) \subset \mathbb{C}^{m+1}$ *for* $i = 0, ..m$.

Proof. we prove this lemma for $i = 0$, same proof works for $i = 1, ..m$.

(2.36)

$$\begin{aligned}
|\frac{\partial}{\partial u_0}G_N(u)| &= |\frac{1}{N^{m+1}}\frac{\partial}{\partial u_0}S_N(z+\frac{u}{N},z)| = |\frac{1}{N^{m+2}}\frac{\partial}{\partial z_0}S_N(z+\frac{u}{N},z)| \\
&= |\frac{1}{N^{m+2}}\sum_{|J|\leq N} c_J j_0 (z_0+\frac{u_0}{N})^{j_0-1}...(z_m+\frac{u_m}{N})^{j_m}\bar{z}^J| \\
&= |\frac{1}{N^{m+2}}\sum_{|J|\leq N} ((1+\frac{u_0}{Nz_0})^{j_0-1}..(1+\frac{u_m}{Nz_m})^{j_m})c_J z_0^{j_0-1}...z_m^{j_m}\bar{z}^J| \\
&\leq \frac{1}{N^{m+2}}\sum_{|J|\leq N} (|1+\frac{u_0}{Nz_0}|^{j_0-1}..|1+\frac{u_m}{Nz_m}|^{j_m})c_J z_0^{j_0-1}...z_m^{j_m}\bar{z}^J| \\
&\leq e^{\sum_{i=0}^{m}|\frac{u_i}{z_i}|}\frac{1}{N^{m+2}}\sum_{|J|\leq N} c_J j_0 z_0^{j_0-1}..z_m^{j_m}\bar{z}^J \\
&= e^{\sum_{i=0}^{m}|\frac{u_i}{z_i}|}\frac{1}{N^{m+2}}\frac{\partial}{\partial z_0}S_N(z,z).
\end{aligned}$$

By using 2.19 we see that $\frac{1}{N^{m+2}}\frac{\partial}{\partial z_0}S_N(z,z)$ converges for any $z \in X$ which each $z_i \neq 0$ so there is a positive constant M_0 such that $|\frac{1}{N^{m+2}}\frac{\partial}{\partial z_0}S_N(z,z)| \leq M_0$ so

$$|\frac{\partial}{\partial u_0}G_N(u)| \leq M_0 e^{\sum_{i=0}^{m}|\frac{u_i}{z_i}|}.$$

□

Now by using lemma 2.14 we see that G_N is an equicontinues sequence of holomorphic functions on $\bar{B}(0,1) \subset \mathbb{C}^{m+1}$ which is also uniformly bounded on $\bar{B}(0,1)$, so by using Arzel Ascoli theorem, there is a subsequence like $\{G_{N_j}\}$ which converges uniformly on $\bar{B}(0,1)$, in the next theorem we compute limit of this subsequence and after that we prove that whole sequence converges to the same limit.

Theorem 2.15. $\lim_{N\to\infty}\frac{1}{N^{m+1}}S_N(z+\frac{u}{N},z) = s_0(z,z)t_0^{m+1}F_m(t_0(d'\rho(z)\cdot u))$.

Proof. As we already proved there is a convergent subsequence of G_N like G_{N_j}, which converges uniformly on $\bar{B}(0,1) \subset \mathbb{C}^{m+1}$, now by writing Taylor series for any $\{G_{N_j}\}$ around origin we will have,

$$G_{N_j}(u) = \sum_{\alpha}\frac{\partial^\alpha}{\partial u^\alpha}G_{N_j}(0)\frac{u^\alpha}{\alpha!}$$

on the other hand if we let,

$$G(u) = lim_{j\to\infty}G_{N_j}(u)$$

then

$$\frac{\partial^\alpha}{\partial u^\alpha}G(0) = \lim_{j\to\infty}\frac{\partial^\alpha}{\partial u^\alpha}G_{N_j}(0)$$

because each G_{N_j} is holomorphic on $\mathbb{C}^{m+1}$ and they converge uniformly on $\bar{B}(0,1)$ to $G(u)$, so

