COEFFICIENT ESTIMATES FOR INVERSES OF STARLIKE FUNCTIONS OF POSITIVE ORDER

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ABSTRACT: In the present paper, the coefficient estimates are found for the class $S^{*-1}(\alpha)$ consisting of inverses of functions in the class of univalent starlike functions of order α in $D = \{z \in C : |z| < 1\}$. These estimates extend the work of Krzyz, Libera and Zlotkiewicz [12] who found sharp estimates on only first two coefficients for the functions in the class $S^{*-1}(\alpha)$. The coefficient estimates are also found for the class $\Sigma^{*-1}(\alpha)$, consisting of inverses of functions in the class $\Sigma^{*}(\alpha)$ of univalent starlike functions of order α in $V = \{z \in C : 1 < |z| < \infty\}$. The open problem of finding sharp coefficient estimates for functions in the class $\Sigma^{*}(\alpha)$ stands completely settled in the present work by our method developed here.

Key words: *Univalent, Starlike, Order, Inverse function, Coefficient estimates*

AMS (MOS) Subject classification: Primary *30C50*; Secondary *30C45*

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1. INTRODUCTION

Let A_0 be the class of functions f, analytic in the unit disc $D=\{z\in C:|z|<1\}$ and having the power series expansion,

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n. \tag{1.1}$$

The class of univalent functions in A_0 is denoted by S. A function $f\in A_0$ is said to be in the class $S^*(\alpha)$, $0\le\alpha<1$, of starlike functions of order α if , for $z\in D$,

$$\mathrm{Re}\left(z\frac{f'(z)}{f(z)}\right)>\alpha\ , \tag{1.2}$$

It is known that $S^*(\alpha)\subseteq S^*(0)\equiv S^*\subset S$ for $0\le\alpha<1$ [5, p.51]. The class of functions

$$g(z)=z+b_0+\frac{b_1}{z}+\frac{b_2}{z^2}+\cdots \tag{1.3}$$

that are analytic and univalent in $V=\{z\in C:1<|z|<\infty\}$ is denoted by Σ and the class of starlike functions of order α, $0\le\alpha<1$, in V is denoted by $\Sigma^*(\alpha)$, i.e. a function $g\in\Sigma^*(\alpha)$ if and only if $g\in\Sigma$ satisfies $\mathrm{Re}\left(\frac{z\,g'(z)}{g(z)}\right)>\alpha$ for $z\in V$. We are primarily concerned here with the investigation of sharp coefficient estimates for the inverse functions in the above classes. Let S^{-1} be the class of inverse functions f^{-1} of functions $f\in S$ with the Taylor series expansion

$$f^{-1}(w)=w+\sum_{n=2}^{\infty}A_n w^n \tag{1.4}$$

in some disc $|w|<r_0(f)$ and Σ^{*-1} be the class of inverse functions g^{-1} of functions $g\in\Sigma^*$ with the series expansion

$$g^{-1}(w)=w+B_0+\frac{B_1}{w}+\frac{B_2}{w^2}+\cdots\cdots \tag{1.5}$$

in some neighbourhood of infinity. The classes $S^{*-1}(\alpha)$ and $\Sigma^{*-1}(\alpha)$ are defined analogously.

The coefficient estimate problem for the class S, known as the Bieberbach conjecture [2], is settled by De-Branges [4], who proved that for a function $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ in the class S, $|a_n|\le n,$ for all $n=2,3,\ldots$, with equality only for the rotations of the Koebe function

$$K_0(z)=\frac{z}{(1-z)^2} \tag{1.6}$$

Loewner, using his parametric method [15; also see 7, p.222] proved that if f^{-1}, given by (1.4), is in the class S^{-1} or S^{*-1}, the sharp estimate

$$|A_n|\le\frac{\Gamma(2n+1)}{\Gamma(n+2)\ \Gamma(n+1)}\ ,\qquad n=2,3,\cdots\cdots$$

