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QUASICONFORMAL HARMONIC MAPPINGS ONTO A CONVEX DOMAIN REVISITED

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ABSTRACT. We give an explicit dependence of quasiconformal constant on its boundary function, provided that the mapping is quasiconformal harmonic and maps the unit disk onto a strictly convex domain. This result refines some earlier results obtain by the first author and Pavlović ([11, 27]).

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

1.0.1. *Harmonic mappings.* The function

$$P(r,t) = \frac{1-r^2}{2\pi(1-2r\cos t + r^2)}, \quad 0 \le r < 1, \ \ t \in [0, 2\pi]$$

is called the Poisson kernel. Let $\mathbf{U} = \{z : |z| < 1\}$ be the unit disk and $\mathbf{T} = \partial\mathbf{U}$ is the unit circle. The Poisson integral of a complex function $F \in L^1(\mathbf{T})$ is a complex harmonic mapping given by

$$w(z) = u(z) + iv(z) = P[F](z) = \int_0^{2\pi} P(r, t-\tau) F(e^{it}) dt, \tag{1.1}$$

where $z = re^{i\tau} \in \mathbf{U}$. If w is a bounded harmonic mapping, then there exists a function $F \in L^\infty(\mathbf{T})$, such that $w(z) = P[F](z)$ (see e.g. [4, Theorem 3.13 b), $p = \infty$]). From now on we will identify $F(t)$ with $F(e^{it})$ and $F'(t)$ with $\frac{dF(e^{it})}{dt}$.

We refer to Axler, Bourdon and Ramey [4] for good setting of harmonic mappings.

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1.0.2. *Quasiconformal mappings.* A sense-preserving injective harmonic mapping $w = u + iv$ is called K-*quasiconformal* (K-q.c), $K \geq 1$, if

$$|w_{\bar{z}}| \leq k|w_z| \tag{1.2}$$

on $\mathbf{U}$ where $k = (K-1)/(K+1)$. Notice that, since

$$|\nabla w(z)| := \max\{|\nabla w(z)h| : |h| = 1\} = |w_z(z)| + |w_{\bar{z}}(z)|,$$

and

$$l(\nabla w(z)) := \min\{|\nabla w(z)h| : |h| = 1\} = \||w_z(z)| - |w_{\bar{z}}(z)\||.$$

The condition (1.2) is equivalent with

$$|\nabla w(z)| \leq Kl(\nabla w(z)). \tag{1.3}$$

For a general definition of quasiregular mappings and quasiconformal mappings we refer to the book of Ahlfors [1].

For a background on the topic of quasiconformal harmonic mappings we refer [5], [8]-[22], [23], [26], [27]. In this paper we obtain some new results concerning a characterization of this class. We will restrict ourselves to the class of q.c. harmonic mappings w between the unit disk $\mathbf{U}$ and a convex Jordan domain D. The unit disk is taken because of simplicity. Namely, if $w : \Omega \to D$ is q.c. harmonic, and $a : \mathbf{U} \to \Omega$ is conformal, then $w \circ a$, is also q.c. harmonic. However the image domain D cannot be replaced by the unit disk.

To state the main result of the paper, we make use of Hilbert transforms formalism. It provides a necessary and a sufficient condition for the harmonic extension of a homeomorphism from the unit circle to a smooth convex Jordan curve γ to be a q.c mapping. It is an extension of the corresponding result [11, Theorem 3.1] related to convex Jordan domains. The Hilbert transformation of a function $\chi \in \mathrm{L}^1(\mathbf{T})$ is defined by the formula

$$\tilde{\chi}(\tau) = H[\chi](\tau) = -\frac{1}{\pi}\int_{0+}^{\pi} \frac{\chi(\tau+t) - \chi(\tau-t)}{2\tan(t/2)}\mathrm{d}t. \tag{1.4}$$

Here $\int_{0+}^{\pi} \Phi(t)dt := \lim_{\epsilon\to 0^+} \int_{\epsilon}^{\pi} \Phi(t)dt$. This integral is improper and converges for a.e. $\tau \in [0, 2\pi]$; this and other facts concerning the operator H used in this paper can be found in the book of Zygmund [31, Chapter VII]. If $f = u + iv$ is a harmonic function defined in the unit disk then a harmonic function $\tilde{f} = \tilde{u} + i\tilde{v}$ is called the harmonic conjugate of f if $u + i\tilde{u}$ and $v + i\tilde{v}$ are analytic functions and $\tilde{u}(0) = \tilde{v}(0) = 0$. Let $\chi, \tilde{\chi} \in L^1(\mathbf{T})$. Then

$$P[\tilde{\chi}] = \widetilde{P[\chi]}, \tag{1.5}$$

where $\tilde{k}(z)$ is the harmonic conjugate of $k(z)$ (see e.g. [28, Theorem 6.1.3]).

