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Stolz angle limit of a certain class of self-mappings of the unit disk

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Abstract

Let f be a mapping of the open unit disk $\mathbf{U}$ onto itself having a non-singular differentiable extension to the boundary point 1 which is a fixed point of f. For $a \in \mathbf{U}$ let p and q be Möbius transformations of the unit disk onto itself such that $p(0) = a$ and $q(f(a)) = 0$. It is proved that the Stolz angle limit of $p \circ f \circ q$ when $a \to 1$ is a diffeomorphic self-mapping g of the unit disk, which is a conjugate of an affine transformation. The convergence is almost uniform in $\mathbf{U}$.

Keywords: Stolz angle limit; Möbius transformations; Linearization

1. Introduction and statement of the main results

Two main generalizations of conformal mappings defined between two domains in the complex plane are quasiconformal mappings and harmonic mappings. Let us recall their definitions. Let D and Ω be subdomains of the complex plane $\mathbf{C}$.

We say that a function $w : D \to \mathbf{C}$ is ACL (absolutely continuous on lines) in the region D, if for every closed rectangle $R \subset D$ with sides parallel to the x and y-axes, u is absolutely continuous on a.e. horizontal and a.e. vertical line segment in R. Such a function has of course, partial derivatives w_x, w_y a.e. in D.

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A sense-preserving homeomorphism $w: D \to \Omega$, where D and Ω are subdomains of the complex plane $\mathbf{C}$, is said to be *k-quasiconformal* (k-q.c), with $0 \le k < 1$, if w is ACL in D and

$$|w_{\bar z}| \le k|w_z| \quad \text{a.e. on } D,$$

(cf. Ahlfors' book [1, pp. 23–24]). See also the book of Lehto and Virtanen [6] for good setting of quasiconformal mappings in the plane and the books of Väisälä [10] and Vuorinen [11] for quasiconformal mappings in the space.

Let $\rho(w)|dw|^2$ be an arbitrary conformal C^1-metric defined on D. If $f : \Omega \to D$ is a C^2 mapping between Jordan domains Ω and D, the energy integral of f is defined by the formula:

$$E[f,\rho] = \int_\Omega \rho \circ f(|f_z|^2 + |f_{\bar z}|^2)dx\,dy.$$

The stationary points of the energy integral satisfy the Euler–Lagrange equation

$$f_{z\bar z} + (\log \rho)_w \circ f f_z\, f_{\bar z} = 0,$$

and a C^2 solution of this equation is called a *harmonic mapping* (more precisely a ρ-harmonic mapping). If $\rho \equiv 1$, then a ρ-harmonic mapping is an Euclidean harmonic mapping.

It is known that f is a harmonic mapping if and only if the mapping

$$\Phi = \rho \circ f f_z \overline{f}_{\bar z}$$

is a holomorphic mapping. We refer to [8, Chapter 1] for the above definition and some basic properties of harmonic mappings.

We need the following definitions in the sequel. We say that a mapping $f = u + iv$ of the unit disk into itself is differentiable at a boundary point $e^{i\alpha}$ if there exist three complex constants f_0, f_1 and f_2 such that

$$f(z) = f_0 + f_1 \mathrm{Re}(z - e^{i\alpha}) + f_2 \mathrm{Im}(z - e^{i\alpha}) + o(z - e^{i\alpha}) \quad \text{for } z \in \mathbf{U}. \tag{1.1}$$

Without loss of generality we may assume that $e^{i\alpha} = 1$. For $z = x + iy \in \mathbf{C}$, we define $f(1) = f_0$, $f_x(1) = f_1$, $f_y(1) = f_2$, $f_z(1) = (f_1 - if_2)/2$ and $f_{\bar z}(1) = (f_1 + if_2)/2$. Further we say that 1 is a non-singular point of f if

$$J_f(1) := \det \begin{pmatrix} u_x(1) & u_y(1) \\ v_x(1) & v_y(1) \end{pmatrix} = |f_z(1)|^2 - |f_{\bar z}(1)|^2 \ne 0.$$

A mapping g said to be *conjugate* to h, w.r.t Möbius transformation, if there exists a Möbius transformation γ such that $g = \gamma^{-1} \circ h \circ \gamma$ (see e.g. [2, p. 3]). For $M > 1$ we define *Stolz angle* with vertex 1 as the points of the unit disk satisfying the condition $|z - 1| < M(1 - |z|)$ (see Fig. 1). We say that, a function $a \mapsto \xi(a)$ has *Stolz angle limit* ξ_0 at 1 if for every $M > 0$ and for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, M)$ such that $|\xi(a) - \xi_0| \le \varepsilon$ for $|a - 1| \le \min\{\delta, M(1 - |a|)\}$.