$$\begin{aligned}
G(u) = \sum_\alpha \frac{\partial^\alpha}{\partial u^\alpha}G(0)\frac{u^\alpha}{\alpha!} &= \sum_\alpha \lim_{j\to\infty}\frac{\partial^\alpha}{\partial u^\alpha}G_{N_j}(0)\frac{u^\alpha}{\alpha!}\\
&= \sum_\alpha \lim_{N_j\to\infty}\frac{1}{N_j^{m+|\alpha|+1}}\frac{\partial^\alpha}{\partial \bar{z}^\alpha}S_{N_j}(z,z)\frac{(u)^\alpha}{\alpha!}\\
&= s_0(z,z)t_0^{m+1}\sum_\alpha \int_0^1 y^m \frac{(t_0 y\frac{\partial\rho}{\partial z_0}u_0 .. \frac{\partial\rho}{\partial z_m}u_m)^\alpha}{\alpha!}dy \qquad (2.37)\\
&= s_0(z,z)t_0^{m+1}\int_0^1 e^{yt_0(d'\rho(z)\cdot u)}y^m dy\\
&= s_0(z,z)t_0^{m+1}F_m(t_0(d'\rho(z)\cdot u)),
\end{aligned}$$

which we let $F_m(x) = \int_0^1 e^{yx}y^m$. As you see any convergent subsequence of $\{\frac{1}{N^{m+1}}S_N(z+\frac{u}{N},z)\}$ converges to $s_0(z,z)t_0^{m+1}F_m(t_0(d'\rho(z)\cdot u))$ and also we showed that $\{\frac{1}{N^{m+1}}S_N(z+\frac{u}{N},z)\}$ is bounded so it means:

$$\lim_{N\to\infty}\frac{1}{N^{m+1}}S_N(z+\frac{u}{N},z) = s_0(z,z)t_0^{m+1}F_m(t_0(d'\rho(z)\cdot u)). \qquad (2.38)$$

□

Theorem 2.16. *If* $z=(z_0,..,z_m)\in X, u=(u_0,..,u_m), v=(v_0,...,v_m)\in\mathbb{C}^{m+1}, z_i\neq 0$ *then:*

$$\lim_{N\to\infty}\frac{1}{N^{m+1}}S_N(z+\frac{u}{N},z+\frac{v}{N}) = s_0(z,z)t_0^{m+1}F_m(t_0(d'\rho(z)\cdot u+d''\rho(z)\cdot\bar{v})) \qquad (2.39)$$

2.3. **Derivatives of partial szegö kernel.** Our main tool for computing scaling limit correlation function is the Kac-Rice formula which for that we need to know derivatives of partial szegö kernel, in this section we put our aim to compute scaling limit of derivative of partial szegö kernel,

Theorem 2.17.

$$\lim_{N\to\infty}\frac{1}{N^{m+2}}\frac{\partial}{\partial z_i}S_N(z+\frac{u}{N},z+\frac{v}{N}) = s_0(z,z)t_0^{m+2}\frac{\partial\rho}{\partial z_i}F_{m+1}(t_0(d'\rho(z)\cdot u+d''\rho(z)\cdot\bar{v})),$$

$$\lim_{N\to\infty}\frac{1}{N^{m+2}}\frac{\partial}{\partial \bar{z}_i}S_N(z+\frac{u}{N},z+\frac{v}{N}) = s_0(z,z)t_0^{m+2}\frac{\partial\rho}{\partial \bar{z}_i}F_{m+1}(t_0(d'\rho(z)\cdot u+d''\rho(z)\cdot\bar{v})),$$

Proof. Let $G_N(u,v) = \frac{1}{N^{m+1}}S_N(z+\frac{u}{N},z+\frac{v}{N})$ then,

$$\begin{aligned}
\frac{\partial}{\partial u_i}G_N(u,v) &= \frac{1}{N^{m+1}}\frac{\partial}{\partial z_i}S_N(z+\frac{u}{N},z+\frac{v}{N})\frac{1}{N} \qquad (2.40)\\
&= \frac{1}{N^{m+2}}\frac{\partial}{\partial z_i}S_N(z+\frac{u}{N},z+\frac{v}{N}).
\end{aligned}$$

(2.41)