holds, K_0^{-1} being the extremal function for all n in the above inequality. The above coefficient estimate problem for the classes S^{-1} and S^{*-1} is also investigated by the methods developed by Shaeffer and Spencer [17], FitzGerald [6] , Baernstein [1], Poole [16] and others. Prior to de-Branges result on the sharp coefficient bounds for the whole class S, the coefficient estimate problem was established for several subclasses of S, e.g. the classes of convex functions, starlike functions of order α, close-to-convex functions, normalized integrals of functions with positive real part etc. However, in contrast, although for the class of inverse functions the coefficient problem for the whole classes S^{-1} and S^{*-1} had been completely solved as early as in 1923 [15], only few results are known on the sharp coefficient estimates for the inverse of functions in the above subclasses [9, 11, 14, 18]. In certain cases, the coefficients of the inverse of the functions in some of these subclasses show unexpected behaviour. For example, it is known that if f is a univalent convex function and f^{-1} is given by (1.4), then $|A_n| \le 1$ for $n = 1,...,8$ [3,13] and equality holds for the inverse of the function $K_{1/2}(z) = z/(1-z)$; while $|A_{10}| > 1$ [10] and the exact bounds on $|A_9|$ and $|A_n|$ for $n > 10$ are still unknown.

Krzyz, Libera and Zlotkiewicz [12] showed that if f^{-1}, given by (1.4), is in $S^{*-1}(\alpha)$, then

$$|A_2| \le 2(1-\alpha) \tag{1.7}$$

and

$$|A_3| \le \begin{cases} (1-\alpha)(5-6\alpha), & 0 \le \alpha \le \frac{2}{3} & (1.8) \\ (1-\alpha), & \frac{2}{3} \le \alpha < 1 & (1.9) \end{cases}$$

The estimates (1.7) and (1.8) are sharp for the function K_α^{-1} and (1.9) is sharp for the function $K_{\alpha,2}^{-1}$, where

$$K_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \quad \text{and} \quad K_{\alpha,n}(z) = \sqrt[n]{K_\alpha(z^n)} \quad , \ n = 2,3,... \tag{1.10}$$

The determination of sharp estimates on $|A_n|$ for $n \ge 4$ is hitherto an open problem for the class $S^{*-1}(\alpha)$.

In the present paper, the estimates on $|A_n|$, $n \ge 2$, for functions in the class $S^{*-1}(\alpha)$ are found. These estimates are sharp for each n when $\alpha \in [0, \frac{2}{n}) \cup [\frac{n-1}{n}, 1)$ extending the work of Krzyz, Libera and Zlotkiewicz [12], who found sharp estimates given by (1.7), (1.8), and (1.9) on only first two coefficients for the functions in the class $S^{*-1}(\alpha)$. More specifically, it is shown that for each $n \ge 3$, the functions K_α^{-1} and $K_{\alpha,n}^{-1}$, given by (1.10), are extremal for $0 \le \alpha < \frac{2}{n}$ and $\frac{n-1}{n} \le \alpha < 1$ respectively; at variance to the existence of a single extremal function K_0^{-1} for the whole class S^{*-1}. We also prove that, for the functions $g^{-1} \in \Sigma^{*-1}(\alpha)$, given by (1.5), the sharp estimate $|B_n| \le 2(1-\alpha)$, $\frac{n-1}{n} \le \alpha < 1$, holds for all $n = 1, 2, \cdots\cdots$. The open problem of finding sharp coefficient estimates for functions in the class $\Sigma^*(\alpha)$ stands completely settled in the present work by our method developed here.

2. SOME AUXILIARY RESULTS

Let Ω be the class of functions

$$f(z)=\sum_{v=1}^{\infty} a_v z^v, \qquad a_1 \neq 0 \tag{2.1}$$

analytic in $|z|<\rho$ for some $\rho>0$. For a fixed $n \in N$ and $f \in \Omega$, let

$$\frac{1}{(f(z))^n}=\frac{a_{-n}^{(-n)}}{z^n}+\frac{a_{-n+1}^{(-n)}}{z^{n-1}}+\cdots+\frac{a_{-1}^{(-n)}}{z}+\sum_{p=0}^{\infty} a_p^{(-n)}\ z^p=\sum_{\vartheta=0}^{\infty} a_{-n+\vartheta}^{(-n)}\ z^{-n+\vartheta}\quad, |z|<\rho \tag{2.2}$$

Note that $a_{-n}^{(-n)}=1$, if $a_1=1$.