In [3] the author proved that, if f is a quasiconformal harmonic diffeomorphism of the unit disk onto itself with respect to a metric ρ and a_n is a sequence of points converging to a boundary point a and if q and p are Möbius transformation of the unit disk onto itself such that $q(0) = a_n$ and $p(f(a_n)) = 0$, then the sequence $f_n = p \circ f \circ q$ has a convergent subsequence which converges to a harmonic self-mapping diffeomorphism g of the unit disk with respect to a metric ρ_0. See also [5,9] for similar approach in multidimensional setting. The question arises, whether we can relax the assumption that f be a quasiconformal harmonic diffeomorphism. Another question is and what can be said about the metric ρ_0 and does g depend on the sequence a_n. The following Theorem 1.1 and its Corollary 1.3 contain partial answers to these questions.

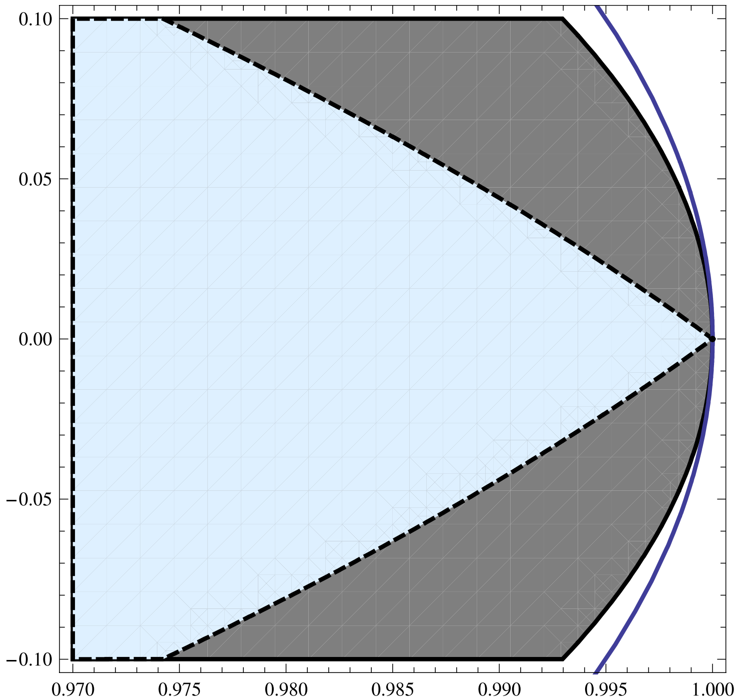

Fig. 1. The Stolz angle $|z-1| < 5(1-|z|)$, inside the domain $|z-1|^2 < 5(1-|z|)$ in the unit disk.

Theorem 1.1. *Let f be a mapping of the open unit disk $\mathbf{U}$ onto itself having a non-singular differentiable extension to the boundary point 1 which is a fixed point of f ($f(1)=1$). Let $a \in \mathbf{U}$, and let q and p be Möbius transformations of the unit disk onto itself such that $q(0)=a$ and $p(f(a))=0$ and define*

$$g_a(t) = p(f(q(t))). \tag{1.2}$$

Then the function

$$g(t) = \sphericalangle \lim_{a \to 1} g_a(t)$$

exists and defines a diffeomorphism of the unit disk onto itself which is conjugate, w.r.t a Möbius transformation, to the linear transformation $h(z) = Bx + C(1+y) + iy$, $B \neq 0$. Here $\sphericalangle \lim$ means the Stolz angle limit and the convergence is uniform at every set $U_n = \mathbf{U} \setminus \{t \in \mathbf{U} : |t+1| \le 1/n\}$, $n \in \mathbf{N}$.

Remark 1.2. Notice first that the mapping f need not to be continuous in $\mathbf{U}$ except at 1. If f is an analytic self-mapping of the unit disk $\mathbf{U}$ having a differentiable extension to its invariant point 1, then by the Schwarz inequality on the boundary of Osserman [7, Lemma 3], $|f'(1)| > 0$ and therefore the condition $J_f(1) \neq 0$ is a priori satisfied. The resulting function g is then the identity. Namely by the proof of Theorem 1.1 we find that

$$B = \frac{\operatorname{Im} f_y(1)}{\operatorname{Re} f_x(1)} = 1 \quad \text{and} \quad C = -\frac{\operatorname{Im} f_x(1)}{\operatorname{Re} f_x(1)} = 0.$$