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N^{m+2}}\frac{\partial}{\partial z_i}S_N(z+\frac{u}{N},z+\frac{v}{N}) &= \lim_{N\to\infty}\frac{\partial}{\partial u_i}G_N(u,v)\\ &= \frac{\partial}{\partial u_i}\lim_{N\to\infty}G_N(u,v)\\ &= \frac{\partial}{\partial u_i}(s_0(z,z)t_0^{m+1}F_m(t_0(d'\rho(z)\cdot u+d''\rho(z)\cdot\bar{v})))\\ &= (s_0(z,z)t_0^{m+1}\frac{\partial}{\partial u_i}F_m(t_0(d'\rho(z)\cdot u+d''\rho(z)\cdot\bar{v})))\\ &= s_0(z,z)t_0^{m+2}\frac{\partial\rho}{\partial z_i}F'_m(t_0(d'\rho(z)\cdot u+d''\rho(z)\cdot\bar{v}))\\ &= s_0(z,z)t_0^{m+2}\frac{\partial\rho}{\partial z_i}F_{m+1}(t_0(d'\rho(z)\cdot u+d''\rho(z)\cdot\bar{v})).\end{aligned}$$

and similarly by following the same proof we can show that

(2.42)

$$\lim_{N\to\infty}\frac{1}{N^{m+2}}\frac{\partial}{\partial \bar{z}_i}S_N(z+\frac{u}{N},z+\frac{v}{N}) = s_0(z,z)t_0^{m+2}\frac{\partial\rho}{\partial \bar{z}_i}F_{m+1}(t_0(d'\rho(z)\cdot u+d''\rho(z)\cdot\bar{v})).$$

□

Theorem 2.18.

$$\lim_{N\to\infty}\frac{1}{N^{m+3}}\frac{\partial^2}{\partial \bar{z}_i\partial z_j}S_N(z+\frac{u}{N},z+\frac{v}{N}) = s_0(z,z)t_0^{m+3}\frac{\partial\rho}{\partial \bar{z}_i}\frac{\partial\rho}{\partial z_j}F_{m+2}(t_0(d'\rho(z)\cdot u+d''\rho(z)\cdot\bar{v})).$$

Proof.

(2.43)

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N^{m+3}}\frac{\partial^2}{\partial \bar{z}_i\partial z_j}S_N(z+\frac{u}{N},z+\frac{v}{N}) &= \lim_{N\to\infty}\frac{1}{N^{m+1}}\frac{\partial^2}{\partial v_i\partial u_j}G_N(u,v)\\ &= \frac{\partial^2}{\partial v_i\partial u_j}\lim_{N\to\infty}\frac{1}{N^{m+1}}G_N(u,v)\\ &= \frac{\partial^2}{\partial v_i\partial u_j}(s_0(z,z)t_0^{m+1}F_m(t_0(d'\rho(z)\cdot u+d''\rho(z)\cdot\bar{v})))\\ &= s_0(z,z)t_0^{m+3}\frac{\partial^2}{\partial v_i\partial u_j}F_m(t_0(d'\rho(z)\cdot u+d''\rho(z)\cdot\bar{v}))\\ &= s_0(z,z)t_0^{m+3}\frac{\partial\rho}{\partial \bar{z}_i}\frac{\partial\rho}{\partial z_j}F''_m(t_0(d'\rho(z)\cdot u+d''\rho(z)\cdot\bar{v}))\\ &= s_0(z,z)t_0^{m+3}\frac{\partial\rho}{\partial \bar{z}_i}\frac{\partial\rho}{\partial z_j}F_{m+2}(t_0(d'\rho(z)\cdot u+d''\rho(z)\cdot\bar{v})).\end{aligned}$$

□

3. Scaling limit Distributions

We now have all the ingredients that we need to compute the Scaling limit distribution functions, we expect the scaling limits to exist and depend only on the m, z, ρ , we are going to use Kac-Rice formula for computing distribution function, if we look at ensemble (P_N, γ_N) and if we fix $z+\frac{u}{N}$ then we can have this random variable,

$$X_N : (P_N, \gamma_N) \to \mathbb{C} \times \mathbb{C}^{m+1},$$

which:

$$X_N(f_N) = (f_N(z + \frac{u}{N}), \frac{\partial f_N}{\partial z_0}(z + \frac{u}{N}), ..., \frac{\partial f_N}{\partial z_m}(z + \frac{u}{N})).$$

and covariance matrix for X_N is equal to

$$\triangle_N = \begin{pmatrix} A_N & B_N \\ B_N^* & C_N \end{pmatrix}, where: \tag{3.1}$$

