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Journal of Functional Analysis 217 (2004) 347–365
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

Quasi-multipliers of operator spaces ☆

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Received 17 December 2003; revised 2 June 2004; accepted 10 June 2004

Communicated by G. Pisier

Abstract

We use the injective envelope to study quasi-multipliers of operator spaces. We prove that all representable operator algebra products that an operator space can be endowed with are induced by quasi-multipliers. We obtain generalizations of the Banach–Stone theorem.

MSC: primary 46L05; secondary 46A22; 46H25; 46M10; 47A20

Keywords: Injective; Multipliers; Operator spaces; Banach–Stone

1. Introduction

We begin with some general algebraic comments, inspired by Brown et al. [7], that make clear the role that quasi-multipliers can play. Let $\mathcal{A}$ be an algebra and let $X \subseteq \mathcal{A}$ be a subspace. We shall call an element $z \in \mathcal{A}$ a *quasi-multiplier of X* (*relative to* $\mathcal{A}$) provided that $XzX \subseteq X$, i.e., $x_1 z x_2 \in X$ for every $x_1, x_2 \in X$. Clearly, the set of quasi-multipliers of X is a linear subspace of $\mathcal{A}$. Moreover, each quasi-multiplier z induces a bilinear map $m_z : X \times X \to X$ defined by $m_z(x_1, x_2) := x_1 z x_2$. The associativity of the product on $\mathcal{A}$, implies that each m_z is an associative bilinear map and hence can

☆ Research was supported in part by a grant from the National Science Foundation.
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0022-1236/$ - see front matter
doi:10.1016/j.jfa.2004.06.002

be regarded as a product on X. This product gives X the structure of an algebra, which we denote by (X, m_z). There are two homomorphisms, π_ℓ and π_{r} from this algebra into $\mathcal{A}$, defined by $\pi_\ell(x) := xz$ and $\pi_{\mathrm{r}}(x) := zx$.

The range of π_ℓ is contained in the subalgebra of *left multipliers of X* (*relative to* $\mathcal{A}$), while the range of π_{r} is contained in the subalgebra of *right multipliers of X* (*relative to* $\mathcal{A}$). Recall that an element $a \in \mathcal{A}$ is called a *left* (*respectively*, *right*) *multiplier* of X provided that $aX \subseteq X$ (respectively, $Xa \subseteq X$). Finally, note that since the quasi-multipliers are a linear subspace of $\mathcal{A}$, the set of "products" on X that one obtains in this manner is a linear subspace of the vector space of bilinear maps from $X \times X$ into X.

In general, linear (or, even convex) combinations of associative bilinear maps need not be associative. For an example of this phenomenon, consider $X = \mathbb{C}^2$ and the associative bilinear maps, $m_1((a,b),(c,d)) := (ac, bd)$ and $m_2((a,b),(c,d)) := (ac, bc)$. Their convex combination, $m := (m_1 + m_2)/2$, is not associative.

One shortcoming of the above representation of quasi-multipliers is that it is *extrinsic*. The quasi-multipliers that one obtains and their induced bilinear maps, could easily depend on the algebra $\mathcal{A}$ and on the particular embedding of X into $\mathcal{A}$ and not on *intrinsic* properties of X. Thus, the totality of bilinear maps that one could obtain in this manner would be a union of linear spaces, taken over all embeddings of X into an algebra, which would no longer need to be a linear space.

In this paper, we develop a theory of quasi-multipliers of operator spaces and then use Hamana's injective envelope [9,10] to give an intrinsic characterization of quasi-multipliers and of their associated bilinear maps. Among the results that we obtain are that an operator space endowed with a completely contractive product can be represented completely isometrically as an algebra of operators on some Hilbert space if and only if the product is a bilinear map that belongs to this space of "bilinear quasi-multipliers". As a corollary, we find that the set of "representable" completely contractive products is a convex set. In fact, it is affinely isomorphic with the unit ball of the space of quasi-multipliers.

We then turn our attention to generalizations of the Banach–Stone theorem. Our basic result is that a linear complete isometry between any two operator algebras induces a quasi-multiplier and that by using the quasi-multiplier, one recovers earlier generalizations of the Banach–Stone theorem.

2. Quasi-multipliers

In this section, we introduce various spaces of quasi-multipliers of an operator space and develop some of their key properties. Let X be an operator space, $\mathcal{H}$ be a Hilbert space, $B(\mathcal{H})$ denote the algebra of bounded, linear operators on $\mathcal{H}$, and let $\phi : X \to B(\mathcal{H})$ be a complete isometry. We set

$$QM_\phi(X) := \{z \in B(\mathcal{H}) : \phi(X) z \phi(X) \subseteq \phi(X)\}$$

and we call $QM_\phi(X)$ the space of *quasi-multipliers of* X *relative to* ϕ. Note that $QM_\phi(X)$ is a norm-closed subspace of $B(\mathcal{H})$.

Each $z \in QM_\phi(X)$ induces a bilinear map, $m_z : X \times X \to X$ defined by $m_z(x_1, x_2) := \phi^{-1}(\phi(x_1) z \phi(x_2))$. The bilinear map m_z is completely bounded in the sense of Christensen–Sinclair, i.e., its linear extension is completely bounded as a map from $X \otimes_h X$ to X with $\|m_z\|_{cb} \leqslant \|z\|$. We say that m_z is *the bilinear map induced by* z.

Definition 2.1. Let $QMB(X)$ denote the set of bilinear maps from $X \times X$ to X, that are of the form m_z for some quasi-multiplier z and some completely isometric map ϕ from X into the bounded linear operators on some Hilbert space. For $m \in QMB(X)$ we set $\|m\|_{qm} := \inf\{\|z\| : m = m_z\}$, where the infimum is taken over all possible completely isometric maps ϕ and z as above.

Note that by the above remarks, every $m \in QMB(X)$ is completely bounded as a bilinear map and $\|m\|_{cb} \leqslant \|m\|_{qm}$. We shall show in Example 2.11 that this inequality can be sharp. For a fixed map ϕ, the set of bilinear maps, $\{m_z : z \in QM_\phi(X)\}$ is a linear subspace of the set $QMB(X)$. However, since $QMB(X)$ is the union of these subspaces, it is not clear that it is a vector subspace of the vector space of bilinear maps from $X \times X$ to X. We shall prove that it is a vector space later.

The above definitions are extrinsic, in the sense that they could depend on the particular embedding. We now seek intrinsic characterizations of these maps by using the injective envelope as in [15,8,5]. A detailed explanation of the injective envelope and the particular construction that we are using can be found in [13].

We begin by recalling a construction used in [5], but we prefer the notation from [13].

Recall that if $X \subseteq B(\mathcal{K}, \mathcal{H})$ is a (concrete) operator space, then we may form the (concrete) operator system, in $B(\mathcal{H} \oplus \mathcal{K})$,

$$\mathcal{S}_X := \left\{ \begin{pmatrix} \lambda I_{\mathcal{H}} & x \\ y^* & \mu I_{\mathcal{K}} \end{pmatrix} : \lambda, \mu \in \mathbb{C}, x, y \in X \right\}.$$

Given a complete isometry $\varphi : X \to B(\mathcal{K}_1, \mathcal{H}_1)$, the operator system

$$\mathcal{S}_{\varphi(X)} := \left\{ \begin{pmatrix} \lambda I_{\mathcal{H}_1} & \varphi(x) \\ \varphi(y)^* & \mu I_{\mathcal{K}_1} \end{pmatrix} : \lambda, \mu \in \mathbb{C}, x, y \in X \right\}$$

is completely order isomorphic to $\mathcal{S}_X$ via the map, $\Phi : \mathcal{S}_X \to \mathcal{S}_{\varphi(X)}$ defined by

$$\Phi\left(\begin{pmatrix} \lambda I_{\mathcal{H}} & x \\ y^* & \mu I_{\mathcal{K}} \end{pmatrix} \right) := \begin{pmatrix} \lambda I_{\mathcal{H}_1} & \varphi(x) \\ \varphi(y)^* & \mu I_{\mathcal{K}_1} \end{pmatrix}.$$

Thus, the operator system $\mathcal{S}_X$ only depends on the operator space structure of X and not on any particular representation of X.

