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Differential subordination related to conic sections

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Abstract

In this paper some problems of the theory of differential subordination are investigated in connection with conic domains. In particular, fundamental conditions for functions mapped the unit disk onto domains bounded by parabolas and hyperbolas are deduced.

Keywords: Differential subordination; Domains bounded by conic sections

1. Introduction and definitions

Following Miller and Mocanu (cf. [8, p. 21]), denote by Q the set of functions q analytic and injective on $\bar{\mathcal{U}} \setminus E(q)$, where

$$E(q) = \left\{\zeta \in \partial\mathcal{U}\colon \lim_{z\to\zeta} = \infty\right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{U} \setminus E(q)$.

Miller and Mocanu [8] formulated for functions in Q the fundamental lemma in the theory of differential subordinations which is the key lemma for numerous problems of analytic and univalent functions, see below.

Lemma 1.1. [8, p. 22] *Let $p(z) = a + a_n z^n + \cdots$ be analytic in $\mathcal{U}$ with $p(z) \not\equiv a$ and $n \geqslant 1$, and let $q \in Q$ with $q(0) = a$. If there exist points $z_0 \in \mathcal{U}$ and $\zeta_0 \in \partial\mathcal{U} \setminus E(q)$ such that $p(z_0) = q(\zeta_0)$ and $p(\mathcal{U}_{r_0}) \subset q(\mathcal{U})$, where $r_0 = |z_0|$, then there exists $m \geqslant n \geqslant 1$ such that*

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(1) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$ *and*

(2) $\Re\dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geqslant m\Re\left(\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right)$.

Lemma 1.1, known also as the extension of the Jack's lemma, have been used by Miller and Mocanu and other mathematicians in order to prove directly some results in the theory of univalent functions. For instance, Miller and Mocanu proved that if $p(z) + zp'(z) + z^2 p''(z) \prec (n^2 + 1)Mz$ then $p(z) \prec Mz$. In this case we obviously have $p(0) = 0$, and the obtained result is sharp. However Miller and Mocanu constructed a special tool which makes proofs very short and easy. Such a tool is called *the admissibility condition*.

Definition 1.1. [8, p. 27] Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The class of admissible functions, $\Psi_n[\Omega, q]$, consists of those functions $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\psi(r, s, w; z) \notin \Omega \tag{1.1}$$

whenever

$$r = q(\zeta), \qquad s = m\zeta q'(\zeta), \quad \text{and} \quad \Re\frac{w}{s} + 1 \geqslant m\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$z \in \mathcal{U}$, $\zeta \in \partial\mathcal{U} \setminus E(q)$ and $m \geqslant n$.

Making use of Lemma 1.1 and the above admissibility condition, Miller and Mocanu formulated and proved the following:

Theorem 1.1. [8, p. 28] *Let* $\psi \in \Psi[\Omega, q]$ *with* $q(0) = a$. *If* $p(z) = a + a_n z^n + \cdots$ *satisfies*

$$\psi\big(p(z), zp'(z), z^2 p''(z); z\big) \in \Omega, \tag{1.2}$$

then $p \prec q$.

Applying Theorem 1.1 it suffices to check the admissibility condition in order to prove $p \prec q$. Miller and Mocanu in their monograph [8] considered in details two special cases of Lemma 1.1; when the function q maps the unit disk onto a disk and onto a half-plane. Their results have been used by many authors and found many applications in the geometric theory of univalent functions.

For $k \in [0, \infty)$, set

$$\Omega_k = \big\{u + iv:\ u^2 > k^2(u-1)^2 + k^2 v^2,\ u > 0\big\}. \tag{1.3}$$

Note that Ω_k is the domain bounded by a conic section: line for $k = 0$, a right branch of a hyperbola when $0 < k < 1$, a parabola when $k = 1$, and finally an ellipse when $k > 1$. Moreover, $1 \in \Omega_k$ for all k and each Ω_k is convex and symmetric about the real axis. The author and Wiśniowska [1,4,5] considered the family Ω_k in their study of k-uniformly convex and k-starlike functions and gave the explicit formulas for conformal mappings $q_k : \mathcal{U} \to \Omega_k$ so that $q_k(0) = 1$ and $q_k'(0) > 0$ as follows:

Theorem 1.2 ([1,4]). *The conformal map* $q_k : \mathcal{U} \to \Omega_k$ *with* $q_k(0) = 1$ *and* $q_k'(0) > 0$ *is given by*

$$q_k(z) = \begin{cases} \frac{1+z}{1-z} & \text{for } k = 0, \\ 1 + \frac{2}{1-k^2} \sinh^2[A(k) \tanh^{-1} \sqrt{z}] & \text{for } k \in (0,1), \\ 1 + \frac{2}{\pi^2} \log^2\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right) & \text{for } k = 1, \\ 1 + \frac{2}{k^2-1} \sin^2\left(\frac{\pi}{2\mathcal{K}(t)} \mathcal{F}(\sqrt{z}/\sqrt{t}, t)\right) & \text{for } k > 1, \end{cases} \tag{1.4}$$

where $A(k) = (2/\pi) \arccos k$, $\mathcal{F}(w,t)$ *is the Jacobi* $\mathcal{F}$*-function*:

$$\mathcal{F}(w,t) = \int_0^w \frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}},$$

$\mathcal{K}(t) = \mathcal{F}(1,t)$ *is the complete elliptic integral of the first kind, and* $t \in (0,1)$ *is such that*

$$k = \cosh\left(\frac{\pi \mathcal{K}'(t)}{2\mathcal{K}(t)}\right) = \cosh \mu(t),$$

$\mathcal{K}'(t) = \mathcal{K}(\sqrt{1-t^2})$. (*The quantity* $\mu(t)$ *is known as the modulus of the Grötzsch ring* $\mathcal{U} \setminus [0,t]$ *for* $t \in (0,1)$.)

We will abbreviate $q := q_1$ and $\Omega := \Omega_1$. The domain Ω is enclosed by the parabola $2u = v^2 + 1$ with focus at 1. Such a domain was also treated in the study of uniformly convex functions due to Rønning [7], and Ma and Minda [6], independently.

2. Main results

The motivation of that paper is the supplement the subordination theory connected with domains Ω_k. We will discuss variations of Lemma 1.1 involving the subordination to functions those map the unit disk onto conic regions. The results given here may be used in solving extremal problems for families of univalent functions. Some ideas of differential subordinations related to domains bounded by conic sections has been developed by the author [1], by the author and Lecko [2], and by Kim and Lecko [3].

First we consider the case when $q(\mathcal{U})$ is a domain bounded by the parabola $2u = v^2 + 1$. Then $E(q) = \{1\}$ and we have:

Theorem 2.1. *Let* $p(z) = 1 + p_n z^n + \cdots$ *be analytic in* $\mathcal{U}$ *with* $p(z) \not\equiv 1$ *and* $n \geqslant 1$. *If there exist points* $z_0 = r_0 e^{i\theta_0} \in \mathcal{U}$ *and* $\zeta_0 \in \partial\mathcal{U} \setminus E(q)$, *such that* $p(\mathcal{U}_{r_0}) \subset q(\mathcal{U})$ *and*

$$p(z_0) = q(\zeta_0) = \frac{1}{2} + \frac{2}{\pi^2} \log^2 x + i \frac{2}{\pi} \log x, \quad x > 0, \tag{2.1}$$

then there exist $m \geqslant n \geqslant 1$ *such that*

$$\text{(1)} \quad z_0 p'(z_0) = \frac{2m}{\pi^2} \sqrt{\frac{\pi^2}{2}\big(p(z_0) - 1\big)} \sinh \sqrt{\frac{\pi^2}{2}\big(p(z_0) - 1\big)}$$
$$= -\frac{m}{2\pi}\left(x + \frac{1}{x}\right) + i \frac{m \log x}{\pi^2}\left(x + \frac{1}{x}\right),$$

(2) $\Re z_0 p'(z_0) \leqslant -\dfrac{1}{\pi}, \qquad |z_0 p'(z_0)| \geqslant \dfrac{1}{\pi},$

(3) $\Re \dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geqslant \dfrac{m\pi}{8} \min_{x>0}\left(\dfrac{x+\frac{1}{x}}{\log^2 x + \pi^2/4}\right) \geqslant \dfrac{1}{\pi}.$