$$\begin{aligned} A_N &= S_N(z + \frac{u}{N}, z + \frac{u}{N}), \\ B_N &= (\frac{\partial}{\partial \bar{z}_i} S_N(z + \frac{u}{N}, z + \frac{u}{N}))_{0 \le i \le m}, \\ C_N &= (\frac{\partial^2 S_N}{\partial z_i \partial \bar{z}_j}(z + \frac{u}{N}, z + \frac{u}{N}))_{0 \le i,j \le m}, \end{aligned} \tag{3.2}$$

$$\Lambda_N = C_N - (B_N)^* A_N^{-1} B_N, \tag{3.3}$$

and by using Kac-Rice formula

$$D^N_{\mu,X}(z + \frac{u}{N}) = \frac{1}{\pi} \frac{\sum_{i=0}^m (\Lambda_N)_{i,i}}{det(A_N)}. \tag{3.4}$$

Our goal is to compute,

$$\lim_{N \to \infty} \frac{D^N_{\mu,X}(z + \frac{u}{N})}{N^2} = \lim_{N \to \infty} \frac{1}{\pi} \frac{\sum_{i=0}^m \frac{(\Lambda)_{i,i}}{N^{m+3}}}{\frac{det(A_N)}{N^{m+1}}}, \tag{3.5}$$

we define

$$\begin{aligned} \alpha &= t_0(d'\rho(z) \cdot u + d''\rho(z) \cdot \bar{u}), \\ P &= (\frac{\partial \rho}{\partial \bar{z}_0}, ..., \frac{\partial \rho}{\partial \bar{z}_m}), \end{aligned} \tag{3.6}$$

so by using definition of α, P we can simplify each formula that we got for the scaling limit of szegö kernel and its derivatives, now if we use theorems 2.16, 2.17, 2.18 then we will have:

$$\lim_{N \to \infty} \frac{A_N}{N^{m+1}} = s_0(z,z) t_0^{m+1} F_m(\alpha), \tag{3.7}$$

$$\begin{aligned} \lim_{N \to \infty} \frac{B_N}{N^{m+2}} &= s_0(z,z) t_0^{m+2} (\frac{\partial \rho(z)}{\partial \bar{z}_0}, .., \frac{\partial \rho(z)}{\partial \bar{z}_m}) F_{m+1}(\alpha) \\ &= s_0(z,z) t_0^{m+2} F_{m+1}(\alpha) P, \end{aligned} \tag{3.8}$$

$$
\begin{aligned}
\lim_{N\to\infty}\frac{C_N}{N^{m+3}} &= s_0(z,z)t_0^{m+3}(\frac{\partial\rho(z)}{\partial z_i}\frac{\partial\rho(z)}{\partial\bar{z}_j}F_{m+2}(\alpha))_{0\le i,j\le m} \\
&= s_0(z,z)t_0^{m+3}F_{m+2}(\alpha)\begin{pmatrix}\frac{\partial\rho(z)}{\partial z_0}\\ .\\ .\\ .\\ \frac{\partial\rho(z)}{\partial z_m}\end{pmatrix}\begin{pmatrix}\frac{\partial\rho(z)}{\partial\bar{z}_0} & . & . & . & \frac{\partial\rho(z)}{\partial\bar{z}_m}\end{pmatrix} \\
&= s_0(z,z)t_0^{m+3}F_{m+2}(\alpha)P^*P,
\end{aligned} \tag{3.9}
$$

Now if we plug results that we got from equations 3.7, 3.8, 3.9 in,

$$
\lim_{N\to\infty}\frac{\Lambda_N}{N^{m+3}} = \lim_{N\to\infty}(\frac{C_N}{N^{m+3}}-(\frac{B_N}{N^{m+2}})^*(\frac{A_N}{N^{m+1}})^{-1}(\frac{B_N}{N^{m+2}})), \tag{3.10}
$$

we will have,

$$
\lim_{N\to\infty}\frac{\Lambda_N}{N^{m+3}} = s_0(z,z)t_0^{m+3}F_{m+2}(\alpha)P^*P - s_0(z,z)t_0^{m+3}\frac{F_{m+1}^2(\alpha)}{F_m(\alpha)}P^*P, \tag{3.11}
$$

So by using 3.11 we get this formula

$$
\lim_{N\to\infty}\frac{(\Lambda_N)_{i,i}}{N^{m+3}} = s_0(z,z)t_0^{m+3}(F_{m+2}(\alpha)-\frac{F_{m+1}^2(\alpha)}{F_m(\alpha)})(P^*P)_{i,i}, \tag{3.12}
$$