Since $\mathbb{C}\oplus\mathbb{C}\cong\left\{\begin{pmatrix}\lambda I_{\mathcal{H}} & 0\\ 0 & \mu I_{\mathcal{K}}\end{pmatrix}:\ \lambda,\mu\in\mathbb{C}\right\}$ is a C^*-subalgebra of $\mathcal{S}_X$, $\mathbb{C}\oplus\mathbb{C}$ will still be a C^*-subalgebra of the C^*-algebra, $I(\mathcal{S}_X)$ with $\begin{pmatrix}I_{\mathcal{H}} & 0\\ 0 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & 0\\ 0 & I_{\mathcal{K}}\end{pmatrix}$ corresponding to orthogonal projections e_1 and e_2, respectively, in the C^*-algebra $I(\mathcal{S}_X)$. We have that e_1+e_2 is equal to the identity and $e_1e_2=0$.

A few words on such a situation are in order. Let $\mathcal{A}$ be any unital C^*-algebra with orthogonal projections e_1 and e_2 satisfying $e_1+e_2=1$, $e_1e_2=0$ and let $\pi:\mathcal{A}\to B(\mathcal{H})$ be a one-to-one unital $*$-homomorphism. Setting $\mathcal{H}_1=\pi(e_1)\mathcal{H}$, $\mathcal{H}_2=\pi(e_2)\mathcal{H}$ we have that $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$ and relative to this decomposition every $T\in B(\mathcal{H})$ has the form $T=(T_{ij})$ where $T_{ij}\in B(\mathcal{H}_j,\mathcal{H}_i)$. In particular, identifying $\mathcal{A}$ with $\pi(\mathcal{A})$ we have that

$$\mathcal{A}=\left\{\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}:\ a_{ij}\in\mathcal{A}_{ij}\right\},$$

where $\mathcal{A}_{ij}=e_i\mathcal{A}e_j$, with $\mathcal{A}_{ii}\subseteq B(\mathcal{H}_i)$ unital C^*-subalgebras and $\mathcal{A}_{21}=\mathcal{A}_{12}^*$. The operator space $\mathcal{A}_{12}\subseteq B(\mathcal{H}_2,\mathcal{H}_1)$ will be referred to as a *corner of* $\mathcal{A}$. Note that $\mathcal{A}_{11}\mathcal{A}_{12}\mathcal{A}_{22}\subseteq\mathcal{A}_{12}$ so that $\mathcal{A}_{12}$ is an $\mathcal{A}_{11}-\mathcal{A}_{22}$-bimodule.

Returning to $I(\mathcal{S}_X)$, relative to e_1 and e_2, we wish to identify each of these 4 subspaces. Note that $X\subseteq e_1I(\mathcal{S}_X)e_2=I(\mathcal{S}_X)_{12}$. As shown in Chapter 16 of [13], we may identify $I(\mathcal{S}_X)_{12}=I(X)$.

We define, $I_{11}(X):=I(\mathcal{S}_X)_{11}$ and $I_{22}(X):=I(\mathcal{S}_X)_{22}$. Thus we have the following picture of the C^*-algebra $I(\mathcal{S}_X)$, namely,

$$I(\mathcal{S}_X)=\left\{\begin{pmatrix}a & z\\ w^* & b\end{pmatrix}:\ a\in I_{11}(X), b\in I_{22}(X), z,w\in I(X)\right\}$$

where $I_{11}(X)$ and $I_{22}(X)$ are injective C^*-algebras and $I(X)$ is an operator $I_{11}(X)-I_{22}(X)$-bimodule. Moreover, the fact that $I(\mathcal{S}_X)$ is a C^*-algebra means that for $z,w\in I(X)$,

$$\begin{pmatrix}0 & z\\ w^* & 0\end{pmatrix}\begin{pmatrix}0 & z\\ w^* & 0\end{pmatrix}=\begin{pmatrix}zw^* & 0\\ 0 & w^*z\end{pmatrix}$$

and consequently there are natural products, $zw^*\in I_{11}(X)$, $w^*z\in I_{22}(X)$.

It is interesting to note that setting $\langle z,w\rangle=zw^*$ defines an $I_{11}(X)$-valued inner product that makes $I(X)$ a Hilbert C^*-module over $I_{11}(X)$, but we shall not use this additional structure.

Definition 2.2. We set $QM(X):=\{z\in I(X)^*: XzX\subseteq X\}$, where the products are all taken in the C^*-algebra $I(S_X)$.

In [7] a theory was developed of quasi-multipliers of Hilbert C^*-bimodules. If X is a Hilbert C^*-bimodule, then they also use the notation, $QM(X)$, for their quasi-multipliers. We warn the reader, and apologize, that although we are using the same notation, our quasi-multipliers and theirs are not the same objects. In the first place their quasi-multipliers are defined using the second dual of the linking algebra and their quasi-multipliers are a subset of the second dual of X.

Briefly, if X is a Hilbert $A-B$-bimodule, then an element t in the second dual of X is a quasi-multiplier in the sense of [7] provided that $AtB \subseteq X$ where the products are defined in the second dual of the linking algebra. Note that if A and B are both unital C^*-algebras, then this forces $t \in X$.

To make our quasi-multiplier theory fit with theirs a bit better, we should have perhaps taken adjoints of elements so that our $QM(X)$ is a subset of $I(X)$. However, this alternate definition would have made several natural maps, that we define later, conjugate linear and we believe that it would have led to the "opposite", i.e., transposed operator space structure.

Theorem 2.3. *Let X be an operator space, $\mathcal{H}$ be a Hilbert space and let $\phi : X \to B(\mathcal{H})$, be a complete isometry. Then there exists a unique, completely contractive map $\gamma : QM_\phi(X) \to QM(X)$ such that $\phi(x_1)z\phi(x_2) = \phi(x_1\gamma(z)x_2)$ for all $x_1, x_2 \in X$ and every $z \in QM_\phi(X)$.*

Proof. The proof is similar to that of Theorem 1.7 in [5]. Let $\mathcal{S}_{\phi(X)} \subseteq B(\mathcal{H} \oplus \mathcal{H})$ be the concrete operator system defined above and let $C^*(S_{\phi(X)})$ be the C^*-subalgebra of $B(\mathcal{H} \oplus \mathcal{H})$ that it generates. The C^*-subalgebra of $I(S_X)$ generated by S_X is known to be the C^*-envelope of S_X, $C^*_e(S_X)$. Consequently, by [10] Corollary 4.2 the identity map on S_X extends to be a surjective $*$-homomorphism $\pi : C^*(S_{\phi(X)}) \to C^*_e(S_X)$.

Let $\Gamma : B(\mathcal{H} \oplus \mathcal{H}) \to I(S_X)$ be a completely positive map that extends this $*$-homomorphism. Since Γ extends π, it will be a π-bimodule map, that is, for $A, B \in C^*(S_{\phi(X)})$ we will have that $\Gamma(ATB) = \pi(A)\Gamma(T)\pi(B)$. This forces Γ to be a matrix of maps, i.e., for

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$$

we will have that

$$\Gamma(T) = \begin{pmatrix} \gamma_{11}(T_{11}) & \gamma_{12}(T_{12}) \\ \gamma_{21}(T_{21}) & \gamma_{22}(T_{22}) \end{pmatrix}.$$

In particular, for $x \in X$ and $z \in QM_\phi(X)$ we will have that

$$\Gamma\left(\begin{pmatrix} 0 & \phi(x) \\ z & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & x \\ \gamma_{21}(z) & 0 \end{pmatrix}.$$

Hence, by the π-bimodule property, for $x_1, x_2 \in X$ and $z \in QM_\phi(X)$ we have

$$\Gamma\left(\begin{pmatrix} 0 & \phi(x_1)z\phi(x_2) \\ 0 & 0 \end{pmatrix}\right) = \Gamma\left(\begin{pmatrix} 0 & \phi(x_1) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}\begin{pmatrix} 0 & \phi(x_2) \\ 0 & 0 \end{pmatrix}\right)$$
$$= \begin{pmatrix} 0 & x_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \gamma_{21}(z) & 0 \end{pmatrix}\begin{pmatrix} 0 & x_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_1\gamma_{21}(z)x_2 \\ 0 & 0 \end{pmatrix}.$$

Thus, it follows that $\gamma_{21}(z) \in QM(X)$ and $\phi(x_1\gamma_{21}(z)x_2) = \phi(x_1)z\phi(x_2)$. Because Γ is a unital, completely positive map, γ_{21} is completely contractive.