Proof. Making use of Lemma 1.1, it suffices to calculate $\zeta_0 q'(\zeta_0)$ and $\zeta_0 q''(\zeta_0)/q'(\zeta_0)$, where $\zeta_0 = q^{-1}[p(z_0)]$. For the function q given by (1.4), we have

$$\zeta_0 = q^{-1}\big[p(z_0)\big] = \tanh^2 \sqrt{\frac{\pi^2}{8}\big(p(z_0)-1\big)}. \tag{2.2}$$

Since

$$q'(z) = \frac{4}{\pi^2}\frac{1}{\sqrt{z}(1-z)}\log\frac{1+\sqrt{z}}{1-\sqrt{z}}$$

and

$$zq'(z) + z^2 q''(z) = \frac{4}{\pi^2}\frac{z}{(1-z)^2}\left[\frac{1+z}{2\sqrt{z}}\log\frac{1+\sqrt{z}}{1-\sqrt{z}} + 1\right],$$

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{1-z}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{-1},$$

we obtain

$$\zeta_0 q'(\zeta_0) = \frac{1}{\pi^2}\sqrt{\frac{\pi^2}{2}\big(p(z_0)-1\big)}\,\frac{e^{\sqrt{2\pi^2(p(z_0)-1)}}-1}{e^{\sqrt{\pi^2(p(z_0)-1)/2}}} \tag{2.3}$$

$$= \frac{2}{\pi^2}\sqrt{\frac{\pi^2}{2}\big(p(z_0)-1\big)}\sinh\sqrt{\frac{\pi^2}{2}\big(p(z_0)-1\big)}. \tag{2.4}$$

Setting $\zeta_0 = e^{it}$ $(t \in (0, 2\pi))$, we have

$$\log\frac{1+\sqrt{\zeta_0}}{1-\sqrt{\zeta_0}} = \log x + i\frac{\pi}{2}, \quad x > 0,$$

so that we immediately obtain (2.1). Combining this, (2.3), (2.4) and Lemma 1.1, we obtain the assertion (1). As a consequence of (1) and the inequalities:

$$m \geqslant n \geqslant 1 \quad \text{and} \quad x + \frac{1}{x} \geqslant 2 \quad \text{for } x > 0,$$

we conclude (2). Observe next, that

$$1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} = \frac{\pi}{8}\frac{x+\frac{1}{x}}{\log^2 x + \pi^2/4} + \frac{i}{4}\left[x - \frac{1}{x} + \frac{(x+\frac{1}{x})\log x}{\log^2 x + \pi^2/4}\right],$$

then, in view of Lemma 1.1

$$\Re\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geqslant \frac{m\pi}{8}\frac{x+\frac{1}{x}}{\log^2 x + \pi^2/4} =: \frac{m\pi}{8} f(x)$$

with $m \geqslant 1$. Note that

$$f'(x) = \frac{\left(1 - \frac{1}{x^2}\right)\left(\log^2 x + \frac{\pi^2}{4}\right) - \frac{2}{x}\left(1 + \frac{1}{x^2}\right)\log x}{(\log^2 x + \pi^2/4)^2}$$

which equals 0 if and only if

$$\left(1 - \frac{1}{x^2}\right)\left(\log^2 x + \frac{\pi^2}{4}\right) - 2\left(1 + \frac{1}{x^2}\right)\log x = 0. \tag{2.5}$$

By the substitution $\log x = t$ ($t \in \mathbb{R}$), equality (2.5) becomes

$$2e^{-t}\cosh t\left[\left(t^2 + \frac{\pi^2}{4}\right)\tanh t - 2t\right] = 0.$$

Denote

$$g(t) = \left(t^2 + \frac{\pi^2}{4}\right)\tanh t - 2t.$$

We will show $g(t) = 0$ if and only if $t = 0$. By the oddness of g we only need verify the case $t \geqslant 0$. For $t > 0$, we have $\tanh t < t$, that implies

$$g(t) < t\left(t^2 + \frac{\pi^2}{4} - 2\right) =: u(t).$$

Moreover, it is easy to check that

$$\tanh t \geqslant s(t) := \begin{cases} 8t/\pi^2 & \text{for } t \in \langle 0, 0.8), \\ t/5 + 2/5 & \text{for } t \in \langle 0.8, 2.5), \\ 2/t & \text{for } t \in \langle 2.5, \infty). \end{cases} \tag{2.6}$$