So

$$
\begin{aligned}
\lim_{N\to\infty}\sum_{i=0}^{m}\frac{(\Lambda_N)_{i,i}}{N^{m+3}} &= s_0(z,z)t_0^{m+3}(F_{m+2}(\alpha)-\frac{F_{m+1}^2(\alpha)}{F_m(\alpha)})(\sum_{i=0}^{m}(P^*P)_{i,i}) \\
&= s_0(z,z)t_0^{m+3}(F_{m+2}(\alpha)-\frac{F_{m+1}^2(\alpha)}{F_m(\alpha)})||P||^2.
\end{aligned} \tag{3.13}
$$

Theorem 3.1. *Let $D_{\mu,X}^N$ be the expected zero density for the probability space (P_N,γ_N) then*

$$
\lim_{N\to\infty}\frac{1}{N^2}D_{\mu,X}^N(z+\frac{u}{N}) = D_{z,X}^{\infty}(u),
$$

where

$$
D_{z,X}^{\infty}(u) = \frac{(t_0||P||)^2}{\pi}(\log F_m)^{''}(\alpha),
$$

where α, P were defined at 3.6

Proof.

(3.14)

$$D^{\infty}_{z,X}(u) = \frac{1}{\pi}\lim_{N\to\infty}\frac{D^N_{\mu,X}(z+\frac{u}{N})}{N^2} = \lim_{N\to\infty}\frac{1}{\pi}\frac{\sum_{i=0}^m \frac{(\Lambda)_{i,i}}{N^{m+3}}}{\frac{det(A_N)}{N^{m+1}}}$$
$$= \frac{s_0(z,z)t_0^{m+3}(F_{m+2}(\alpha)-\frac{F^2_{m+1}(\alpha)}{F_m(\alpha)})}{s_0(z,z)t_0^{m+1}F_m(\alpha)}||P||^2$$
$$= \frac{1}{\pi}t_0^2\frac{F_{m+2}(\alpha)F_m(\alpha)-F^2_{m+1}(\alpha)}{F_m(\alpha)^2}||P||^2$$
$$= \frac{1}{\pi}(t_0||P||)^2(\log F_m)^{''}(\alpha).$$

□

4. The scaling limit zero correlation function

let $z\in X$ and $u,v\in\mathbb{C}^{m+1}$ now we define the new random variable

$$X_N : (P_N,\gamma_N)\to\mathbb{C}^2\times\mathbb{C}^{2(m+1)},$$

Such that if f_N is a random polynomial in P_N then

$$X_N(f_N) = (f_N(z+\frac{u}{N}), f_N(z+\frac{v}{N}), \frac{\partial f_N}{\partial z_0}(z+\frac{u}{N}),...,\frac{\partial f_n}{\partial z_m}(z+\frac{u}{N}), \frac{\partial f_N}{\partial z_0}(z+\frac{v}{N}),...,\frac{\partial f_N}{\partial z_m}(z+\frac{v}{N})).$$

At first we are going to compute covariance matrix for X_N

$$\triangle_N = \begin{pmatrix} A_N & B_N \\ B_N^* & C_N\end{pmatrix}, \tag{4.1}$$

where,

$$A_N = \begin{pmatrix} S_N(z+\frac{u}{N}, z+\frac{u}{N}) & S_N(z+\frac{u}{N}, z+\frac{v}{N}) \\ S_N(z+\frac{v}{N}, z+\frac{u}{N}) & S_N(z+\frac{v}{N}, z+\frac{v}{N})\end{pmatrix}, \tag{4.2}$$

$$B_N = \begin{pmatrix} B_N^1 & B_N^2 \\ B_N^3 & B_N^4\end{pmatrix}, \tag{4.3}$$

such that,

$$\begin{aligned} B_N^1 &= (\frac{\partial S_N(z+\frac{u}{N}, z+\frac{u}{N})}{\partial \bar z_i})_{0\le i\le m}, \\ B_N^2 &= (\frac{\partial S_N(z+\frac{u}{N}, z+\frac{v}{N})}{\partial \bar z_i})_{0\le i\le m}, \\ B_N^3 &= (\frac{\partial S_N(z+\frac{v}{N}, z+\frac{u}{N})}{\partial \bar z_i})_{0\le i\le m}, \\ B_N^4 &= (\frac{\partial S_N(z+\frac{v}{N}, z+\frac{v}{N})}{\partial \bar z_i})_{0\le i\le m}, \end{aligned} \tag{4.4}$$