Finally, the uniqueness of the map comes from the following observation. Suppose $q_1, q_2 \in QM(X)$ have the property that $x_1q_1x_2 = x_1q_2x_2$ for every $x_1, x_2 \in X$. This implies that $(x_1q_1 - x_1q_2)X = 0$ and so by Corollary 1.3 of [5], we have that $x_1(q_1 - q_2) = 0$ for every $x_1 \in X$. Applying the corollary again, we see that $X(q_1 - q_2)(q_1 - q_2)^* = 0$ and so $q_1 = q_2$. □

Remark 2.4. Let A be a C^*-algebra and let $\pi_u : A \to B(\mathcal{H}_u)$ be its universal representation. The classical definition of the quasi-multiplier space of A as given in [14] is the set $QM_{\pi_u}(A)$. In [11,12] the first author proves that the map $\gamma : QM_{\pi_u}(A) \to QM(A)$ is an onto complete isometry, and preserves quasi-multiplication. Thus, at least in the case of a C^*-algebra our definition and the classical definition agree.

Corollary 2.5. *Let X be an operator space. The map $z \mapsto m_z$ from $QM(X)$ to $QMB(X)$ equipped with $\|\cdot\|_{qm}$ is an onto isometry, where $m_z : X \times X \to X$ is defined by $m_z(x_1, x_2) := x_1zx_2$. Consequently, $QMB(X)$ is a linear subspace of the vector space of bilinear maps from $X \times X$ to X.*

Let X be an operator space. Given an associative, bilinear map $m : X \times X \to X$, we let (X, m) denote the resulting algebra. We let $CCP(X)$ denote the set of associative bilinear maps on X that are completely contractive in the sense of Christensen–Sinclair, i.e., $CCP(X)$ denotes the set of *completely contractive products on X*. We let $OAP(X) \subseteq CCP(X)$ denote those maps such that the algebra (X, m) has a completely isometric homomorphism into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. That is, $OAP(X)$ denotes the set of *operator algebra products on X*.

We let $CBP(X)$ denote those associative, bilinear maps from $X \times X$ to X (i.e., products on X), that are completely bounded in the sense of Christensen–Sinclair, that is the set of *completely bounded algebra products*. By a result of [1], $m \in CBP(X)$ if and only if there exists a Hilbert space $\mathcal{H}$ and a completely bounded homomorphism, $\pi : (X, m) \to B(\mathcal{H})$ with completely bounded inverse, $\pi^{-1} : \pi(X) \to (X, m)$. In fact, by [1] one may choose π such that $\|\pi\|_{cb}\|\pi^{-1}\|_{cb} \leqslant 1 + \varepsilon$ for any $\varepsilon \geqslant 0$. Finally, we let $SOAP(X) \subseteq CBP(X)$ denote those associative, bilinear maps for which one can choose, π satisfying, $\|\pi\|_{cb}\|\pi^{-1}\|_{cb} = 1$, i.e., such that π is a scalar multiple of a complete isometry. These are the *scaled operator algebra products*.

The following theorem illustrates the importance of quasi-multipliers.

Theorem 2.6. *Let X be an operator space. Then $QMB(X) = SOAP(X)$ and $OAP(X) = \{m \in QMB(X) : ||m||_{qm} \leqslant 1\}$.*

Proof. We prove the second equality first. Let $m \in OAP(X)$ and let $\pi : (X, m) \to B(\mathcal{H})$ be a completely isometric homomorphism. Then $I_{\mathcal{H}}$ is a quasi-multiplier of $\pi(X)$ that induces the bilinear map m. Thus, $m \in QMB(X)$ and $||m||_{qm} \leqslant 1$. Conversely, let $m \in QMB(X)$ with $||m||_{qm} \leqslant 1$. Then there exists $z \in QM(X)$ with $||z|| \leqslant 1$ such that $m = m_z$.

As in [6] Remark 2, define $\pi : (X, m) \to I(S_X)$ by

$$\pi(x) := \begin{pmatrix} xz & x\sqrt{1 - zz^*} \\ 0 & 0 \end{pmatrix}.$$

It is easily seen that, $\pi(x_1)\pi(x_2) = \pi(x_1 z x_2) = \pi(m(x_1, x_2))$ and that $||\pi(x)||^2 = ||\pi(x)\pi(x)^*|| = ||xx^*|| = ||x||^2$, where all products take place in $I(S_X)$. Thus, π is an isometric homomorphism. The proof that π is completely isometric is similar and thus $m \in OAP(X)$.

If $m \in SOAP(X)$ and $\pi : (X, m) \to B(\mathcal{H})$ is a completely bounded homomorphism such that $\phi = \pi / ||\pi||_{cb}$, is a complete isometry, then $I_{\mathcal{H}}$ is a quasi-multiplier of $\phi(X)$. We have that

$$\phi(x_1) I_{\mathcal{H}} \phi(x_2) = ||\pi||_{cb}^{-2} \pi(x_1)\pi(x_2) = ||\pi||_{cb}^{-2} \pi(m(x_1, x_2)) = ||\pi||_{cb}^{-1} \phi(m(x_1, x_2)).$$

Hence, $m \in QMB(X)$ and $||m||_{qm} \leqslant ||\pi||_{cb}$.

Conversely, if $m(x_1, x_2) = x_1 z x_2$ for $z \in QM(X)$ with $||z|| = r$, then consider

$$\pi(x) = \begin{pmatrix} xz & x\sqrt{r^2 - zz^*} \\ 0 & 0 \end{pmatrix}$$

and argue as above to prove that π is a homomorphism and is r times a complete isometry. □

Corollary 2.7. *Let X be an operator space, then $OAP(X)$ is a convex set, and $SOAP(X)$ is a vector space.*

The following example shows that, in general, $CCP(X)$ and $CBP(X)$ are not convex sets.

Example 2.8. The following example shows that, in general, $OAP(X) \neq CCP(X)$, $SOAP(X) \neq CBP(X)$, and that the sets $CCP(X)$ and $CBP(X)$ need not be convex.

Let $X = C_2$, i.e., the subspace of the two-by-two matrices, M_2 consisting of those matrices that are 0 in the second column and whose first column is arbitrary. Let $m_1(\binom{a}{b}, \binom{c}{d}) := \binom{ac}{bd}$ and $m_2(\binom{a}{b}, \binom{c}{d}) := \binom{ac}{bc}$ be as in the introduction.

Since m_2 is the product on C_2 induced by the inclusion of C_2 into M_2, we have that $m_2 \in OAP(C_2) \subseteq CCP(C_2)$.

We claim that $m_1 \in CCP(C_2)$. To see this claim, note that if we identify $M_n(C_2) = \left\{ \binom{A}{B} : A, B \in M_n \right\}$, then $m_1^{(n)}\left(\binom{A}{B}, \binom{C}{D}\right) = \binom{AC}{BD}$. If $\left\|\binom{A}{B}\right\| \leqslant 1$ and $\left\|\binom{C}{D}\right\| \leqslant 1$, then $\left\|\binom{AC}{BD}\right\|^2 = \|C^*A^*AC + D^*B^*BD\|$. However, $0 \leqslant C^*A^*AC + D^*B^*BD \leqslant C^*C + D^*D \leqslant I$ and so $\|m_1\|_{cb} \leqslant 1$. Thus, $m_1 \in CCP(X)$ as claimed.

Since $(m_1 + m_2)/2$ is not even associative, we see that $CCP(C_2)$ is not convex and hence cannot be equal to $OAP(C_2)$. Since both m_1 and m_2 are also in $CBP(C_2)$, we have that this set is also not convex and hence it is not equal to $SOAP(C_2)$. Also, since $SOAP(C_2)$ is convex and $m_2 \in SOAP(C_2)$, it must be the case that m_1 is not in $SOAP(C_2)$.