Then the function

$$\begin{aligned} r(t) &:= \left(t^2 + \frac{\pi^2}{4}\right)s(t) - 2t \\ &= \begin{cases} 8t^3/\pi^2 & \text{for } t \in \langle 0, 0.8), \\ t^3/5 + 2t^2/5 + (\pi^2/20 - 2)t + \pi^2/10 & \text{for } t \in \langle 0.8, 5.2), \\ \pi^2 t/2 & \text{for } t \in \langle 5.2, \infty), \end{cases} \end{aligned}$$

satisfies inequalities

$$0 \leqslant r(t) \leqslant g(t) \leqslant u(t), \quad t \geqslant 0.$$

Both functions $r(t)$ and $u(t)$ attain its zeros at the only point $t = 0$ so does $g(t)$. That is equivalent to the fact that g attains its only zero at $t = 0$, or equivalently $f'(x) = 0$ if and only if $x = 1$. Moreover $f''(1) = 8(\pi^2 - 8)/\pi^4 > 0$, thus

$$f(x) \geqslant f(1) = \frac{8}{\pi^2},$$

and the third assertion follows. □

Now, we concentrate on the case when $q(\mathcal{U})$ is the region bounded by hyperbola. In this case we also have $E(q) = \{1\}$. We first prove some lemma.

Lemma 2.1. *Let $k \in (0,1)$ and $A(k) = (2/\pi)\arccos k$. Also set*

$$\varphi(k) = 1 - k^2 - 2A^2(k). \tag{2.7}$$

Then $\varphi(k) > 0$ for $k \in (1/\sqrt{2}, 1)$ and $\varphi(k) < 0$ in $(0, 1/\sqrt{2})$.

Proof. Observe that

$$\varphi'(k) = -2\frac{\pi^2 k\sqrt{1-k^2} - 8\arccos k}{\pi^2\sqrt{1-k^2}},$$

and set $\psi(k) := \pi^2 k\sqrt{1-k^2} - 8\arccos k$. Then $\varphi'(k) > 0$ in $(0,1)$ if $\psi(k) < 0$. We have

$$\psi'(k) = \frac{8 + \pi^2(1-2k^2)}{\sqrt{1-k^2}} > 0$$

$$\text{if and only if} \quad k \in \left(0, \sqrt{\frac{4}{\pi^2} + \frac{1}{2}}\right) \quad \text{and} \quad \psi'(k) < 0 \quad \text{in} \left(\sqrt{\frac{4}{\pi^2} + \frac{1}{2}}, 1\right).$$

Since $\psi(0) = -4\pi < 0$ and $\psi(1) = 0$ then there exists the only $k_1 \in (0, \sqrt{4/\pi^2 + 1/2})$ such that $\psi(k_1) = 0$, $\psi(k) < 0$ in $(0, k_1)$ and $\psi(k) > 0$ in $(k_1, 1)$. Equivalently, we have $\varphi'(k) > 0$ in $(0, k_1)$ and $\varphi'(k) < 0$ in $(k_1, 1)$ with $\varphi'(k_1) = 0$. It means that φ increases from $\varphi(0) = -1$ to $\varphi(k_1)$ and next φ decreases to $\varphi(1) = 0$ with the only zero at some $k_0 \in (0, k_1)$. Since $\varphi(1/\sqrt{2}) = 0$ thus $k_0 = 1/\sqrt{2}$, and the assertion follows. □