$$C_N = \begin{pmatrix} C_N^{1,1} & C_N^{1,2} \\ C_N^{2,1} & C_N^{2,2}\end{pmatrix}, \tag{4.5}$$

where

$$C_N^{1,1} = (\frac{\partial^2 S_N}{\partial z_i \partial \bar{z}_j}(z+\frac{u}{N}, z+\frac{u}{n}))_{0\le i,j\le m},$$
$$C_N^{1,2} = (\frac{\partial^2 S_N}{\partial z_i \partial \bar{z}_j}(z+\frac{u}{N}, z+\frac{v}{n}))_{0\le i,j\le m},$$
$$C_N^{2,1} = (\frac{\partial^2 S_N}{\partial z_i \partial \bar{z}_j}(z+\frac{v}{N}, z+\frac{u}{n}))_{0\le i,j\le m}, \tag{4.6}$$
$$C_N^{2,2} = (\frac{\partial^2 S_N}{\partial z_i \partial \bar{z}_j}(z+\frac{v}{N}, z+\frac{v}{n}))_{0\le i,j\le m}.$$

we let $v = 0$ and $\alpha = t_0(d'\rho(z)\cdot u + d''\rho(z)\cdot \bar{u})$ and $\beta = t_0(d'\rho(z)\cdot u)$ then

$$A_\infty = \lim_{N\to\infty} \frac{1}{N^{m+1}} A_N = s_0(z,z)t_0^{m+1}\begin{pmatrix} F_m(\alpha) & F_m(\beta) \\ F_m(\bar{\beta}) & F_m(0)\end{pmatrix},$$
$$B_\infty = \lim_{N\to\infty} \frac{1}{N^{m+2}} B_N = s_0(z,z)t_0^{m+2}\begin{pmatrix} F_{m+1}(\alpha)\bar{P} & F_m(\beta)\bar{P} \\ F_{m+1}(\bar{\beta})\bar{P} & F_{m+1}(0)\bar{P}\end{pmatrix}, \tag{4.7}$$
$$C_\infty = \lim_{N\to\infty} \frac{1}{N^{m+3}} C_n = s_0(z,z)t_0^{m+3}\begin{pmatrix} F_{m+2}(\alpha)P^*P & F_{m+2}(\beta)P^*P \\ F_{m+2}(\bar{\beta})P^*P & F_{m+2}(0)P^*P\end{pmatrix}.$$

Now wee define two matrices $G_m(x)$, $Q_m(x)$ which make our computation easier,

$$G_m(x) = \begin{pmatrix} F_m(x+\bar{x}) & F_m(x) \\ F_m(\bar{x}) & F_m(0)\end{pmatrix}, \tag{4.8}$$

and for the $Q_m(x)$ we assume $x \neq 0$,

$$Q_m(x) = G_{m+2}(x) - G_{m+1}(x)G_m(x)^{-1}G_{m+1}(x). \tag{4.9}$$

Because if $x = 0$ then $G_m(x)$ is not invertible, so $Q_m(x)$ is a two by two matrix and also we choose $u \in \mathbb{C}^{m+1}$ such that $\beta = d'\rho(z)\cdot u \neq 0$ then it means that $G_m(\beta)^{-1}$ is a well-defined matrix so we have

$$B_\infty^* A_\infty^{-1} B_\infty = s_0(z,z)t_0^{m+3}\begin{pmatrix} (G_{m+1}(\beta)G_m(\beta)^{-1}G_{m+1}(\beta))_{1,1}P^*P & (G_{m+1}(\beta)G_m(\beta)^{-1}G_{m+1}(\beta))_{1,2}P^*P \\ (G_{m+1}(\beta)G_m(\beta)^{-1}G_{m+1}(\beta))_{2,1}P^*P & (G_{m+1}(\beta)G_m(\beta)^{-1}G_{m+1}(\beta))_{2,2}P^*P\end{pmatrix}, \tag{4.10}$$

$$\Lambda_\infty = s_0(z,z)t_0^{m+3}\begin{pmatrix} Q_{1,1}P^*P & Q_{1,2}P^*P \\ Q_{2,1}P^*P & Q_{2,2}P^*P\end{pmatrix}. \tag{4.11}$$