Let D be a strictly convex domain with C^2 Jordan boundary γ. By κ_z we denote the curvature of γ at $z \in \gamma$. We now state a theorem that concerns with quasiconformal harmonic mappings between the unit disk and strictly convex domains.

Theorem 1.1. (I) *Let γ be a $C^{1,\alpha}$ convex Jordan curve and let F be an arbitrary absolutely continuous parametrization.*

Then $w = P[F]$ is a quasiconformal mapping if and only if

$$0 < m = \operatorname{ess}\inf_{\tau} |F'(\tau)|, \tag{1.6}$$

$$M = \|F'\|_\infty := \operatorname{ess}\sup_{\tau} |F'(\tau)| < \infty \tag{1.7}$$

and

$$H = \|H(F')\|_\infty := \operatorname{ess\,sup}_\tau |H(F')(\tau)| < \infty. \tag{1.8}$$

(II) *Let γ be a C^2 convex Jordan curve and κ_z be the curvature of γ at $z \in \gamma$. Further let $\kappa_0 = \min_{z\in\gamma} \kappa_z$ and $\kappa_1 = \max_{z\in\gamma} \kappa_z$. If F satisfies the conditions* (1.6)*,* (1.7) *and* (1.8)*, and γ is strictly convex, then $w = P[F]$ is K quasiconformal, where*

$$K \le \frac{\kappa_1(M^2 + H^2) + \sqrt{(\kappa_1(M^2 + H^2))^2 - (2\kappa_0^2 m^3)^2}}{2\kappa_0^2 m^3}. \tag{1.9}$$

The constant K is the best possible in the following sense, if w is the identity or it is a mapping close to the identity, then $K = 1$ or K close to 1 (respectively).

2. Preliminaries

Suppose γ is a rectifiable, directed, differentiable curve given by its arc-length parametrization $g(s)$, $0 \le s \le l$, where $l = |\gamma|$ is the length of γ. Then $|g'(s)| = 1$ and $s = \int_0^s |g'(t)|dt$, for all $s \in [0, l]$. We say that $\gamma \in C^{1,\alpha}$ if $g \in C^{1,\alpha}$.

If γ is a twice-differentiable curve, then the curvature of γ at a point $p = g(s)$ is given by $\kappa_\gamma(p) = |g''(s)|$. Let

$$K(s,t) = \text{Re}\,[\overline{(g(t) - g(s))} \cdot ig'(s)] \tag{2.1}$$

be a function defined on $[0, l] \times [0, l]$. By $K(s \pm l, t \pm l) = K(s, t)$ we extend it on $\mathbb{R} \times \mathbb{R}$. Note that $ig'(s)$ is the unit normal vector of γ at $g(s)$ and therefore, if γ is convex then

$$K(s,t) \ge 0 \text{ for every } s \text{ and } t. \tag{2.2}$$

Suppose now that $F : \mathbb{R} \mapsto \gamma$ is an arbitrary 2π periodic Lipschitz function such that $F|_{[0,2\pi)} : [0, 2\pi) \mapsto \gamma$ is an orientation preserving bijective function.

Then there exists an increasing continuous function $f : [0, 2\pi] \mapsto [0, l]$ such that

$$F(\tau) = g(f(\tau)). \tag{2.3}$$

In the remainder of this paper we will identify $[0, 2\pi)$ with the unit circle S^1, and $F(s)$ with $F(e^{is})$. In view of the previous convention we have

$$F'(\tau) = g'(f(\tau)) \cdot f'(\tau),$$

and therefore

$$|F'(\tau)| = |g'(f(\tau))| \cdot |f'(\tau)| = f'(\tau).$$

Along with the function K we will also consider the function K_F defined by

$$K_F(t,\tau) = \text{Re}\,[\overline{(F(t) - F(\tau))} \cdot iF'(\tau)].$$

It is easy to see that

$$K_F(t,\tau) = f'(\tau)K(f(t), f(\tau)). \tag{2.4}$$

Lemma 2.1. [12] *If $w = P[F]$ is a harmonic mapping, such that F is a Lipschitz homeomorphism from the unit circle onto a Jordan curve of the class $C^{1,\alpha}$ ($0 < \alpha < 1$), then for almost every $\tau \in [0, 2\pi]$ there exists*