We need the following lemmas in the sequel for our main results:

Lemma1. *Let* $0 \le \alpha < 1$ *and* $n \in N$ *be fixed. Then,*

$$4n\ (1-\alpha)\left[n\ (1-\alpha)+\sum_{m=1}^{\vartheta-1}\{n(1-\alpha)-m\}(\prod_{j=0}^{m-1}\frac{2n(1-\alpha)-j}{j+1})^2\right]$$
$$=\frac{1}{((\vartheta-1)!)^2}\prod_{j=0}^{\vartheta-1}(2n(1-\alpha)-j)^2 \tag{2.3}$$

Proof. For $\vartheta=1$, (2.3) trivially holds. Assume that

$$4n\ (1-\alpha)\left[n(1-\alpha)+\sum_{m=1}^{\vartheta-2}\{n(1-\alpha)-m\}(\prod_{j=0}^{m-1}\frac{2n(1-\alpha)-j}{j+1})^2\right]= \frac{1}{((\vartheta-2)!)^2}\prod_{j=0}^{\vartheta-2}(2n(1-\alpha)-j)^2 .$$

Then,

$$4n(1-\alpha)\left[n(1-\alpha)+\sum_{m=1}^{\vartheta-1}\{n(1-\alpha)-m\}(\prod_{j=0}^{m-1}\frac{2n(1-\alpha)-j}{j+1})^2\right]$$

$$=\frac{1}{((\vartheta-2)!)^2}\prod_{j=0}^{\vartheta-2}(2n(1-\alpha)-j)^2+4n(1-\alpha)\ \{n(1-\alpha)-(\vartheta-1)\}\frac{1}{((\vartheta-1)!)^2}(\prod_{j=0}^{\vartheta-2}(2n(1-\alpha)-j))^2$$
$$=\frac{1}{((\vartheta-2)!)^2}\prod_{j=0}^{\vartheta-2}(2n(1-\alpha)-j)^2\left[1+4n(1-\alpha)\{n(1-\alpha)(-(\vartheta-1)\}\frac{1}{(\vartheta-1)^2}\right]$$
$$=\frac{1}{((\vartheta-1)!)^2}\prod_{j=0}^{\vartheta-2}(2n(1-\alpha)-j)^2\left[2n(1-\alpha)-(\vartheta-1)\right]^2=\frac{1}{((\vartheta-1)!)^2}\prod_{j=0}^{\vartheta-1}(2n(1-\alpha)-j)^2 .$$

The identity (2.3) therefore follows by induction on ϑ. This completes the proof of Lemma 1.

Throughout in the sequel, let $I_k(n)$ denote the semiclosed interval $[\frac{k}{n},\frac{k+1}{n})$, *k = 0,1,….,n-1.*

Lemma 2. *Let* the function $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ *be in the class* $S^*(\alpha)$ *and* $a_{-n+\vartheta}^{(-n)}$ *be given by (2.2). Then, for* $\alpha\in I_k(n)$,

$$\left|a_{-n+\vartheta}^{(-n)}\right|\le\begin{cases}\dfrac{\Gamma(2n(1-\alpha)+1)}{\Gamma(\vartheta+1)\ \Gamma(2n(1-\alpha)+1-\vartheta)} & ,\vartheta=1,\cdots\cdots,n-k \qquad (2.4)\\ \dfrac{\Gamma(2n(1-\alpha)+1)}{\vartheta\ \Gamma(n-k)\ \Gamma\big(2n(1-\alpha)+1-(n-k)\big)} & ,\vartheta=n-k+1,\cdots \qquad (2.5)\end{cases}$$

In particular, if $\alpha\in I_{n-1}(n)$, *then*

$$\left|a_{-n+\vartheta}^{(-n)}\right|\le\frac{2n(1-\alpha)}{\vartheta},\quad \vartheta=1,2,\cdots \tag{2.6}$$

The estimates (2.4) and (2.6) are sharp.