Our goal is to compute scaling limit normalized correlation function,

$$\tilde{K}_{z,X}^{\infty}(u) = \lim_{N\to\infty} \frac{K_{\mu,X}^N(z+\frac{u}{N}, z)}{D_{\mu,X}^N(z+\frac{u}{N})D_{\mu,X}^N(z)}, \tag{4.12}$$

where

$$E_{X,\mu}^N(|Z_{f_N}|^2 \wedge \frac{\omega^{2m}}{(2m)!}) = K_{\mu,X}^N(z,w)\frac{\omega^{2m+2}}{(2m+2)!}. \tag{4.13}$$

Theorem 4.1. *let $\tilde{K}^N_{\mu,X}(z,w)$ be the normalized correlation function for the probability space (P_N,γ_N) and choose $u\in\mathbb{C}^{m+1}$ such that $u\notin T^h_zX$ then,*

$$\frac{1}{N^4}K^N_{\mu,X}(z+\frac{u}{N},z)=K^\infty_{z,X}(u),$$
$$\lim_{N\to\infty}\tilde{K}^N_{\mu,X}(z+\frac{u}{N},z)=\tilde{K}^\infty_{z,X}(u),$$

where

$$K^\infty_{z,X}(u)=\frac{1}{\pi^2}\frac{perm(Q_m(\beta))}{det(G_m(\beta))}(||P||t_0)^2,$$
$$\tilde{K}^\infty_{z,X}(u)=\frac{1}{(\log F_m)''(\alpha)(\log F_m)''(0)}\frac{perm(Q_m(\beta))}{det(G_m(\beta))},$$

which $K^N_{\mu,X}(z,w)$, $\tilde{K}^N_{\mu,X}(z,w)$ are defined in 4.13, 4.12 .

Proof. At first by using Kac-Rice formula we compute $\frac{1}{N^4}K^N_{\mu,X}(z+\frac{u}{N},z)$ and then by using theorem 3.1 we compute scaling limit for the normalized correlation function,

(4.14)
$$\frac{1}{N^4}K^N_{\mu,X}(z+\frac{u}{N},z)\to\frac{(\sum_{i=0}^m\frac{\Lambda^N_{i,i}}{N^{m+3}})(\sum_{i=m+1}^{2m}\frac{\Lambda^N_{i,i}}{N^{m+3}})+\sum_{i=m+1}^{2m}\frac{\Lambda^N_{1,i}}{N^{m+3}}\frac{\Lambda^N_{i,1}}{N^{m+3}}+..+\sum_{i=m+1}^{2m}\frac{\Lambda^N_{m,i}}{N^{m+3}}\frac{\Lambda^N_{i,m}}{N^{m+3}}}{\pi^2\frac{det(A^N)}{N^{2m+2}}}$$
$$=\frac{(Q_{1,1}Q_{2,2}+Q_{1,2}Q_{2,1})||P||^4t_0^4}{\pi^2det(G(\beta,m))}$$
$$=\frac{1}{\pi^2}\frac{perm(Q_m(\beta))}{det(G_m(\beta))}(||P||t_0)^4.$$

Now we are ready to give a general formula for $\tilde{K}^\infty_{z,X}(u)$, if we use equation 4.14 then,

(4.15)
$$\tilde{K}^\infty_{z,X}(u)=\lim_{N\to\infty}\frac{K^N_{\mu,X}(z+\frac{u}{N},z)}{D^N_{\mu,X}(z+\frac{u}{N})D^N_{\mu,X}(z)}=\lim_{N\to\infty}\frac{\frac{K^N_{\mu,X}(z+\frac{u}{N},z)}{N^4}}{\frac{D^N_{\mu,X}(z+\frac{u}{N})}{N^2}\frac{D^N_{\mu,X}(z)}{N^2}}$$
$$=\frac{\frac{perm(Q_m(\beta))(||P||t_0)^4}{\pi^2det(G_m(\beta))}}{(\frac{||P||^2t_0^2}{\pi}F_m(\alpha))(\frac{||P||^2t_0^2}{\pi}F_m(0))}$$
$$=\frac{1}{(\log F_m)''(\alpha)(\log F_m)''(0)}\frac{perm(Q_m(\beta))}{det(G_m(\beta))}.$$

□

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