This last fact can also be seen by using the injective envelope and Theorem 2.6. Since C_2 is already an injective operator space, we have that $I(C_2)^* = C_2^* = R_2$, where R_2 denotes the corresponding row space. Since $C_2R_2C_2 \subseteq C_2$, we have that $QM(C_2) = R_2$. Now it is easily checked that there is no $z = (e, f) \in R_2$ such that $m_1(x_1, x_2) = x_1 z x_2$ and so m_1 is not in $QMB(C_2) = SOAP(C_2)$.

Finally, note that $\pi : (C_2, m_1) \to M_2$ defined by $\pi(\binom{a}{b}) := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is a completely contractive homomorphism with completely bounded inverse. This gives a direct way, independent of Blecher's theorem [1], to see that (C_2, m_1) is completely boundedly representable.

It is also interesting to note that for the natural inclusion $\phi : C_2 \to M_2$, we have that $QM_\phi(C_2)$ is all of M_2, while $QM(C_2) = R_2$ is two-dimensional. Thus, we see that the map γ of Theorem 2.3 need not be one-to-one.

Remark 2.9. Although, $OAP(X)$ is a convex set and $CCP(X)$ is not in general, we know little else about the structure of $CCP(X)$ or about the subset of $CCP(X)$ consisting of those products that can be induced by a completely contractive, but not completely isometric representation. The product m_1 from Example 2.8 is one such product.

For a finite-dimensional vector space the set of associative bilinear maps is an algebraic set. Thus, when X is a finite-dimensional operator space, $CCP(X)$ is the intersection of this algebraic set with the set of completely contractive bilinear maps. Generally, the set of completely contractive bilinear maps need not even be a semialgebraic set. But it is still possible that the intersection, $CCP(X)$, is a semialgebraic set.

The following result illustrates how some of the results of [4] on operator algebras with one-sided identities can be deduced from the theory of quasi-multipliers.

Proposition 2.10. *Let X be an operator space and let $m \in OAP(X)$. Then (X, m) has a right contractive identity e if and only if $m = m_z$ where $z \in QM(X)$ satisfies $z^* = e$, and zz^* is the identity of $I_{22}(X)$. In this case the map $x \to xz$ defines a completely isometric homomorphism of (X, m) into $M_\ell(X)$.*

Proof. Since $m \in OAP(X)$ we have that $m = m_z$ for some $z \in QM(X)$ with $||z|| \leqslant 1$.

Assume that (X, m) has a contractive right identity e. Then we have that for every $x \in X, x = m(x, e) = xze$. Hence, $x(1_{22} - ze) = 0$ for every $x \in X$. By [5] Corollary 1.3, this implies that $1_{22} - ze = 0$. But since both z and e are contractions, $e = z^*$ must hold.

Conversely, if $z^* = e$ and $zz^* = 1_{22}$, then clearly, $m(x, e) = xze = x$ and so e is a contractive, right identity.

Finally, since $||x|| = ||xzz^*|| \leqslant ||xz||$, we see that the completely contractive homomorphism of (X, m) into $M_\ell(X)$ given by $x \to xz$ is a complete isometry. □

Note that by the above result, the relationship between the product m and the product in $I(S_X)$ is that $m(x_1, x_2) = x_1 e^* x_2$.

There is an analogous result for left identities.

It is possible for a concrete algebra of operators to have a two-sided identity e of norm greater than one. For an example see [13], p. 279. In this case the multiplication will still be given by a contractive quasi-multiplier z and one has $ze = 1_{22}, ez = 1_{11}$ but one no longer has that $e = z^*$.

We close this section with a number of examples of spaces of quasi-multipliers that illustrate the limits of some of the above results.

Example 2.11. This example shows that it is possible to have $||z|| > ||m_z||_{cb}$ for a quasi-multiplier. Thus, for $m \in QMB(X), ||m||_{qm} \neq ||m||_{cb}$, in general. It is inspired by Example 4.4 of [2].

Let $\mathcal{A} \subseteq M_2$ denote the subalgebra that is the span of $\{E_{12}, I_2\}$, where I_2 denotes the identity matrix. Let $Q := I_2 + J$ where J is the matrix whose entries are all 1s, and set $P := Q^{1/2}$. A little calculation shows that $P = I_2 - \frac{1+\sqrt{3}}{2} J$ and that $P^{-1} = I_2 - \frac{3+\sqrt{3}}{6} J$.

We let $\mathcal{X} = \mathcal{A}P$. Since the C^*-subalgebra of M_2 generated by $\mathcal{X}$ is all of M_2, which is irreducible, one finds that $I_{11}(\mathcal{X}) = I(\mathcal{X}) = M_2$, with the usual product. From this we see quite easily that $M_\ell(\mathcal{X}) = \mathcal{A}$ and $QM(\mathcal{X}) = P^{-1}\mathcal{A}$.

Let $Z := P^{-1}E_{12}$, so that $||Z|| = \sqrt{2/3}$. Writing $X, Y \in \mathcal{X}$ as $X = (aI_2 + N)P$, $Y = (bI_2 + M)P$ where N and M are scalar multiples of E_{12}, we have that, $m_Z(X, Y) := XZY = abE_{12}P$.

Since $||E_{12}P|| = \sqrt{2}$, we have that $||m_Z|| = \kappa\sqrt{2}$, where $\kappa = \sup\{|ab| : ||X|| \leqslant 1, ||Y|| \leqslant 1\}$ and the bilinear map $m_Z : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is equipped with the usual norm $||m_Z|| := sup\{||m_Z(X, Y)|| : ||X|| \leqslant 1, ||Y|| \leqslant 1\}$. The first inequality implies that $XX^* \leqslant I_2$. From the $(2, 2)$-entries of these matrices, we obtain that $2|a|^2 \leqslant 1$, which leads to the conclusion that $|a| \leqslant \frac{1}{\sqrt{2}}$. Similarly, $|b| \leqslant \frac{1}{\sqrt{2}}$.

Thus, we are led to conclude that $||m_Z|| \leqslant \frac{1}{\sqrt{2}} < \sqrt{2/3}$. In fact, $||m_Z|| = \frac{1}{\sqrt{2}}$ is attained if you take $a = b = \frac{1}{\sqrt{2}}$, $N = M = -\frac{1}{2\sqrt{2}} E_{12}$, for example.

The same calculation, using matrix coefficients shows that $||m_Z||_{cb} \leqslant \frac{1}{\sqrt{2}}$ too, and so the result follows. Indeed, if we write $X \in M_{n,m}(\mathcal{X})$, $Y \in M_{m,n}(\mathcal{X})$ as $X = A \otimes E_{11}P + C \otimes E_{12}P + A \otimes E_{22}P$ and $Y = B \otimes E_{11}P + D \otimes E_{12}P + B \otimes E_{22}P$, where $A, C \in M_{n,m}$ and $B, D \in M_{m,n}$, then it is easily seen that $m_Z^{(n)}(X, Y) = AB \otimes E_{12}P$. Now $||X|| \leqslant 1$ implies that $XX^* \leqslant I_n \otimes I_2$. From $(I_n \otimes (0 \oplus 1))XX^*(I_n \otimes (0 \oplus 1)) \leqslant I_n \otimes (0 \oplus 1)$, we obtain that $2AA^* \leqslant I_n$, which leads to $||A|| \leqslant \frac{1}{\sqrt{2}}$. Similarly, $||B|| \leqslant \frac{1}{\sqrt{2}}$. Hence, $||AB \otimes E_{12}P|| \leqslant \sqrt{2}||A||||B|| \leqslant \frac{1}{\sqrt{2}}$, so that $||m_Z^{(n)}|| \leqslant \frac{1}{\sqrt{2}}, \forall n \in \mathbb{N}$, thus $||m_Z||_{cb} \leqslant \frac{1}{\sqrt{2}}$. In fact, the equality is attained since $\frac{1}{\sqrt{2}} = ||m_Z|| \leqslant ||m_Z||_{cb} \leqslant \frac{1}{\sqrt{2}}$.