Theorem 2.2. *Let $k \in (0,1)$ and $A(k) = (2/\pi)\arccos k$. Also, let $p(z) = 1 + p_n z^n + \cdots$ be analytic in $\mathcal{U}$ with $p(z) \not\equiv 1$. If there exist points $z_0 = r_0 e^{i\theta_0} \in \mathcal{U}$ and $\zeta_0 \in \partial\mathcal{U} \setminus E(q)$, such that $p(\mathcal{U}_{r_0}) \subset q(\mathcal{U})$ and*

$$p(z_0) = q_k(\zeta_0) = \frac{\cosh[A(k)\log x] - k^2}{1-k^2} + i\frac{1}{\sqrt{1-k^2}}\sinh\bigl[A(k)\log x\bigr], \tag{2.8}$$

with $x > 0$, then there exists $m \geqslant n \geqslant 1$ such that

(1) $$z_0 p'(z_0) = -\frac{m}{4A(k)\sqrt{1-k^2}}\left(x + \frac{1}{x}\right)\cosh\bigl[A(k)\log x\bigr] + i\frac{mkA(k)}{4(1-k^2)}\left(x + \frac{1}{x}\right)\sinh\bigl[A(k)\log x\bigr],$$

(2) $$\Re z_0 p'(z_0) \leqslant -\frac{A(k)}{2\sqrt{1-k^2}}, \qquad \bigl|z_0 p'(z_0)\bigr| \geqslant \frac{A(k)}{2\sqrt{1-k^2}},$$

(3) $$\Re\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geqslant \frac{m}{4}A(k)k\sqrt{1-k^2}\min_{x>0}\frac{x + \frac{1}{x}}{\cosh^2[A(k)\log x] - k^2}.$$

Moreover for $k \in (1/\sqrt{2}, 1)$, it holds

$$\Re\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geqslant \frac{A(k)k}{2\sqrt{1-k^2}}.$$

Proof. Reasoning along the same line as in the proof of Theorem 2.1, we obtain

$$\zeta_0 = q_k{}^{-1}\big[p(z_0)\big] = \tanh^2\left[\frac{\sinh^{-1}\sqrt{\frac{1-k^2}{2}[p(z_0)-1]}}{A(k)}\right],$$

$$q_k'(z) = \frac{A(k)}{1-k^2}\frac{\sinh[2A(k)\tanh^{-1}\sqrt{z}]}{\sqrt{z}(1-z)},$$

and

$$zq_k'(z) + z^2 q_k''(z) = \frac{A(k)}{1-k^2}\frac{z}{(1-z)^2}\left\{\frac{1+z}{2\sqrt{z}}\sinh\big[2A(k)\tanh^{-1}\sqrt{z}\big] + A(k)\cosh\big[2A(k)\tanh^{-1}\sqrt{z}\big]\right\},$$

so that

$$1 + \frac{zq_k''(z)}{q_k'(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{1-z}A(k)\coth\big[2A(k)\tanh^{-1}\sqrt{z}\big].$$

Setting

$$\log\frac{1+\sqrt{\zeta_0}}{1-\sqrt{\zeta_0}} = \log(ix), \quad x > 0,$$

we have

$$\zeta_0 q_k'(\zeta_0) = \frac{A(k)}{4}\left(x + \frac{1}{x}\right)\left[-\frac{1}{\sqrt{1-k^2}}\cosh\big[A(k)\log x\big] + i\frac{k}{1-k^2}\sinh\big[A(k)\log x\big]\right],$$

and

$$1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} = \frac{A(k)}{4}\left(x + \frac{1}{x}\right)\frac{k\sqrt{1-k^2}}{\cosh^2[A(k)\log x] - k^2} + \frac{i}{4}\left[x - \frac{1}{x} + \frac{A(k)}{2}\left(x + \frac{1}{x}\right)\frac{\sinh[2A(k)\log x]}{\cosh^2[A(k)\log x] - k^2}\right].$$

Making use inequalities $x + 1/x \geqslant 2$ and $\cosh[A(k)\log x] \geqslant 1$ for $x > 0$ we easily obtain the assertion (2), that holds for each $k \in (0,1)$. Now we prove the inequality (3). Observe that

$$\Re\frac{\zeta_0 q_k''(\zeta_0)}{q_k'(\zeta_0)} + 1 =: \frac{A(k)k\sqrt{1-k^2}}{4}L(x)$$

and

$$L'(x) = \frac{\left(1 - \frac{1}{x^2}\right)\{\cosh^2[A(k)\log x] - k^2\} - A(k)\left(x + \frac{1}{x}\right)\sinh[2A(k)t]}{\{\cosh^2[A(k)\log x] - k^2\}^2}.$$