$$J_w(e^{i\tau}) := \lim_{r\to 1^-} J_w(re^{i\tau})$$

and there hold the formula

$$J_w(e^{i\tau}) = f'(\tau)\int_0^{2\pi} \frac{\mathrm{Re}\,[\overline{(g(f(t)) - g(f(\tau)))}\cdot ig'(f(\tau))]}{2\sin^2\frac{t-\tau}{2}}dt. \tag{2.5}$$

Lemma 2.2. *If $\varphi : \mathbf{R} \to \mathbf{R}$ is a $(\ell, \mathcal{L})$ bi-Lipschitz mapping, such that $\varphi(x + a) = \varphi(x) + b$ for some a and b and every x, then there exists a sequence of $(\ell, \mathcal{L})$ bi-Lipschitz diffeomorphisms (respectively a sequence of diffeomorphisms) $\varphi_n : \mathbf{R} \to \mathbf{R}$ such that φ_n converges uniformly to φ, and $\varphi_n(x + a) = \varphi_n(x) + b$.*

Proof. We introduce appropriate mollifiers: Fix a smooth function $\rho : \mathbb{R} \to [0, 1]$ which is compactly supported in the interval $(-1, 1)$ and satisfies $\int_{\mathbb{R}} \rho = 1$. For $\varepsilon = 1/n$ consider the mollifier

$$\rho_\varepsilon(t) := \frac{1}{\varepsilon}\,\rho\left(\frac{t}{\varepsilon}\right). \tag{2.6}$$

It is compactly supported in the interval $(-\varepsilon, \varepsilon)$ and satisfies $\int_{\mathbb{R}} \rho_\varepsilon = 1$. Define

$$\varphi_\varepsilon(x) = \varphi * \rho_\varepsilon = \int_{\mathbf{R}} \varphi(y)\frac{1}{\varepsilon}\rho(\frac{x-y}{\varepsilon})dy = \int_{\mathbf{R}} \varphi(x - \varepsilon z)\rho(z)dz,$$

then

$$\varphi'_\varepsilon(x) = \int_{\mathbf{R}} \varphi'(x - \varepsilon z)\rho(z)dz.$$

It follows that

$$\ell\int_{\mathbf{R}} \rho(z)dz = \ell \le |\varphi'_\varepsilon(x)| \le \mathcal{L}\int_{\mathbf{R}} \rho(z)dz = \mathcal{L}.$$

The fact that $\varphi_\varepsilon(x)$ converges uniformly to φ follows by Arzela-Ascoli theorem. □

Lemma 2.3. *For every bi-Lipschitz mapping $\phi : [0, \pi] \to [0, \pi]$, $\phi'(0) = \phi'(\pi)$ we have*

$$\operatorname{ess\,inf}(\phi'(x))^2 \le \frac{\sin^2\phi(x)}{\sin^2 x} \le \operatorname{ess\,sup}(\phi'(x))^2.$$

Proof. Assume first that, ϕ is a diffeomorphism such that $\phi'(0) = \phi'(\pi)$. Let

$$h(x) = \frac{\sin\phi(x)}{\sin x}.$$

Then h is differentiable in $[0, \pi]$. The stationary points of h satisfy the equation

$$\phi'\frac{\cos\phi(x)}{\sin x} - \frac{\cos x}{\sin x}h = 0.$$

Therefore

$$h^2(x) = (\phi'(x))^2\cos^2\phi(x) + \sin^2\phi(x).$$

Since

$$\phi(2\pi) - \phi(0) = \int_0^{2\pi} \phi'(x)dx,$$

we have that $\min_x(\phi'(x)) \le 1 \le \max_x(\phi'(x))$. It follows that

$$\min_x(\phi'(x))^2 \le h^2(x) \le \max_x(\phi'(x))^2.$$

The general case follows from Lemma 2.2. □

3. The proof of Theorem 1.1

We begin by the following lemma

Lemma 3.1. *Let γ be a C^2 strictly convex Jordan curve and let F be an arbitrary parametrization. Let $m = \min_{\tau\in[0,2\pi]} |F'(\tau)|$ and $M = \max_{\tau\in[0,2\pi]} |F'(\tau)|$. Then we have the following double inequalities:*