Example 2.12. Let $\{E_{ij}\}$ denote the canonical matrix units and let $X = span\{E_{11}, E_{12}, E_{21}, E_{32}\} \subseteq M_{3,2}$. We compute $QM(X)$ for this space and illustrate some of its properties.

It is not difficult to show that $I(S_X) = \begin{pmatrix} M_3 & M_{3,2} \\ M_{2,3} & M_2 \end{pmatrix} = M_5$, with the obvious identifications. To see this one first shows that since X is a $\mathcal{D}_3 - \mathcal{D}_2$-bimodule, where $\mathcal{D}_n$ denotes the $n \times n$ diagonal matrices, then any completely contractive map Φ from $M_{3,2}$ into itself that fixes X must be a bimodule map. From this it follows that Φ must be given as a Shur product map, but then the fact that Φ is completely contractive forces Φ to be the identity map.

Now a direct calculation shows that $QM(X) = span\{E_{12}, E_{23}\} \subseteq M_{2,3}$, that $M_\ell(X) = span\{E_{11}, E_{12}, E_{13}, E_{22}, E_{33}\} \subseteq M_3$ and that $M_{\mathrm{r}}(X) = span\{E_{11}, E_{22}\} \subseteq M_2$.

Note that the span of the products $XQM(X)$ is not dense in $M_\ell(X)$ but that the span of the products $QM(X)X$ is all of $M_{\mathrm{r}}(X)$.

For the contractive quasi-multiplier $z = E_{12} + E_{23}$, we see that the induced homomorphism $\pi_\ell(x) = xz$ into $M_\ell(X)$ is a complete isometry, but that $\pi_{\mathrm{r}}(x) = zx$ is not even one-to-one. For the quasi-multipliers E_{12} and E_{23} neither π_ℓ nor π_{r} is one-to-one.

Example 2.13. Let $X = span\{E_{11} + E_{32}, E_{21} + E_{33}\} \subseteq M_3$. This space can be identified as a concrete representation of the maximum of C_2 and R_2, that is, as the least operator space structure on $\mathbb{C}^2$ that is greater than both C_2 and R_2. We will show that $QM(X) = (0)$ and consequently, there are no non-trivial operator algebra products on this operator space, i.e., $OAP(X) = (0)$. However, since the natural maps from X to the concrete operator algebras $C_2 \subseteq M_2$ and $R_2 \subseteq M_2$ are both complete contractions, we see that there are at least four different products (up to scaling) in $CCP(X)$ that have completely contractive representations whose inverses are completely bounded.

To see these claims, one first shows that if one regards $S_X \subseteq M_6$, then $C^*(S_X) = I(S_X)$. From this it follows that $I_{11}(X) = M_2 \oplus \mathbb{C}$, $I_{22}(X) = \mathbb{C} \oplus M_2$, $I(X) = span\{E_{11}, E_{32}, E_{21}, E_{33}\}$ and that $M_\ell(X)$ and $M_{\mathrm{r}}(X)$ are both just the scalar multiples of the identity. Once these things are seen, it is straightforward to check that $QM(X) = (0)$.

To prove that $C^*(S_X) = I(S_X)$, first note that there is a $*$-homomorphism of $C^*(S_X)$ onto the C^*-subalgebra of $I(S_X)$ generated by the copy of S_X, i.e., onto the boundary C^*-algebra. But the original C^*-algebra has only 2 ideals that could be the kernel of this map. Now argue that if you mod out by either ideal then you will not have a 2-isometry on S_X. Hence, this homomorphism must be 1-1. But $C^*(S_X)$ is injective so we are done.

3. A non-commutative Banach–Stone theorem

In this section, we use quasi-multipliers to describe linear complete isometries from one operator algebra onto another. Our theorem needs no assumptions concerning the existence of units or approximate units.

To understand the statement of the theorem, it is perhaps instructive to keep the following example in mind. Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a unital C^*-subalgebra and let $\mathcal{B} := \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in \mathcal{A} \right\} \subseteq B(\mathcal{H} \oplus \mathcal{H})$. These are both algebras of operators, although the product of any two elements of $\mathcal{B}$ is 0. The identification of $\mathcal{A}$ with $\mathcal{B}$ is a complete isometry, onto, but clearly the only possible homomorphism between these two algebras is the 0 map. However, in this example one sees that the left multipliers of $\mathcal{B}$ can be identified with the C^*-algebra $\mathcal{A}$.

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of operators and let $\psi : \mathcal{A} \to \mathcal{B}$ be a linear complete isometry that is onto. Then we have the following*:

(1) *there exists a unique $z \in QM(\mathcal{A})$, with $\|z\| \leqslant 1$ such that $\psi(a_1)\psi(a_2) = \psi(a_1 z a_2)$ for every $a_1, a_2 \in \mathcal{A}$;*
(2) *there exists a unique $w \in QM(\mathcal{B})$, with $\|w\| \leqslant 1$ such that $\psi(a_1 a_2) = \psi(a_1) w \psi(a_2)$ for every $a_1, a_2 \in \mathcal{A}$;*
(3) *setting $\pi_\ell(a) = \psi(a)w$ and $\pi_r(a) = w\psi(a)$ defines completely contractive homomorphisms of $\mathcal{A}$ into $M_\ell(\mathcal{B})$ and $M_r(\mathcal{B})$, respectively;*
(4) *if $\mathcal{A}$ has a contractive right (respectively, left) approximate identity, then π_ℓ (respectively, π_r) is a completely isometric homomorphism;*
(5) *if $\mathcal{A}$ has a contractive right (respectively, left) identity, e, then $w = \psi(e)^*$, ww^* is the identity of $I_{22}(\mathcal{B})$, and $\psi(a) = \pi_\ell(a)w^*$ (respectively, w^*w is the identity of $I_{11}(\mathcal{B})$ and $\psi(a) = w^*\pi_r(a)$).*

Proof. Set $\gamma := \psi^{-1}$ and define $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by $m(a_1, a_2) := \gamma(\psi(a_1)\psi(a_2))$. It is easily checked that

$$m(a_1, m(a_2, a_3)) = \gamma(\psi(a_1)\psi(a_2)\psi(a_3)) = m(m(a_1, a_2), a_3),$$

so that m is an associative bilinear map on $\mathcal{A}$ and defines a new product on $\mathcal{A}$. Moreover, because the product on $\mathcal{B}$ is completely contractive this new product on $\mathcal{A}$ is completely contractive and the map $\psi : (\mathcal{A}, m) \to \mathcal{B}$ is a completely isometric algebra isomorphism.

Thus, since $\mathcal{B}$ is an algebra of operators, we see that $m \in OAP(\mathcal{A})$ and hence by Theorem 2.6 there exists a unique $z \in QM(\mathcal{A})$ such that $m(a_1, a_2) = a_1 z a_2$. Hence, $\psi(a_1 z a_2) = \psi(m(a_1, a_2)) = \psi(a_1)\psi(a_2)$ and (1) follows.

Applying (1) to γ yields $w \in QM(\mathcal{B})$ such that

$$\gamma(\psi(a_1) w \psi(a_2)) = \gamma(\psi(a_1))\gamma(\psi(a_2)) = a_1 a_2.$$

Thus, $\psi(a_1) w \psi(a_2) = \psi(a_1 a_2)$ and so (2) follows:

To see (3), note that $\pi_\ell(a_1)\pi_\ell(a_2) = \psi(a_1) w \psi(a_2) w = \psi(a_1 a_2) w = \pi_\ell(a_1 a_2)$ with a similar calculation for $\pi_{\rm r}$. Since $\pi_\ell(a) b = \psi(a) w \psi(\gamma(b)) = \psi(a\gamma(b)) \in \mathcal{B}$ for every $b \in \mathcal{B}$, we see that $\pi_\ell(a) \in M_\ell(\mathcal{B})$ for every $a \in \mathcal{A}$, with a similar calculation for $\pi_{\rm r}$.