Remark. *By allowing k to vary from 0 to n-1, the estimates (2.4), (2.5) and (2.6) give estimates on the all the coefficients* $\left|a_{-n+\vartheta}^{(-n)}\right|$ *for a function* $f\in S^*(\alpha)$, $0\le\alpha<1$.

Proof . By a direct calculation, $\left(z(1/(f(z))^n)'\right)/\left(-n(1/(f(z))^n)\right)=z\,f'(z)/f(z)$. Thus, $\left(z(1/(f(z)^n)')\right)/\left(-n(1/f(z)^n)\right)=\left(1+(1-2\alpha)w(z)\right)/\left(1-w(z)\right)$, $z\in D$, for a function $w(z)=\sum_{m=1}^{\infty}w_m z^m$ analytic in D and satisfying the conditions of Schwarz Lemma. Equivalently, $z\left(1/(f(z))^n\right)'+\left(n/(f(z))^n\right)=\left[\ z\left(1/(f(z))^n\right)'-\left(n(1-2\alpha)/(f(z))^n\right)\right]w(z)$. The Substitution of the corresponding series expansions of the functions in this identity and a simplification gives

$$\sum_{m=1}^{\infty}m\ a_{-n+m}^{(-n)}\ z^m=\left[-2n(1-\alpha)+\sum_{m=1}^{\infty}\{m-2n(1-\alpha)\}\ a_{-n+m}^{(-n)}\ z^m\right]w(z) \tag{2.7}$$

Equating coefficients on both sides of (2.7), it is observed that, for every $\vartheta=1,2,\cdots$, the coefficient $a_{-n+\vartheta}^{(-n)}$ depends only on $a_{-n+1}^{(-n)},\ a_{-n+2}^{(-n)},\ \cdots,\ a_{-n+\vartheta-1}^{(-n)}$. Hence (2.7) can be rearranged as $\sum_{m=1}^{\vartheta}m\ a_{-n+m}^{(-n)}\ z^m+\sum_{m=\vartheta+1}^{\infty}b_m z^m=\left[-2n(1-\alpha)+\sum_{m=1}^{\vartheta-1}\{m-2n(1-\alpha)\}\,a_{-n+m}^{(-n)}\ z^m\right]\sum_{p=1}^{\infty}w_p\ z^p,\quad \vartheta=1,2,3,\cdots$ the second sum in the left hand side being convergent in D. The inequality $|w(z)|<1$and Parseval's theorem give

$$\sum_{m=1}^{\vartheta}m^2\left|a_{-n+m}^{(-n)}\right|^2\le 4n^2(1-\alpha)^2+\sum_{m=1}^{\vartheta-1}\{m-2n(1-\alpha)\}^2\left|a_{-n+m}^{(-n)}\right|^2\ .$$

Equivalently,

$$\vartheta^2 \left|a_{-n+\vartheta}^{(-n)}\right|^2 \le 4n(1-\alpha)\left[n(1-\alpha)+\sum_{m=1}^{\vartheta-1}\{n(1-\alpha)-m\}\left|a_{-n+m}^{(-n)}\right|^2\right] \quad . \tag{2.8}$$

The sign of each term inside the summation symbol on the right hand side of (2.8) depends on the sign of the expression $n(1-\alpha)-m,\ m=1,\cdots\cdots,\vartheta-1$. To determine the sign of this expression, we need to partition the interval $0\le\alpha<1$ into n semi-closed intervals $I_k(n),\ k=0,1,\cdots,n-1$. For any fixed k, if $\alpha\in I_k(n)$, $k=0,1,\cdots,n-1$, then $n-k-1<n(1-\alpha)\le n-k$ so that $n(1-\alpha)-m>0$ *if* $m=1,...,n-k-1$ and $n(1-\alpha)-m\le 0$ *if* $m=n-k,...$.Considering only nonnegative contributions in the right hand summation in (2.8), it follows by using the above inequalities that, for $\vartheta=1,\ldots\ldots,n-k$,