Instead of the normalization $f(1) = 1$ we can take $f(e^{i\alpha}) = e^{i\beta}$, but the last case can be reduced to the previous case, by composing by appropriate rotations. If the Jacobian $J_f(1) > 0$, then the corresponding limit g is a sense-preserving diffeomorphism. If f is not differentiable at 1, then the resulting limit sometimes is neither a diffeomorphism nor a conjugate to a linear transformation. Namely for $f(z) = z\left(1 - \sqrt{1-|z|^2}\right)$, $z = x + iy$ and $a_n = 1 - 1/n$, under notation of Theorem 1.1 the limit of g_{a_n}, where g_{a_n} is defined in (1.2), when $n \to \infty$ is

$$g(z) = \frac{-\sqrt{1-x^2-y^2} + \sqrt{1+2x+x^2+y^2}}{\sqrt{1-x^2-y^2} + \sqrt{1+2x+x^2+y^2}}.$$

Notice that g is conjugate to the function $h(z) = \sqrt{y}$. Although f is a diffeomorphic self-mapping of the unit disk onto itself with $f(1) = 1$, f is not differentiable at 1 and thus the hypotheses of Theorem 1.1 are not satisfied. If f is a quasiconformal mapping of the unit disk onto itself, then g_a is quasiconformal mapping of the unit disk onto itself such that $g_a(0) = 0$. Then for a given convergent sequence $a_n \to 1$, the sequence g_{a_n}, up to some subsequence, converges to a quasiconformal self-mapping of the unit disk (see e.g. [10, pp. 69–77]). This does not contradict the previous example, because $f(z) = z\left(1 - \sqrt{1-|z|^2}\right)$ is not quasiconformal.

Corollary 1.3. *Under the conditions of Theorem 1.1 let*

$$k = \min\left\{\frac{|f_{\bar z}(1)|}{|f_z(1)|}, \frac{|f_z(1)|}{|f_{\bar z}(1)|}\right\}$$

and let $\rho(w) = 2/|1+w|^2$. Then the function g is a ρ-harmonic k-quasiconformal diffeomorphism of the unit disk **U** *onto itself.*

Proof. The proof follows from Theorem 1.1 and the following proposition

Proposition 1.4 (*[4, Corollary 4.3]*). *Let h be an Euclidean harmonic mapping, let ψ be a conformal mapping on the image domain of h; and let $\psi = ((\varphi^{-1})')^2$. Then the mapping $\hat h = \varphi \circ h$ is harmonic with respect to the metric $\rho(w) = |\psi(w)|$.*

Namely as

$$h(z) = \varphi^{-1} \circ g \circ \varphi(z) = Bx + C(1+y) + iy,$$

is an Euclidean harmonic mapping, we obtain that $\varphi \circ h$ is harmonic w.r.t the metric

$$\rho(z) = |\psi'(z)| = \left|\frac{d}{dz}\frac{i(z-1)}{z+1}\right| = \frac{2}{|z+1|^2}. \quad \square$$

2. Proof of Theorem 1.1

We begin by the following lemma

Lemma 2.1. *Under conditions of Theorem 1.1 we have* $\mathrm{Re}\, f_y(1) = 0$.

Proof of Lemma 2.1. Let $f(z) = \alpha(z) + i\beta(z)$. Define a function γ by

$$\gamma(\lambda) = |f((1-\lambda^2)e^{i\lambda})|^2 = \alpha^2((1-\lambda^2)e^{i\lambda}) + \beta^2((1-\lambda^2)e^{i\lambda}), \quad \lambda \in (-1, 1).$$

Then the function γ is differentiable at $\lambda = 0$ and achieves its maximum at $\lambda = 0$. Thus by Fermat's theorem $\gamma'(0) = 0$. On the other hand for $s = (1-\lambda^2)e^{i\lambda}$,

$$\begin{aligned}\gamma'(\lambda) = {} & 2\alpha(s)\left(\left(\cos\lambda - \lambda^2\cos\lambda - 2\lambda\sin\lambda\right)\alpha_y(s)\right.\\ & \left. - \left(2\lambda\cos\lambda + \left(1-\lambda^2\right)\sin\lambda\right)\alpha_x(s)\right)\\ & + 2\beta(s)\left(\left(\cos\lambda - \lambda^2\cos\lambda - 2\lambda\sin\lambda\right)\beta_y(s)\right.\\ & \left. - \left(2\lambda\cos\lambda + \left(1-\lambda^2\right)\sin\lambda\right)\beta_x(s)\right)\end{aligned}$$

and therefore $\gamma'(0) = 2\alpha(1)\alpha_y(1) + 2\beta(1)\beta_y(1) = 2\alpha_y(1)$. □