Now let $\{e_\alpha\}$ be a contractive approximate right identity for $\mathcal{A}$. We then have that $\|\psi(a)\| = \lim \|\psi(a e_\alpha)\| = \lim \|\psi(a) w \psi(e_\alpha)\| = \lim \|\pi_\ell(a)\psi(e_\alpha)\| \leqslant \|\pi_\ell(a)\|$. Thus, π_ℓ is an isometry. The proof that π_ℓ is a complete isometry and the case for $\pi_{\rm r}$ are similar.

Finally, if $\mathcal{A}$ has a right identity e, then $\psi(a) = \psi(ae) = \psi(a) w \psi(e)$. This shows that $bw\psi(e) = b$ for every $b \in \mathcal{B}$ and hence $w\psi(e)$ is a right identity for $M_{\rm r}(\mathcal{B})$. By [5] Corollary 1.3, we have that $w\psi(e)$ is the identity of $I_{22}(\mathcal{B})$. Since $\|w\| \leqslant 1$ and $\|\psi(e)\| \leqslant 1$, we have that $\psi(e) = w^*$ and (5) follows. □

Remark 3.2. (1) When $\mathcal{B}$ has a contractive right identity, then one may identify $\mathcal{B} \subseteq M_\ell(\mathcal{B})$, but it is not clear if the image of π_ℓ maps onto this copy of $\mathcal{B}$. However, in this case it is clear how to define a homomorphism into $\mathcal{B}$. Let $\psi(a_0) = e_{\mathcal{B}}$ and define $\rho : \mathcal{A} \to \mathcal{B}$ by setting $\rho(a) = e_{\mathcal{B}} w \psi(a) = \psi(a_0 a)$. Letting the product in $\mathcal{B}$ be denoted by $\odot$ to avoid confusion, we have that $b_1 \odot b_2 = b_1 e_{\mathcal{B}}^* b_2$, where the latter product is taken in $I(S_{\mathcal{B}})$. Since $e_{\mathcal{B}}^* e_{\mathcal{B}} = 1_{22}$, by Proposition 2.10, we have that $\rho(a_1) \odot \rho(a_2) = e_{\mathcal{B}} w \psi(a_1) e_{\mathcal{B}}^* e_{\mathcal{B}} w \psi(a_2) = e_{\mathcal{B}} w \psi(a_1 a_2) = \rho(a_1 a_2)$, and so ρ is a homomorphism. Note that ρ is onto $\mathcal{B}$ if and only if $a_0 \mathcal{A} = \mathcal{A}$.

(2) If one considers $\mathcal{A} = \mathcal{B} = C_2 \subseteq M_2$ and lets ψ be the identity map, then we are in the situation of the last remark. Thus, π_ℓ is a complete isometry, but since $M_{\rm r}(C_2) = \mathbb{C}$, we have that $\pi_{\rm r}$ is not a complete isometry. In fact, it is the compression to the $(1, 1)$-entry.

4. Further results on QMB(X)

In [2,3] various characterizations are given of the linear maps of an operator space X into itself that are given as left multiplication by an element from the left multiplier algebra of $X, M_\ell(X)$. In this section, we present characterizations of the bilinear maps of an operator space into itself that are in $QMB(X)$. Among the results that we obtain is a characterization of when a linear map from X into $M_\ell(X)$ is given as right multiplication by a quasi-multiplier. We also identify a subspace of $QM(X)$, related to ternary structures on X, that we denote by $TER(X)^*$ for which we have $||z|| = ||m_z||_{qm} = ||m_z||_{cb}$.

In [3] it was shown that one could determine whether or not a linear map from X into X was given by a contractive left multiplier by determining whether or not an associated linear map was completely contractive. The following is an analogous result for determining when a map is given as multiplication by a quasi-multiplier.

Recall that given any operator space $X, R_2(X)$ denotes the operator subspace of $M_2(X)$ consisting of 1×2 matrices.

Theorem 4.1. *Let X be an operator space and let $\gamma : X \to I_{11}(X)$ be a linear map. There exists $y \in I(X)^*$ with $||y|| \leqslant 1$ such that $\gamma(x) = xy$ for every $x \in X$ if and only if the map $\beta : R_2(X) \to I(S_X)$ defined by $\beta((x_1, x_2)) := \begin{pmatrix} \gamma(x_1) & x_2 \\ 0 & 0 \end{pmatrix}$ is completely contractive.*

Proof. Note that if such an element y exists, then β is given, at least formally, as right multiplication by the matrix $\begin{pmatrix} y & 0 \\ 0 & 1_{22} \end{pmatrix}$ and since this matrix has norm 1, β should be a complete contraction. To complete this argument, we create a C^*-algebra where these products occur.

To this end consider the following C^*-algebra,

$$\mathcal{B} := \begin{pmatrix} I_{11}(X) & I(X) & I(X) \\ I(X)^* & I_{22}(X) & I_{22}(X) \\ I(X)^* & I_{22}(X) & I_{22}(X) \end{pmatrix},$$

where the products are all induced from the products in $I(S_X)$. Identifying $R_2(X)$ with the subspace $\begin{pmatrix} 0 & X & X \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \subseteq \mathcal{B}$, we see that β is given as right multiplication in the C^*-algebra $\mathcal{B}$ by the matrix $\begin{pmatrix} 0 & 0 & 0 \\ y & 0 & 0 \\ 0 & 1_{22} & 0 \end{pmatrix}$.

For the converse, we must assume that β is a complete contraction and produce the element y. To this end we create a second C^*-algebra, $\mathcal{C}$ and an operator system $\mathcal{S}$.

Let

$$\mathcal{C} := \begin{pmatrix} I_{11}(X) & I_{11}(X) & I(X) \\ I_{11}(X) & I_{11}(X) & I(X) \\ I(X)^* & I(X)^* & I_{22}(X) \end{pmatrix},$$

where the products are all induced from the products in $I(S_X)$ and let

$$\mathcal{S} = \begin{pmatrix} \mathbb{C}1_{11} & X & X \\ X^* & \mathbb{C}1_{22} & 0 \\ X^* & 0 & \mathbb{C}1_{22} \end{pmatrix} \subseteq \mathcal{B}.$$

We define $\Phi : \mathcal{S} \to \mathcal{C}$ by

$$\Phi\left(\begin{pmatrix} \lambda 1_{11} & x_1 & x_2 \\ x_3^* & \mu 1_{22} & 0 \\ x_4^* & 0 & v 1_{22} \end{pmatrix}\right) := \begin{pmatrix} \lambda 1_{11} & \gamma(x_1) & x_2 \\ \gamma(x_3)^* & \mu 1_{11} & 0 \\ x_4^* & 0 & v 1_{22} \end{pmatrix}.$$

Since β is completely contractive, Φ is completely positive and since $\mathcal{C}$ is clearly an injective C^*-algebra, we may extend Φ to a completely positive map on all of $\mathcal{B}$, which we still denote by Φ. Because Φ fixes the diagonal, it will be a bimodule map over the diagonal. Also note that the compression of $\mathcal{S}$ to the span of the first and third entries is a copy of S_X and that Φ fixes this operator system.

By the rigidity properties of the injective envelope, we see that necessarily

$$\Phi\left(\begin{pmatrix} a & 0 & b \\ 0 & \mu 1_{22} & 0 \\ c^* & 0 & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 & b \\ 0 & \mu 1_{11} & 0 \\ c^* & 0 & d \end{pmatrix},$$

for every $a \in I_{11}(X), b, c \in I(X), d \in I_{22}(X)$ and $\mu \in \mathbb{C}$.

These matrices that are fixed by Φ form a common C^*-subalgebra of $\mathcal{B}$ and $\mathcal{C}$ and hence Φ will necessarily be a bimodule map over this C^*-subalgebra.

Thus, we will have for any $x \in X$ that

$$\begin{aligned}\begin{pmatrix} 0 & \gamma(x) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} &= \Phi\left(\begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \Phi\left(\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1_{22} & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Phi\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1_{22} & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix},\end{aligned}$$

where y is the image of 1_{22} under the restriction of Φ to the subspaces corresponding to the (3,2)-entries.

Equating entries of the matrices occurring in the first and last expressions, we see that $\gamma(x) = xy$, and the proof is complete. □

We state the corresponding result for maps into the right multiplier algebra without proof.