$$\vartheta^2 \left|a_{-n+\vartheta}^{(-n)}\right|^2 \le \quad 4n\ (1-\alpha)\left[n\ (1-\alpha)+\sum_{m=1}^{\vartheta-1}\{n\ (1-\alpha)-m\}\left|a_{-n+m}^{(-n)}\right|^2\right] \tag{2.9}$$

while, if $\vartheta=n-k+1,\ldots..$, then

$$\vartheta^2 \left|a_{-n+\vartheta}^{(-n)}\right|^2 \le 4n\ (1-\alpha)\left[n\ (1-\alpha)+\sum_{m=1}^{n-k-1}\{n\ (1-\alpha)-m\}\left|a_{-n+m}^{(-n)}\right|^2\right]+\sum_{m=n-k}^{\vartheta-1}\{n\ (1-\alpha)-m\}\left|a_{-n+m}^{(-n)}\right|^2$$

$$\le 4n\ (1-\alpha)\left[n\ (1-\alpha)+\sum_{m=1}^{n-k-1}\{n\ (1-\alpha)-m\}\left|a_{-n+m}^{(-n)}\right|^2\right] \tag{2.10}$$

We now use induction on ϑ. For $\vartheta=1$, it follows from (2.9) that $\left|a_{-n+1}^{(-n)}\right|\le 2n\ (1-\alpha)$, giving the estimate (2.4) in this case. Now let, for $\vartheta=1,2,\cdots,n-k-1$, the estimate

$$\left|a_{-n+\vartheta}^{(-n)}\right|\le\prod_{j=0}^{\vartheta-1}\frac{2n\ (1-\alpha)-j}{j+1} \tag{2.11}$$

hold. Then, using (2.9), (2.11) and Lemma 1, it follows that, for $\vartheta=1,\cdots\cdots,n-k$,

$$\vartheta^2 \left|a_{-n+\vartheta}^{(-n)}\right|^2 \le 4n\ (1-\alpha)\left[n\ (1-\alpha)+\sum_{m=1}^{\vartheta-1}\{n\ (1-\alpha)-m\}\left(\prod_{j=o}^{m-1}\frac{2n\ (1-\alpha)-j}{j+1}\right)^2\right]$$

$$=\frac{1}{\left((\vartheta-1)!\right)^2}\prod_{j=0}^{\vartheta-1}\left(2n\ (1-\alpha)-j\right)^2 \tag{2.12}$$

Thus, for $\vartheta=1,\cdots\cdots,n-k$,

$$\left|a_{-n+\vartheta}^{(-n)}\right| \le \prod_{j=0}^{\vartheta-1} \frac{2n\,(1-\alpha)-j}{j+1} = \frac{\Gamma\left(2n\,(1-\alpha)+1\right)}{\Gamma(\vartheta+1)\,\Gamma\left(2n(1-\alpha)+1-\vartheta\right)} \tag{2.13}$$

This establishes the inequality (2.4).

Next, if $\vartheta = n-k+1, n-k+2, \cdots\cdots,$ using (2.10), the induction hypothesis (2.11) and Lemma 1, we get

$$\vartheta^2 \left|a_{-n+\vartheta}^{(-n)}\right|^2 \le 4n\,(1-\alpha)\left[n\,(1-\alpha) + \sum_{m=1}^{n-k-1} \{n\,(1-\alpha)-m\} \left(\prod_{j=0}^{m-1} \frac{2n\,(1-\alpha)-j}{j+1} \right)^2 \right]$$

$$= \frac{1}{\left((n-k-1)!\right)^2} \prod_{j=0}^{n-(k+1)} \left(2n(1-\alpha)-j\right)^2 = \left(\frac{\Gamma\left(2n\,(1-\alpha)+1\right)}{\Gamma(n-k)\,\Gamma\left(2n\,(1-\alpha)+1-(n-k)\right)} \right)^2$$

The above inequality yields the estimate (2.5).