Proof of Theorem 1.1. Consider

$$q(t) = \frac{t+a}{1+t\overline{a}}$$

and

$$p(u) = \frac{u - f(a)}{1 - u\overline{f(a)}}.$$

Then

$$h_a(t) = p(f(q(t))).$$

We will translate our problem to the lower half-plane. In order to do so take the conformal mapping

$$t = \varphi(z) = \frac{1 - iz}{1 + iz} \tag{2.1}$$

of the lower half-plane $\mathbf{H} = \{z : \operatorname{Im} z < 0\}$ onto the unit disk $\mathbf{U}$. Then

$$k_a(z) = \frac{f(q(t)) - f(a)}{1 - f(q(t))\overline{f(a)}}$$

is a mapping of $\mathbf{H}$ into $\mathbf{U}$. By composing by the Möbius transformation

$$w = \varphi^{-1}(p) = i\frac{p-1}{p+1}$$

we arrive at the mapping

$$G_a(z) = \frac{i\left(-1 - f(a) + (1 + \overline{f(a)})f\left(\frac{i+z-(-i+z)a}{i-z+(i+z)\overline{a}}\right)\right)}{1 - f(a) + (1 - \overline{f(a)})f\left(\frac{i+z-(-i+z)a}{i-z+(i+z)\overline{a}}\right)}$$

which maps the lower half-plane into itself. We will linearize the function f in the function G_a near the point 1. The resulting function will be H_a. We want to point out that the non-singular nature of the point 1 is important for the proof.

Since f is differentiable at 1 we obtain

$$f(1+h) = f(1) + f_x(1)\operatorname{Re}(h) + f_y(1)\operatorname{Im}(h) + o(h).$$

As $f(1) = 1$, we have

$$1 - f(a) = f_x(1)\mathrm{Re}(1-a) + f_y(1)\mathrm{Im}(1-a) + o(1-a)$$

and for

$$h = -1 + \frac{i+z-(-i+z)a}{i-z+(i+z)\overline{a}},$$

because $o(h) = o(1-a)$, we obtain

$$1 - f\left(\frac{i+z-(-i+z)a}{i-z+(i+z)\overline{a}}\right) = f_x(1)\mathrm{Re}\left(-1 + \frac{i+z-(-i+z)a}{i-z+(i+z)\overline{a}}\right) + f_y(1)\mathrm{Im}\left(\frac{i+z-(-i+z)a}{i-z+(i+z)\overline{a}}\right) + o(1-a).$$

Let $U = f_x(1)$ and $V = f_y(1)$. Then for

$$A = \left(1 - U + V\mathrm{Im}\left(\frac{i+z-a(-i+z)}{i-z+(i+z)\overline{a}}\right) + U\mathrm{Re}\left(\frac{i+z-a(-i+z)}{i-z+(i+z)\overline{a}}\right)\right)$$

and

$$H_a(z) = \frac{i(-2+U-V\mathrm{Im}\,a - U\mathrm{Re}\,a + (2+V\mathrm{Im}\,a + U(-1+\mathrm{Re}\,a))A)}{V\mathrm{Im}\,a + U(-1+\mathrm{Re}\,a) + (V\mathrm{Im}\,a + U(-1+\mathrm{Re}\,a))A}$$

we have

$$\sphericalangle \lim_{a\to 1} H_a(z) = Bx + C(1+y) + iy \tag{2.2}$$

and

$$\sphericalangle \lim_{a\to 1} G_a(z) = \sphericalangle \lim_{a\to 1} H_a(z), \tag{2.3}$$

where

$$B = \frac{\mathrm{Im}\,V}{\mathrm{Re}\,U} = \frac{J_f(1)}{|\mathrm{Re}\,U|^2} \neq 0, \quad \text{and} \quad C = \frac{-\mathrm{Im}\,U}{\mathrm{Re}\,U}.$$

2.1. *Proof of (2.2)*

Let $\varphi(z) = (1-iz)/(1+iz)$ be as defined in (2.1). Fix a positive number r and let $\mathcal{K} = \{t \in \mathbf{U} : |t+1| > r\}$. Then $\mathcal{K}' = \varphi^{-1}(\mathcal{K})$ is a bounded set of the lower half-plane $\mathbf{H}$. Let $a = 1 + \tau$. Then $a \to 1$ if and only if $\tau \to 0$. Further

$$H_a(z) = \frac{K_\tau}{L_\tau},$$

where

$$K_\tau = -i\left(-2 - V\mathrm{Im}\,\tau - U\mathrm{Re}\,\tau + (2 + \overline{V}\mathrm{Im}\,\tau + \overline{U}\mathrm{Re}\,\tau)A\right),$$
$$L_\tau = V\mathrm{Im}\,\tau + U\mathrm{Re}\,\tau + (\overline{V}\mathrm{Im}\,\tau + \overline{U}\mathrm{Re}\,\tau)A$$

and

$$A = \left(1 + V\mathrm{Im}\left[\frac{-\tau(-i+z)-(i+z)\overline{\tau}}{2i+(i+z)\overline{\tau}}\right] + U\mathrm{Re}\left[\frac{-\tau(-i+z)-(i+z)\overline{\tau}}{2i+(i+z)\overline{\tau}}\right]\right).$$