$$\frac{\kappa_0^2}{\kappa_1} \le \frac{K(t,\tau)}{2\sin^2\frac{\tau-t}{2}} \le \frac{\kappa_1^2}{\kappa_0}, \tag{3.1}$$

and

$$\frac{\kappa_0^2}{\kappa_1} m^3 \le \frac{K_F(t,\tau)}{2\sin^2\frac{\tau-t}{2}} \le \frac{\kappa_1^2}{\kappa_0} M^3, \tag{3.2}$$

where K and K_F are defined in (2.1) and (2.4). If γ is in addition a symmetric Jordan curve then we have the better estimates

$$\kappa_0 \le \frac{K(t,\tau)}{2\sin^2\frac{\tau-t}{2}} \le \kappa_1, \tag{3.3}$$

and

$$\kappa_0 m^3 \le \frac{K_F(t,\tau)}{2\sin^2\frac{\tau-t}{2}} \le \kappa_1 M^3. \tag{3.4}$$

Proof. Let $\tilde{g}$ be a arch length parametrization function of the curve $\tilde{\gamma} = \frac{1}{|\gamma|}\gamma$, where $|\gamma|$ is the length of γ. Let $\tilde{\kappa}_0 = \min_{z\in\tilde{\gamma}} \tilde{\kappa}_z$ and $\tilde{\kappa}_1 = \max_{z\in\tilde{\gamma}} \tilde{\kappa}_z$, where $\tilde{\kappa}_z$ is the curvature of $\tilde{\gamma}$ at z. It is clear that

$$|\gamma|\kappa_{|\gamma|z} = \tilde{\kappa}_z. \tag{3.5}$$

Let

$$G(\sigma,\varsigma) := \frac{\langle \tilde{g}(\sigma) - \tilde{g}(\varsigma), i\tilde{g}'(\varsigma)\rangle}{2\sin^2\frac{\sigma-\varsigma}{2}}.$$

Since $\tilde{g}'(\varsigma)$ is a unit vector and γ is a C^2 strictly convex curve, there exists a diffeomorphism $\beta : \mathbb{R} \to \mathbb{R}$, $\beta(0) = 0$, $\beta(2\pi + \sigma) = 2\pi + \beta(\sigma)$ such that

$$\tilde{g}'(\sigma) = e^{i\beta(\sigma)}. \tag{3.6}$$

Therefore

$$G(\sigma,\varsigma) = \frac{\int_\varsigma^\sigma \sin(\beta(\tau) - \beta(\varsigma))d\tau}{2\sin^2\frac{\sigma-\varsigma}{2}}. \tag{3.7}$$

On the other hand from

$$\tilde{g}''(\tau) = i\beta'(\tau)e^{i\beta(\tau)}$$

it follows that

$$\kappa_{\tilde{g}(\tau)} = \beta'(\tau). \tag{3.8}$$

According to (3.6), we obtain first that

$$\int_0^{2\pi} e^{i\beta(\sigma)} d\sigma = \tilde{g}(0) - \tilde{g}(2\pi) = 0. \tag{3.9}$$

Thus

$$\int_0^{2\pi}\sin(\beta(\sigma))d\sigma=\int_0^{2\pi}\cos(\beta(\sigma))d\sigma=0. \tag{3.10}$$

Therefore

$$\int_\varsigma^\sigma \sin(\beta(\tau)-\beta(\varsigma))d\tau=\int_{[0,2\pi]\setminus[\varsigma,\sigma]}\sin(\beta(\varsigma)-\beta(\tau))d\tau.$$

As β is a diffeomorphism it follows that at least one of the following relations hold

$$\sin(\beta(\tau)-\beta(\varsigma))\geq 0 \text{ for } \tau\in[\varsigma,\sigma] \tag{3.11}$$

or

$$\sin(\beta(\varsigma)-\beta(\tau))\geq 0 \text{ for } \tau\in[0,2\pi]\setminus[\varsigma,\sigma]. \tag{3.12}$$