For $k = n-1$, the estimates (2.4) and (2.5) respectively reduce to $\left|a_{-n+1}^{(-n)}\right| \le 2n\,(1-\alpha)$ and $\left|a_{-n+\vartheta}^{(-n)}\right| \le \frac{2n\,(1-\alpha)}{\vartheta}$ for $\vartheta = 2, 3, \cdots\cdots$. Combining the above inequalities, (2.6) follows.

Equality holds in (2.4) for every $\vartheta = 1, \cdots\cdots, n-k$ for $(-n)^{th}$ power of the function $K_\alpha(z)$ defined in (1.10). On the other hand for each $\vartheta = n-k+1, \ldots,$ $(-n)^{th}$ power of the function $K_{\alpha,\vartheta}(z)$, defined in (1.10), provides the sharpness for the estimate (2.6). This completes the proof of Lemma 2.

We also need in the sequel the following result of Jabotinsky [8]:

Lemma 3. *If the function* f *, given by* (2.1), *is in* Ω *then* $f^{-1} \in \Omega$. *Further, if*

$$f^{-1}(w) = \sum_{n=1}^{\infty} A_n w^n$$

then,

$$A_n^{(p)} = \frac{p}{n} a_{-p}^{(-n)}, \quad n = 1, 2, \cdots; \quad p = \pm 1, \pm 2, \cdots \tag{2.14}$$

and $A_0^{(p)}$ *is defined by*

$$\sum_{p=-\infty}^{\infty} A_0^{(p)} z^{-p-1} = \frac{f'(z)}{f(z)} \quad . \tag{2.15}$$

3. MAIN RESULTS

The following theorem gives the coefficient estimates for the inverse of a function in the class $S^*(\alpha)$:

Theorem 1. *Let* $f \in S^*(\alpha)$, $0 \le \alpha < 1$ *and, for* $|w| < \frac{1}{4}$,

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n \tag{3.1}$$

Then,

(a) *for* $\alpha \in I_0(n) \cup I_1(n)$,

$$|A_n| \le \frac{\Gamma\left(2n\ (1-\alpha)+1\right)}{\Gamma(n+1)\ \Gamma\left(2n\ (1-\alpha)+2-n\right)} \tag{3.2}$$

(b) for $\alpha \in I_k(n),\ k = 2, \cdots\cdots, n-2$,

$$|A_n| \le \frac{\Gamma\left(2n\ (1-\alpha)+1\right)}{n(n-1)\ \Gamma(n-k)\ \Gamma\left(2n\ (1-\alpha)+1+k-n\right)} \tag{3.3}$$

(c) for $\alpha \in I_{n-1}$,

$$|A_n| \le \frac{2(1-\alpha)}{n-1} \tag{3.4}$$

where, $I_k(n) \equiv [\frac{k}{n}, \frac{k+1}{n})$, $k = 0,1,...,n-1$. *The estimates (3.2) and (3.4) are sharp.*

Proof. It is known (see e.g. [12]) that $A_n = (1/2\pi in) \int\limits_{|z|=r} (1/\left(f(z)\right)^n)\ dz = (1/n)\ a_{-1}^{(-n)}$**,** $0 < r < 1$**.** Therefore, it is sufficient to find suitable estimates for $\left|a_{-1}^{(-n)}\right|$. To this end, taking $\vartheta = n-1$ in Lemma 2, using the appropriate inequality (2.4), (2.5) or (2.6) for different values of $k = 0,1,...,n-1$ and observing that only for *k=0* and *k=1* the inequality (2.4) is applicable, the following estimate is obtained for $\alpha \in [0, \frac{2}{n})$,

$$\left|a_{-1}^{(-n)}\right| \le \frac{\Gamma\left(2n(1-\alpha)+1\right)}{\Gamma(n)\ \Gamma\left(2n(1-\alpha)+2-n\right)} \tag{3.5}$$

which gives (a). Similarly, for $\alpha \in I_k(n),\ k = 2, \ldots, n-2$, the inequality (2.5) yields

$$\left|a_{-1}^{(-n)}\right| \le \frac{\Gamma\left(2n\ (1-\alpha)+1\right)}{(n-1)\ \Gamma(n-k)\ \Gamma\left(2n\ (1-\alpha)+1+k-n\right)} . \tag{3.6}$$

This gives (b). Finally, for $\alpha \in I_{n-1}(n)$, the inequality (2.6) gives

$$\left|a_{-1}^{(-n)}\right| \leq \frac{2n\,(1-\alpha)}{n-1}. \tag{3.7}$$

Consequently, (c) follows.