Furthermore

$$\frac{-\tau(-i+z)-(i+z)\overline{\tau}}{2i+(i+z)\overline{\tau}} = i(\operatorname{Im}\tau + z\operatorname{Re}\tau) - \frac{(i\operatorname{Re}\tau + \operatorname{Im}\tau)(\operatorname{Im}\tau + z\operatorname{Re}\tau)(i+z)}{2i+(i+z)\overline{\tau}},$$

and thus

$$A = 1 + V(\operatorname{Im}\tau + \operatorname{Re} z\operatorname{Re}\tau) - U\operatorname{Re}\tau\operatorname{Im} z + O(|\tau|^2).$$

The quantity $O(|\tau|^2)$ depends on $\mathcal{K}$ as well. After some elementary transformations we obtain

$$\begin{aligned} K_\tau &= -2\operatorname{Re}\tau(\operatorname{Im} U - x\operatorname{Im} V + y\operatorname{Im} U + xi\operatorname{Re} V - yi\operatorname{Re} U) \\ &\quad - 2i\operatorname{Im}\tau\operatorname{Re} V + O(|\tau|^2). \end{aligned} \tag{2.4}$$

Similarly we obtain

$$L_\tau = 2\operatorname{Re} U\operatorname{Re}\tau + 2\operatorname{Re} V\operatorname{Im}\tau + O(|\tau|^2). \tag{2.5}$$

From Lemma 2.1 we infer that $\operatorname{Re} V = 0$. Thus for $V = i\upsilon$

$$H_a(z) = \frac{\operatorname{Re}\tau(-\operatorname{Im} U + \upsilon x + Uy) + O(|\tau|^2)}{\operatorname{Re}\tau\operatorname{Re} U + O(|\tau|^2)}.$$

If $a = 1+\tau$ belongs to a certain Stolz angle $\left\{b : \frac{|b-1|}{1-|b|} \le M\right\}$ inside the unit disk $\mathbf{U}$ with the vertex 1, then

$$1 - |1+\tau| \ge \frac{|\tau|}{M}$$

and since $|\operatorname{Re}\tau| \ge 1 - |1+\tau|$ it follows that

$$|\operatorname{Re}\tau| \ge \frac{|\tau|}{M}. \tag{2.6}$$

This implies (2.2).

2.2. *Proof of (2.3)*

First we observe that

$$G_a(z) = \frac{K_\tau + o(|\tau|)}{L_\tau + o(|\tau|)}. \tag{2.7}$$

By combining (2.7) and (2.4)–(2.6), we obtain (2.3).

To conclude the proof of Theorem 1.1 observe that for $z = x+iy$ we have

$$\varphi^{-1}\circ g\circ\varphi(z) = Bx + C(1+y) + iy,$$

which maps the lower half-plane onto itself because $B \ne 0$. □

Example 2.2. If $a = e^{ir} - r^2$ and $f(z) = \frac{(2z+\overline{z})}{3}$, then under notation of Theorem 1.1 and of its proof, we have

$$\lim_{r\to 0^+} \varphi^{-1}(p(f(q(\varphi(z))))) = \frac{1}{13}(-4i + 6z - 3\overline{z}),$$

but the corresponding Stolz limit is $\frac{1}{3}(2z - \overline{z})$. Thus the Stolz angle limit in Theorem 1.1 cannot be replaced by uniform limit.

Remark 2.3. If instead of (1.1) we require that f is two times differentiable at 1, or more generally if

$$f(z) = f_0 + f_1 \mathrm{Re}(z-1) + f_2 \mathrm{Im}(z-1) + O((z-1)^2) \quad \text{for } z \in \mathbf{U},$$

then instead of Stolz convergence we can obtain a little bit better convergence w.r.t a, i.e. for $|a-1|^{\mu} \le M(1-|a|)$, where $\mu < 2$. We expect that Theorem 1.1 has its n-dimensional generalization with respect to Möbius transformations of the unit ball.

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