Theorem 4.2. *Let X be an operator space and let $\psi : X \to I_{22}(X)$ be a linear map. There exists $y \in I(X)^*$ such that $\psi(x) = yx$ for every $x \in X$ if and only if the map $\alpha : C_2(X) \to I(S_X)$ defined by $\alpha\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) := \begin{pmatrix} 0 & x_1 \\ 0 & \psi(x_2) \end{pmatrix}$ is completely contractive.*

The above theorem yields a method for determining whether or not a bilinear map $m : X \times X \to X$ is a contractive quasi-multiplier, i.e., whether or not $m \in OAP(X)$.

Corollary 4.3. *Let X be an operator space and let $m : X \times X \to X$ be a bilinear map. Then the following are equivalent*:

(1) $m \in OAP(X)$;
(2) *there exists a linear map* $\gamma : X \to M_\ell(X)$ *such that* $m(x, y) = \gamma(x)y$ *and the map* $\beta : R_2(X) \to I(S_X)$ *defined by* $\beta((x_1, x_2)) = \begin{pmatrix} \gamma(x_1) & x_2 \\ 0 & 0 \end{pmatrix}$ *is completely contractive*;
(3) *there exists a linear map* $\psi : X \to M_r(X)$ *such that* $m(x_1, x_2) = x_1\psi(x_2)$ *and the map* $\alpha : C_2(X) \to I(S_X)$ *defined by* $\alpha\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 0 & x_1 \\ 0 & \psi(x_2) \end{pmatrix}$ *is completely contractive.*

Recall that by the results of [3] (see also [13]) to determine whether or not $x \to m(x_1, x)$ is a contractive left multiplier it is necessary and sufficient that the map $\tau_{x_1} : C_2(X) \to C_2(X)$ defined by $\tau_{x_1}\left(\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}\right) := \begin{pmatrix} m(x_1, x_2) \\ x_3 \end{pmatrix}$ be a complete contraction. There is a similar result involving $R_2(X)$ for determining when a map is a contractive right multiplier.

Thus, by combining Corollary 4.3 with the characterizations of multipliers, one obtains a *bootstrap method* for determining whether or not a bilinear map $m : X \times X \to X$ is a contractive quasi-multiplier, i.e., whether or not $m \in OAP(X)$.

Remark 4.4. Since we have obtained necessary and sufficient conditions for a bilinear map to be in $OAP(X)$, we have in some sense given a full generalization of the Blecher–Ruan–Sinclair characterization of unital operator algebras [6] to arbitrary operator algebras. However, to prove the original BRS theorem by applying Corollary 4.3, one still needs to use the theory of multipliers and the proof that one obtains in this fashion is not really different from the proof given in [13].

In the thesis of the first author [11] (also see [12]), a new direct characterization of the bilinear maps in $OAP(X)$ is given that is independent of the theory of multipliers and is sufficiently simple that the BRS theorem can be deduced directly from this characterization.

The space $QMB(X)$ currently is endowed with two generally different norms, $\|\cdot\|_{cb}$ and $\|\cdot\|_{qm}$. The first norm comes from its natural inclusion into the space of completely bounded bilinear maps from X into X and the second from its identification with the space $QM(X)$. The next results allow us to prove that for a subspace of $QM(X)$, that is related to ternary structures on X, these two norms are the same.

Theorem 4.5. *Let X be an operator space and let* $Y = \begin{pmatrix} X \\ M_r(X) \end{pmatrix}$ *where we give Y the operator space structure that comes from its identification as a subspace of* $I(S_X)$. *Then* $I(S_Y)$ *can be identified with the injective* C^**-algebra* $\mathcal{B} := \begin{pmatrix} I_{11}(X) & I(X) & I(X) \\ I(X)^* & I_{22}(X) & I_{22}(X) \\ I(X)^* & I_{22}(X) & I_{22}(X) \end{pmatrix}$ *in such a way that* $I_{11}(Y) = I(S_X)$, $I(Y) = \begin{pmatrix} I(X) \\ I_{22}(X) \end{pmatrix}$ *and* $I_{22}(Y) = I_{22}(X)$.

Proof. We identify $S_Y = \begin{pmatrix} \mathbb{C} & Y \\ Y^* & \mathbb{C} \end{pmatrix}$ with an operator system in $\mathcal{B}$ via the map that sends $\begin{pmatrix} \alpha & y_1 \\ y_2^* & \beta \end{pmatrix}$ where $y_i = \begin{pmatrix} x_i \\ t_i \end{pmatrix}$ to $\begin{pmatrix} \alpha 1_{11} & 0 & x_1 \\ 0 & \alpha 1_{22} & t_1 \\ x_2^* & t_2^* & \beta 1_{22} \end{pmatrix}$. The proof of the theorem will be complete if we can show that any completely positive map $\Phi : \mathcal{B} \to \mathcal{B}$ that is the identity on S_Y must be the identity on $\mathcal{B}$.

To this end let $\gamma : I(S_X) \to \mathcal{B}$ be defined by $\gamma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ c & 0 & d \end{pmatrix}$ and let $\delta : \mathcal{B} \to I(S_X)$ be defined by $\delta((a_{i,j})) := \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}$. Since $\delta \circ \Phi \circ \gamma$ is the identity on S_X by rigidity it must be the identity on $I(S_X)$.

To simplify notation, we define elements of $\mathcal{B}$ by $E_{11} := \begin{pmatrix} 1_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1_{22} \\ 0 & 0 & 0 \end{pmatrix}$ and give similar definitions to E_{22}, E_{32}, E_{33}. Note that because $I(X)$ need not contain an identity we do not attempt to define E_{12}, E_{13}, E_{21} and E_{31}.

We first prove that Φ fixes the five "matrix units" defined above. Note that since Φ fixes S_Y we already have that $\Phi(E_{11} + E_{22}) = E_{11} + E_{22}, \Phi(E_{33}) = E_{33}$ and $\Phi(E_{23}) = E_{23}$.

Since $\delta \circ \Phi \circ \gamma\left(\begin{pmatrix} 1_{11} & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1_{11} & 0 \\ 0 & 0 \end{pmatrix}$, it follows that $\Phi(E_{11}) =: P = (P_{ij})$ with $P_{11} = 1_{11}$. Since Φ is contractive and positive, it follows that $P_{ij} = 0$ when (i,j) is $(1,2)$, $(1,3)$, $(2,1)$, or $(3,1)$ and that $P_{33} \geqslant 0$. But since $\Phi(E_{33}) = E_{33}$ and $\|\Phi(E_{11} + E_{33})\| \leqslant 1$, we have that $P_{33} = 0$. Now the positivity of P implies that $P_{23} = P_{32} = 0$.

Since $E_{23}^* E_{23} = E_{33} = \Phi(E_{23}^* E_{23})$ and $\Phi(E_{23}) = E_{23}$, we have that E_{23} is in the right multiplicative domain of Φ, that is $\Phi(BE_{23}) = \Phi(B)E_{23}, \forall B \in \mathcal{B}$. If we let $\Phi(E_{22}) =: Q = (Q_{ij})$, then $E_{23} = \Phi(E_{22}E_{23}) = QE_{23}$ and it follows that $Q_{22} = 1_{22}$. But since $\Phi(E_{11} + E_{22}) = E_{11} + E_{22}$, it follows that $P_{22} = 0$ and $Q = E_{22}$.

Thus, we have shown that these five matrix units are fixed by Φ as was claimed. Since the span of these matrix units are a C^*-subalgebra of $\mathcal{B}$, we have that Φ must be a bimodule map over this C^*-subalgebra.

Since this subalgebra contains the diagonal matrices we see that there exist maps ϕ_{ij} such that $\Phi((B_{ij})) = (\phi_{ij}(B_{ij}))$. To prove that Φ is the identity map, it will be enough to show that each ϕ_{ij} is the identity map on its respective domain.

Using the fact that $\delta \circ \Phi \circ \gamma$ is the identity on $I(S_X)$ yields that ϕ_{11} is the identity map on $I_{11}(X)$ and similarly ϕ_{13}, ϕ_{31} and ϕ_{33} are the identities on their respective domains.