It is easily verified that equality holds in (3.5) and (3.7) for the $(-n)^{th}$ power of the function $K_n(z)$ and $(-n)^{th}$ power of the function $K_{\alpha,n-1}(z)$ respectively. Thus, the estimates (3.2) and (3.4) are sharp. This completes the proof of Theorem 1.

Remark. *The sharp coefficient bounds of Krzysz, Libera and Zlotkiewicz [12] for* $|A_2|$ *and* $|A_3|$ *follow as a particular case of Theorem 1.*

Remark: *The sharp coefficient bounds of Krzyz, Libera and Zlotkiewicz [12] for* $|A_2|$ *and* $|A_3|$ *follow as a particular case of Theorem 1.*

The sharp coefficient estimates for functions in $\Sigma^*(\alpha)$, $0 \leq \alpha < 1$, are described by the following:

Theorem 2. *Let the function* $g \in \Sigma^*(\alpha)$, $\quad 0 \leq \alpha < 1$, be *given by the series*

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots\cdots, \qquad z \in V.$$

Then,

$$|b_m| \leq \frac{2\,(1-\alpha)}{m+1}, \qquad m = 0,1\ldots \tag{3.8}$$

The estimate (3.8) is sharp.

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Proof. The mapping $f(z) \to g(z) = 1/f(1/z)$ establishes a one-to-one correspondence between $S^*(\alpha)$ and $\Sigma^*(\alpha)$. Since, $\left(zg'(z)/g(z)\right) = \left(z\left(1/f(1/z)\right)'\right)/\left(1/f(1/z)\right) = \left((1/z)f'(1/z)\right)/f(1/z)$, this mapping too is one-to-one from $S^*(\alpha)$ onto $\Sigma^*(\alpha)$ .We note that the coefficient expansion of $1/f(z)$ around origin is same as the coefficient expansion of $g(z)$ about infinity. Therefore,

$$\max_{g\,\in\,\Sigma^*(\alpha)} |b_m| = \max_{f\,\in\,S^*(\alpha)} \left|a_m^{(-1)}\right|, \quad m = 0,1... \tag{3. 9}$$

Thus, by Lemma 2,

$$\left|a_{-1+\vartheta}^{(-1)}\right| \leq \frac{2\,(1-\alpha)}{\vartheta}, \;\; 0 \leq \alpha < 1; \;\; \vartheta = 1,\ldots \tag{3.10}$$

The inequality (3.10) can be equivalently expressed as

$$\left|a_m^{(-1)}\right| \leq \frac{2(1-\alpha)}{m+1}, \quad 0 \leq \alpha < 1; \;\; m = 0,1,\ldots \tag{3.11}$$

and the inequality (3.11) together with (3.9) gives (3. 8). It is easily seen that the function $g(z) = z\left(1-(1/z^{m+1})\right)^{2(1-\alpha)/(m+1)}$ belongs to the class $\Sigma^*(\alpha)$ and its m^{th} coefficient equals $2(1-\alpha)/(m+1)$. Therefore, the estimate (3.8) is sharp. This completes the proof of Theorem 2.