Introducing the change $a=\beta(\tau)$ we obtain in the case (3.11) that

$$\begin{aligned}\int_\varsigma^\sigma \sin(\beta(\tau)-\beta(\varsigma))d\tau &= \int_{\beta(\varsigma)}^{\beta(\sigma)}\sin(a-\beta(\varsigma))\frac{da}{\beta'(\tau)}\\ &\geq(\leq)\frac{1}{\max_\tau(\min_\tau)\beta'(\tau)}\int_{\beta(\varsigma)}^{\beta(\sigma)}\sin(a-\beta(\varsigma))da\\ &=\frac{2}{\max_\tau(\min_\tau)\beta'(\tau)}\sin^2(\frac{\beta(\sigma)-\beta(\varsigma)}{2}).\end{aligned} \tag{3.13}$$

Therefore

$$\frac{1}{\max_\tau\beta'(\tau)}\frac{\sin^2(\frac{\beta(\sigma)-\beta(\varsigma)}{2})}{\sin^2\frac{\sigma-\varsigma}{2}}\leq G(\sigma,\varsigma)\leq\frac{1}{\min_\tau\beta'(\tau)}\frac{\sin^2(\frac{\beta(\sigma)-\beta(\varsigma)}{2})}{\sin^2\frac{\sigma-\varsigma}{2}}. \tag{3.14}$$

The case (3.12) can be consider similarly. In this case we apply the fact that $\beta(2\pi+\sigma)=2\pi+\beta(\sigma)$ and in the same way obtain (3.14).

By taking $u=\frac{\sigma-\varsigma}{2}$ and $\phi(u)=\frac{\beta(2u+\varsigma)-\beta(\varsigma)}{2}$, and using Lemma 2.3 we obtain that

$$\frac{(\min_\tau\beta'(\tau))^2}{\max_\tau\beta'(\tau)}\leq G(\sigma,\varsigma)\leq\frac{(\max_\tau\beta'(\tau))^2}{\min_\tau\beta'(\tau)}. \tag{3.15}$$

From (3.15) we obtain

$$\frac{\tilde\kappa_0^2}{\tilde\kappa_1}\leq G(\sigma,\varsigma)\leq\frac{\tilde\kappa_1^2}{\tilde\kappa_0}. \tag{3.16}$$

On the other hand there exists a diffeomorphism $\sigma:[0,2\pi]\to[0,2\pi]$ such that

$$F(\tau)=|\gamma|\tilde g(\sigma(\tau)).$$

Thus

$$F'(\tau)=|\gamma|\sigma'(\tau)g'(\sigma(\tau)) \tag{3.17}$$

and

$$|F'(\tau)|=|\gamma|\sigma'(\tau). \tag{3.18}$$

Thus

$$K_F(t,\tau) = \left\langle \overline{F(t)-F(\tau)}, iF'(\tau)\right\rangle \tag{3.19}$$
$$= |\gamma|^2\sigma'(\tau)\left\langle \overline{\tilde g(\sigma(\tau))-\tilde g(\sigma(t))}, i\tilde g'(\sigma(\tau))\right\rangle$$
$$= |\gamma|^2\sigma'(\tau)G(\sigma(t),\sigma(\tau))\cdot 2\sin^2\frac{\sigma(\tau)-\sigma(t)}{2}.$$

By applying again Lemma 2.3 we obtain

$$\min_t(\sigma'(t))^2 \le \frac{2\sin^2\frac{\sigma(\tau)-\sigma(t)}{2}}{2\sin^2\frac{\tau-t}{2}} \le \max_t(\sigma'(t))^2. \tag{3.20}$$

Combining (3.16), (3.19) and (3.20) we obtain

$$\min_t(\sigma'(t))^2\frac{|\gamma|^2\sigma'(t)\tilde\kappa_0^2}{\tilde\kappa_1} \le \frac{K_F(t,\tau)}{2\sin^2\frac{\tau-t}{2}} \le \max_t(\sigma'(t))^2\frac{|\gamma|^2\sigma'(t)\tilde\kappa_1^2}{\tilde\kappa_0}. \tag{3.21}$$

Combining (3.21), (3.5) and (3.18) we obtain

$$\frac{\kappa_0^2m^3}{\kappa_1} \le \frac{K_F(t,\tau)}{2\sin^2\frac{\tau-t}{2}} \le \frac{\kappa_1^2M^3}{\kappa_0}.$$