To see that ϕ_{12} is the identity on its domain, note that for any $u \in I(X)$ we have that

$$\Phi\left(\begin{pmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \Phi\left(\begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} E_{32}\right) = \Phi\left(\begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) E_{32}$$
$$= \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} E_{32} = \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that what was used in this argument was the bimodularity property of the matrix units and the fact that certain maps were the identity maps. A similar argument shows that ϕ_{21}, ϕ_{23} and ϕ_{32} are all the identity maps on their respective domains. Finally, that ϕ_{22} is the identity follows from the rigidity of the upper left corner $I(\mathcal{S}_X)$ of $\mathcal{B}$. This completes the proof of the theorem. $\square$

Given any operator space X the sets $M_\ell(X) \cap M_\ell(X)^*$ and $M_{\rm r}(X) \cap M_{\rm r}(X)^*$ are C^*-subalgebras of $I_{11}(Y)$ and $I_{22}(X)$, respectively. In [5] these sets were denoted by $IM_\ell^*(X)$ and $IM_{\rm r}^*(X)$, respectively. They were shown to be equal to the sets $\mathcal{A}_\ell(X)$ and $\mathcal{A}_{\rm r}(X)$ of adjointable left and right multipliers, respectively, introduced in [2]. We shall use the latter notation for these sets.

Definition 4.6. Given an operator space X, we set $TER(X) := X \cap QM(X)^*$ and we call this the *ternary subspace of X*.

Note that in the multiplication inherited from $I(S_X)$ we have that $TER(X) \cdot TER(X)^* TER(X) \subseteq TER(X)$. The following results give further properties of this subspace.

Corollary 4.7. *Let X be an operator space, let* $Y := \begin{pmatrix} X \\ M_{\rm r}(X) \end{pmatrix}$ *where we give Y the operator space structure that comes from its identification as a subspace of* $I(S_X)$ *and let* $I(S_Y)$ *be identified with the* C^**-algebra* $\mathcal{B}$ *as above. Then* $M_\ell(Y) = \begin{pmatrix} M_\ell(X) & X \\ QM(X) & M_{\rm r}(X) \end{pmatrix}$ *and* $\mathcal{A}_\ell(Y) = \begin{pmatrix} \mathcal{A}_\ell(X) & TER(X) \\ TER(X)^* & \mathcal{A}_{\rm r}(X) \end{pmatrix}$.

Proof. One simply checks that $M_\ell(X)$ is exactly the matrix in $I_{11}(Y) = I(S_X)$ that leaves Y invariant under left multiplication. $\square$

Proposition 4.8. *Let X be an operator space, let* $x \in TER(X)$, *set* $z = x^* \in QM(X)$ *and let* $m_z : X \times X \to X$ *be the associated bilinear map, then* $||z|| = ||m_z||_{qm} = ||m_z||_{cb} = ||m_z||$.

Proof. We have that $||z|| = ||m_z||_{qm}$ by definition and clearly, $||z|| \geqslant ||m_z||_{cb} \geqslant ||m_z||$. To show that these are actually equal, without loss of generality, we may assume that $||z|| = ||x|| = 1$. Then $||m_z(x,x)|| = ||xx^*x|| \geqslant ||x^*xx^*x|| = ||x||^4 = 1$. Thus $||m_z|| = 1$, and the result follows. $\square$

5. Quasi-centralizers and quasi-homomorphisms

We introduce a new family of bilinear maps that we call the *quasi-centralizers* of an operator space and a set of maps that we call the *quasi-homomorphisms* and explore the relationships between these maps and the space $QMB(X)$.

Definition 5.1. Let X be an operator space and let $m : X \times X \to X$ be a bilinear map. We call m a *quasi-centralizer*[1] provided that there exists completely bounded maps, $\gamma : X \to M_\ell(X)$ and $\psi : X \to M_r(X)$ such that $m(x,y) = \gamma(x)y = x\psi(y)$ for every $x, y \in X$. We let $QC(X)$ denote the set of quasi-centralizers. We call a linear map $\gamma : X \to M_\ell(X)$ (respectively, $\psi : X \to M_r(X)$) a *left* (*right*) *quasi-homomorphism*[2] provided that $\gamma(x)\gamma(y) = \gamma(\gamma(x)y)$ (respectively, $\psi(x)\psi(y) = \psi(x\psi(y))$) for every $x, y \in X$.

These definitions are motivated by the following observations. If $m = m_z \in QMB(X)$ for some $z \in QM(X)$, then $m(x,y) = \pi_\ell(x)y = x\pi_r(y)$ where $\pi_\ell(x) = xz$ and $\pi_r(y) = zy$. Thus, $QMB(X) \subseteq QC(X)$. Moreover, the maps π_ℓ and π_r are left and right quasi-homomorphisms, respectively.

Note that $QC(X)$ is a linear subspace of the space of bilinear maps from X to X.

We begin with a few elementary observations about the relationships between these concepts.

Proposition 5.2. *Let X be an operator space. A linear map $\gamma : X \to M_\ell(X)$ (respectively, $\psi : X \to M_r(X)$) is a left (respectively, right) quasi-homomorphism if and only if the bilinear map $m(x,y) = \gamma(x)y$ (respectively, $m(x,y) = x\psi(y)$) is associative. In this case, γ (respectively, ψ) is a homomorphism of the algebra (X, m) into $M_\ell(X)$ (respectively, $M_r(X)$).*

Proof. The proof is straightforward. □

We shall refer to m as the *product associated with the quasi-homomorphism.*

Proposition 5.3. *Let X be an operator space, let $\gamma : X \to M_\ell(X)$ (respectively, $\psi : X \to M_r(X)$) be a linear map and let $m(x,y) := \gamma(x)y$ (respectively, $m(x,y) := x\psi(y)$), then $\|m\|_{cb} = \|\gamma\|_{cb}$ (respectively, $\|m\|_{cb} = \|\psi\|_{cb}$).*

Proof. By [5], we have that the norm of a left (respectively, right) multiplier is given by the *cb*-norm of its action as a left (respectively, right) multiplication. Thus, given $\|(x_{ij})\| \leqslant 1$ and $\|(y_{ij})\| \leqslant 1$, we have that

$$\left\| \left(\sum_k m(x_{ik}, y_{kj}) \right) \right\| = \left\| \left(\sum_k \gamma(x_{ik}) y_{kj} \right) \right\| \leqslant \|(\gamma(x_{ij})\| \|(y_{ij})\| \leqslant \|\gamma\|_{cb}$$

and it follows that $\|m\|_{cb} \leqslant \|\gamma\|_{cb}$.

The other inequalities follow similarly. □

[1] This is a generalization of a *quasi-centralizer* defined for an operator algebra with a two-sided contractive approximate identity in [11] Definition 3.2.1. and [12]

[2] This is different from a *quasi-homomorphism* which M. Kaneda defined in [11] Definition 3.1.1 (2) and [12].

By the above results, the product associated with a completely contractive quasi-homomorphism is completely contractive, i.e., is in $CCP(X)$.

Example 5.4. This is an example of a product associated with a completely contractive quasi-homomorphism that is in $CCP(X)$ but not in $OAP(X)$.

Recall the product m_1 on C_2 of Example 2.8 that is not in $OAP(C_2)$. We have that $M_\ell(C_2) = M_2$ and that $\gamma : C_2 \to M_2$ defined by $\gamma\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is a completely contractive quasi-homomorphism with m_1 the associated product. Thus, γ is a completely contractive homomorphism of (C_2, m_1) into an operator algebra, but since m_1 is not an operator algebra product on C_2, there can be no completely isometric homomorphism of (C_2, m_1) into an operator algebra. It is interesting to note that m_1 is also not a quasi-centralizer. In fact, it is not hard to show by a direct calculation that $QC(C_2) = QMB(C_2)$.

Remark 5.5. Is every quasi-centralizer, automatically an associative bilinear map? Is every associative quasi-centralizer in $QMB(X)$? We conjecture that the answer is no to both of these questions, but we do not know of an example.

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