The coefficient estimates for the inverse of a function in the class $\Sigma^*(\alpha)$ are found in the following:

Theorem 3. *Let the function* $g \in \Sigma^*(\alpha), 0 \le \alpha < 1$, have the series expansion $g^{-1}(w) = w + \sum_{n=0}^{\infty} B_n w^{-n}$ *in some neighbourhood of infinity.* Then,

(a) $$|B_0| \le 2\,(1-\alpha), \quad 0 \le \alpha < 1 \qquad (3.12)$$

(b) *For* $\alpha \in I_k(n), \quad k = 0, \cdots\cdots n-2$,

$$|B_n| \le \frac{\Gamma\left(2n\,(1-\alpha)+1\right)}{n(n+1)\,\Gamma(n-k)\,\Gamma\left(2n\,(1-\alpha)+1-(n-k)\right)} \qquad (3.13)$$

(c) *For* $\alpha \in I_{n-1}(n)$,

$$|B_n| \le \frac{2\,(1-\alpha)}{n+1} \qquad (3.14)$$

where, $I_k(n) \equiv [\frac{k}{n}, \frac{k+1}{n})$, $k = 0,1,\ldots,n-1$. T*he estimates* (3.12) and (3.14) *are sharp*.

Proof. For any $g \in \Sigma^*(\alpha), \quad 0 \le \alpha < 1$, there exists $f \in S^*(\alpha)$ such that $g(z) = 1/f(1/z)$. It can be easily verified that $g^{-1}(w) = 1/f^{-1}(1/w)$**.** Since the coefficients in the expansion of $1/f^{-1}(w)$ around origin and those of $1/f^{-1}(1/w)$ about infinity are the same, we have

$$B_n = A_n^{(-1)}, \quad n = 0,1,2,\cdots. \qquad (3.15)$$

By (2.15), with $\operatorname{Re} Q(z) > \alpha$, $Q(0) = 1$ *and* $z \in D$,

$$\sum_{p=-\infty}^{\infty} A_0^{(p)} z^{-p-1} = \frac{f'(z)}{f(z)} = \frac{1}{z}Q(z) \equiv \frac{1}{z}[1 + \sum_{n=1}^{\infty} q_n z^n]$$

Now, using the well known sharp estimate

$$|q_n| \le 2\,(1-\alpha), \quad n = 1,2,\cdots \qquad (3.16)$$

we get,

$$\max_{g^{-1} \in \Sigma^{*-1}(\alpha)} |B_0| = \max_{f^{-1} \in S^{*-1}(\alpha)} \left|A_0^{-1}\right| = Max|q_1| \le 2\,(1-\alpha),$$

This establishes (3.12). Since the bound in (3.16) is sharp, it follows that the estimate (3.12) is also sharp.

For *n=1, 2,…,* (3.15) together with (2.14) gives

$$\max_{g^{-1}\in\Sigma^{*-1}(\alpha)}\left|B_n\right| = \max_{f^{-1}\in S^{*-1}(\alpha)}\left|A_n^{(-1)}\right| = (1/n)\max_{f^{-1}\in S^{*}(\alpha)}\left|a_1^{(-n)}\right| \tag{3.17}$$

It is easily verified that $a_1^{(-n)} = a_{-n+(n+1)}^{(-n)}$. Since the sharp estimate (2.4) in Lemma 2 is not applicable for any value of $k = 0,\cdots\cdots,n-1$, in order to get (3.13), the estimate (2.5) with $\vartheta = n+1$ has to be used for $k = 0,1,...,n-2$. This gives

$$\left|a_1^{(-n)}\right| \le \frac{\Gamma\left(2n\ (1-\alpha)+1\right)}{(n+1)\ \Gamma(n-k)\ \Gamma\left(2n\ (1-\alpha)+1-(n-k)\right)} \tag{3.18}$$

The estimate (3.13) now easily follows from (3.17) and (3.18). Similarly, the estimate (2.6) with $\vartheta = n+1$ gives,

$$\left|a_1^{(-n)}\right| \le \frac{2n\ (1-\alpha)}{n+1} \tag{3.19}$$

By combining (3.17) and (3.19), the estimate (3.14) follows. Since the inequality (2.6) is sharp, it follows that the estimate (3.14) is sharp. This completes the proof of Theorem 3.

Remark. *The construction of a suitable example to exhibit the sharpness of inequality (2.5) seems to be quite involved and the sharpness of estimates (3.3) and (3.13) depend on the sharpness of the inequality (2.5).*

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