This yields (3.2). In particular, if $F=g$, where g is natural parametrization of γ we obtain (3.1). In order to prove the statement for symmetric domain, we differentiate (3.7). Then we have

$$G_\sigma(\sigma,\varsigma) = \frac{\sin(\beta(\sigma)-\beta(\varsigma))}{2\sin^2\frac{\sigma-\varsigma}{2}} - \frac{\int_\varsigma^\sigma \sin(\beta(\tau)-\beta(\varsigma))d\tau}{2\sin^2\frac{\sigma-\varsigma}{2}}\cdot\cot\frac{\sigma-\varsigma}{2}. \tag{3.22}$$

So $G_\sigma(\tilde\sigma,\tilde\varsigma)=0$ if and only if

$$G(\tilde\sigma,\tilde\varsigma) = \frac{\sin(\beta(\tilde\sigma)-\beta(\tilde\varsigma))}{\sin(\tilde\sigma-\tilde\varsigma)}.$$

Define the function

$$H(\sigma,\varsigma) = \frac{\sin(\beta(\sigma)-\beta(\varsigma))}{\sin(\sigma-\varsigma)}, 0<|\sigma-\varsigma|\neq\pi.$$

Then it can be extended in $[0,2\pi]\times[0,2\pi]$ because of symmetry of γ. Namely if $\sigma-\varsigma=\pi$, we have $\beta(\sigma)-\beta(\varsigma)=\pi$. Thus by L'Hopital's rule we have $H(\sigma,\sigma+\pi)=\beta'(\sigma)=H(\sigma,\sigma)$. By putting $x=\sigma-\varsigma\in[0,\pi]$ and $\phi(x)=\beta(x+\varsigma)-\beta(\varsigma)$ and applying Lemma (2.3), instead of (3.16) we obtain

$$\tilde\kappa_0 \le H(\sigma,\varsigma) \le \tilde\kappa_1, \tag{3.23}$$

and consequently

$$\tilde\kappa_0 \le G(\sigma,\varsigma) \le \tilde\kappa_1. \tag{3.24}$$

By repeating the previous proof we obtain (3.3) and (3.4). □

From Lemma 3.1 it follows at once the following theorem.

Theorem 3.2. *If $w=P[F]$ is a harmonic diffeomorphism of the unit disk onto a (symmetric) convex Jordan domain $D=\mathrm{int}\gamma\in C^2$, such that F is (m,M) bi-Lipschitz, then*

$$\left(\kappa_0m^3 \le J_w(e^{i\tau}) \le \kappa_1M^3\right), \frac{\kappa_0^2m^3}{\kappa_1} \le J_w(e^{i\tau}) \le \frac{\kappa_1^2M^3}{\kappa_0}. \tag{3.25}$$

Proof. From (2.5) we obtain

$$J_w(e^{i\tau}) = \int_0^{2\pi} \frac{K_F(t,\tau)}{2\sin^2\frac{\tau-t}{2}}\frac{dt}{2\pi}. \tag{3.26}$$

From (3.2) and (3.4) we obtain (3.25). □

Proof of Theorem 1.1. The part (I) of this theorem coincides with [11, Theorem 3.1]. Prove the part (II). We have to prove that under the conditions (1.6), (1.7) and (1.8) w is $K-$ quasiconformal, where K is given by (1.9). This means that, according to (1.3), we need to prove that the function

$$K(z) = \frac{|w_z| + |w_{\bar z}|}{|w_z| - |w_{\bar z}|} = \frac{1+|\mu|}{1-|\mu|} \tag{3.27}$$

is bounded by K.

It follows from (1.1) that w_φ is equals to the Poisson-Stieltjes integral of F':

$$w_\varphi(re^{i\tau}) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\tau-t)dF(t).$$

Hence, by Fatou's theorem, the radial limits of F_τ exist almost everywhere and $\lim\limits_{r\to 1-} F_\tau(re^{i\tau}) = F_0'(\tau)$ a.e., where F_0 is the absolutely continuous part of F.