Substituting $t = \log x$ ($t \in \mathbb{R}$), we have that $L'(x) = 0$ if and only if

$$w(t) = \sinh t\big[\cosh^2\big[A(k)t\big] - k^2\big] - A(k)\cosh t\sinh\big[2A(k)t\big]$$

attains its zero for $t \in \mathbb{R}$. Let us denote the function v by the condition $w(t) = v(t)\sinh t$. Then $w(0) = 0$, and similarly as for the function g in the proof of Theorem 2.1 we may prove that

$v(t) > 0$ for $t > 0$ and $k \in (1/\sqrt{2}, 1)$. The function w is odd, so that $w(t) = 0$ if and only if $t = 0$, or equivalently $L'(x) = 0$ if and only if $x = 1$.

Moreover,

$$L''(1) = \frac{1 - k^2 - 2A^2(k)}{(1 - k^2)^2}$$

is nonnegative by Lemma 2.1 for $k \in (1/\sqrt{2}, 1)$. In conclusion the function $L(x)$ attains its only minimum at $x_0 = 1$ for $k \in (1/\sqrt{2}, 1)$ and maximum if $k \in (0, 1/\sqrt{2})$, so that for $k \in (1/\sqrt{2}, 1)$

$$\Re \frac{\zeta_0 q_k''(\zeta_0)}{q_k'(\zeta_0)} + 1 \geqslant \frac{A(k)k\sqrt{1-k^2}}{4} L(1) = \frac{A(k)k}{2\sqrt{1-k^2}}. \qquad \square$$

We now describe the class of admissible functions for this particular q as given in Theorems 2.1 and 2.2. Since, for $k = 1$, $q(\zeta)$, $\zeta \in \partial\mathcal{U}$ is given by (2.1) that we can simplify as

$$q(v) = \frac{v^2 + 1}{2} + iv, \quad v \in \mathbb{R}, \tag{2.9}$$

then the admissibility condition (1.1) becomes

$$\psi(u + iv, s, w; z) \notin \Omega, \quad \text{when } v \in \mathbb{R},\ z \in \mathcal{U}, \tag{2.10}$$
$$u = \frac{v^2 + 1}{2}, \qquad \Re s = -\frac{m\cosh(\pi v/2)}{\pi}, \qquad \Im s = mv\cosh(\pi v/2), \qquad \Re\frac{w}{s} + 1 \geqslant \frac{m}{\pi},$$

and $m \geqslant 1$.

In the case, when $q = q_k$ ($k \in (0, 1)$), in view of (2.8) the admissibility condition (1.1) has the form

$$\psi(u + iv, s, w; z) \notin \Omega, \quad \text{when } t \in \mathbb{R},\ z \in \mathcal{U}, \tag{2.11}$$
$$u = \frac{\cosh(At)}{1 - k^2}, \qquad v = \frac{\sinh(At)}{\sqrt{1 - k^2}}, \qquad \Re s = -\frac{m\cosh t\cosh(At)}{2A\sqrt{1 - k^2}},$$
$$\Im s = m\frac{Ak\cosh t\sinh(At)}{2(1 - k^2)}, \qquad \Re\frac{w}{s} + 1 \geqslant \frac{mAk}{2\sqrt{1 - k^2}} \quad \big(\text{for } k \in (1/\sqrt{2}, 1)\big),$$

and $m \geqslant 1$, $A = A(k) = (2/\pi)\arccos k$.

In spite of simplifying conditions to be checked in proofs of differential subordinations for domains bounded by conic sections they remain incomparably more difficult than in the case when q maps the unit disk onto a disk or a half-plane. The author has proved [1] that the function $\psi(p(z), zp'(z)) = p(z) + zp'(z)/p(z)$ satisfies the admissibility condition for $q = q_1$ and $\Omega = \{w\colon \Re(w - a) > |w - 1 - a|\}$ with $a \geqslant a_0 = -1/\pi$. Also, under additional assumptions, the function $\psi(p(z), zp'(z)) = p(z) + zp'(z)/[\beta p(z) + \gamma]$ satisfies that condition with the same q and Ω [1].

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