As rw_r is harmonic conjugate of w_τ, it turns out that if F is absolutely continuous, then

$$\lim_{r\to 1-} F_r(re^{i\tau}) = H(F')(\tau)\ (a.e.),$$

and

$$\lim_{r\to 1-} F_\varphi(re^{i\tau}) = F'(\tau).$$

As

$$|w_z|^2 + |w_{\bar z}|^2 = \frac{1}{2}\left(|w_r|^2 + \frac{|w_\varphi|^2}{r^2}\right)$$

it follows that

$$\lim_{r\to 1^-} \left(|w_z|^2 + |w_{\bar z}|^2\right) \le \frac{1}{2}(\|F'\|_\infty^2 + \|H(F')\|_\infty^2). \tag{3.28}$$

On the other hand, by (3.25)

$$\lim_{r\to 1^-} \left(|w_z|^2 - |w_{\bar z}|^2\right) \ge \frac{\kappa_0^2 m^3}{\kappa_1}. \tag{3.29}$$

From (3.28) and (3.29) we obtain

$$\lim_{r\to 1-} \frac{|w_z|^2 + |w_{\bar z}|^2}{|w_z|^2 - |w_{\bar z}|^2} \le C := \frac{\kappa_1(\|F'\|_\infty^2 + \|H(F')\|_\infty^2)}{2\kappa_0^2 m^3}, \tag{3.30}$$

i.e.

$$\lim_{r\to 1-} \frac{|w_{\bar z}|}{|w_z|} \le \sqrt{\frac{C-1}{C+1}}. \tag{3.31}$$

By Lewy' theorem, $\frac{|w_{\bar z}|}{|w_z|}$ is a subharmonic function bounded by 1. From (3.31) it follows that

$$\frac{|w_{\bar z}|}{|w_z|} \le \sqrt{\frac{C-1}{C+1}}.$$

Further

$$K = \frac{\sqrt{C+1}+\sqrt{C-1}}{\sqrt{C+1}-\sqrt{C-1}} = C+\sqrt{C^2-1}$$
$$= \frac{\kappa_1(\|F'\|_\infty^2+\|H(F')\|_\infty^2)+\sqrt{(\kappa_1(\|F'\|_\infty^2+\|H(F')\|_\infty^2))^2-(2\kappa_0^2m^3)^2}}{2\kappa_0^2m^3}.$$

The last quantity is equal to 1 for F being identity because all the constants appearing at the quantity are 1 in this special case. Moreover, if F is close to identity in C^2 norm, then the quantity is close to 1. □

Remark 3.3. For symmetric domains, in view of Theorem 3.2, instead of (1.9) we can obtain the following estimate

$$K \le \frac{\|F'\|_\infty^2+\|H(F')\|_\infty^2+\sqrt{(\|F'\|_\infty^2+\|H(F')\|_\infty^2)^2-(2\kappa_0m^3)^2}}{2\kappa_0m^3}.$$

Example 3.4. If F is the arc-parametrization of a C^2 convex Jordan curve γ, then $m = \|F'\|_\infty = 1$. We assume w.l.g. that the length of γ is 2π. Furthermore since $F'(s) = e^{i\beta(s)}$, by applying Lemma 2.3 again we obtain

$$\begin{aligned}
|H[F'](\tau)| &= \left|-\frac{1}{\pi}\int_{0+}^{\pi}\frac{F'(\tau+t)-F'(\tau-t)}{2\tan(t/2)}\mathrm{d}t\right| \\
&\le \frac{1}{\pi}\int_{0+}^{\pi}\frac{|e^{i\beta(\tau+t)}-e^{i\beta(\tau-t)}|}{2\tan(t/2)}\mathrm{d}t \\
&= \frac{1}{\pi}\int_{0+}^{\pi}\frac{2\left|\sin(\frac{\beta(\tau+t)-\beta(\tau-t)}{2})\right|}{2\tan(t/2)}\mathrm{d}t \\
&\le \sup|F''(s)|\frac{1}{\pi}\int_0^{\pi}\frac{\sin t}{\tan(t/2)}dt = \kappa_1.
\end{aligned}$$

So

$$K \le \frac{\kappa_1(1+\kappa_1^2)+\sqrt{(\kappa_1(1+\kappa_1^2))^2-4\kappa_0^4}}{2\kappa_0^2}$$

and for symmetric domains

$$K \le \frac{1+\kappa_1^2+\sqrt{(1+\kappa_1^2)^2-4\kappa_0^2}}{2\kappa_0}.$$

If γ is the unit circle, then $\kappa_0 = 1 = \kappa_1$. Both estimates are asymptotically sharp; if the curve γ approaches in C^2 topology to the unit circle centered at origin, then the quasiconformal constant tends to 1.

In particular if γ is the ellipse $\gamma = \{(x,y) : x^2/a^2 + y^2/b^2 = 1\}$, $a \le b$, $|\gamma| = 2\pi$, then $\kappa_0 = 1/b$ and $\kappa_1 = 1/a$.

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