QUASICONFORMAL HARMONIC MAPPINGS AND CLOSE TO CONVEX DOMAINS

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Abstract. Let $f = h + \overline{g}$ be a univalent sense preserving harmonic mapping of the unit disk $\mathbb{U}$ onto a convex domain Ω. It is proved that: for every a such that $|a| < 1$ ($|a| = 1$) the mapping $f_a = h + a\overline{g}$ is $|a|$ quasiconformal (univalent) close to convex harmonic mapping. This gives an answer to a question posed by Chuaqui and Hernández (J. Math. Anal. Appl. (2007)).

1. Introduction and notation

A planar harmonic mapping is a complex-valued harmonic function $w = f(z)$, $z = x + iy$, defined on some domain $\Omega \subset \mathbb{C}$. When Ω is simply connected, the mapping has a canonical decomposition $f = h + \overline{g}$, where h and g are analytic in Ω. Since the Jacobian of f is given by $|h'|^2 - |g'|^2$, by Lewy's theorem ([14]), it is locally univalent and orientation-preserving if and only if $|g'| < |h'|$, or equivalently if $h'(z) \neq 0$ and the second dilatation $\mu = \frac{g'}{h'}$ has the property $|\mu(z)| < 1$ in Ω. A univalent harmonic mapping is called $k-$quasiconformal ($k < 1$) if $|\mu(z)| \leq k$. For general definition of quasiconformal mappings see [1]. Following the first pioneering work by O. Martio ([15]), the class of quasiconformal harmonic mappings (QCH) has been extensively studied by various authors in the papers [6], [7], [8], [16], [12], [17], [13].

In this short note, by using some results of Clunie and Sheil-Small ([4]) we improve a result by Chuaqui and Hernández ([2]) and answer a question posed there. In addition, for given harmonic diffeomorphism (quasiconformal harmonic mapping) we produce a large class of harmonic diffeomorphisms (quasiconformal harmonic mappings). The main result (Theorem 2.1) can be considered as a partial extension of the fundamental theorem of Choquet-Rado-Kneser ([3] and [5]) which states that: the mapping $f : \mathbb{U} \to \Omega$, between the unit disk $\mathbb{U}$ and a convex domain Ω is univalent if its boundary function $f|_{S^1} : S^1 \to \partial\Omega$ is a homeomorphism.

2000 *Mathematics Subject Classification.* Primary 30C55, Secondary 31C05.

Key words and phrases. Planar harmonic mappings; Quasiconformal mapping; Convex domains, Close to convex domains.

Following Kaplan ([9]), an analytic mapping $f : \mathbb{U} \to \mathbb{U}$ is called close to convex if there exists a univalent convex function ϕ defined in $\mathbb{U}$ such that

$$\operatorname{Re} \frac{f'(z)}{\phi'(z)} > 0.$$

A domain Ω is close to convex if $\mathbb{C} \setminus \Omega$ can be represent as a union of non crossing half-lines. Let f be analytic in $\mathbb{U}$. Then f is close to convex if f is univalent and $f(\mathbb{U})$ is a close to convex domain. It is evident that for $F(z) = f \circ \phi^{-1}$, $F'(z) = \frac{f'(z)}{\phi'(z)}$. Therefore if f is close to convex, then according to the Lemma 2.3 F is univalent; that is f is also univalent.

A harmonic mapping $f : \mathbb{U} \to \mathbb{C}$ is close to convex if it is injective and $f(\mathbb{U})$ is a close to convex domain.

2. The main result

The aim of this paper is to prove the following theorem.

Theorem 2.1. *Let $f = h + \overline{g}$ be a univalent sense preserving harmonic mapping of the unit disk $\mathbb{U}$ onto a convex domain Ω. Then for every a such that $|a| < 1$ ($|a| = 1$) the mapping $f_a = h + a\overline{g}$ is $|a|$ quasiconformal close to convex harmonic mapping ((univalent) close to convex harmonic mapping).*

Theorem 2.1 gives an answer to the question posed by Chuaqui and Hernández in [2], where they proved Theorem 2.1 (see [2, Theorem 3], the convex case) under the condition $|\mu(z)| = |\frac{g'}{h'}| \leq \frac{1}{3}$, and asked if this is the best possible condition. It is shown that, no restriction is needed on the dilatation μ. This result can be considered as an extension of Choquet-Rado-Kneser theorem mentioned in the introduction of this paper.

The proof of Theorem 2.1 depends on the following proposition which we prove for sake of completeness.

Proposition 2.2. [4] *If $f = h + \overline{g} : \mathbb{U} \to \Omega$ is a univalent harmonic mapping of the unit disk onto the convex domain Ω, then*

(i) for every $\varepsilon \in \overline{\mathbb{U}}$ the mapping

$$F_\varepsilon = h(z) + \varepsilon g(z) \tag{2.1}$$

is close to convex;

(ii) for every $z_1, z_2 \in \mathbb{U}$, $z_1 \neq z_2$

$$|g(z_1) - g(z_2)| < |h(z_1) - h(z_2)|. \tag{2.2}$$

Let us first prove the following lemma.

Lemma 2.3. *If f is an analytic mapping defined in a convex domain Ω, such that for some $\alpha \in [0, 1]$*

$$\alpha \operatorname{Re} f'(z) + (1 - \alpha) \operatorname{Im} f'(z) > 0, \quad z \in \Omega,$$

then f is univalent.

Proof. For $z_1, z_2 \in \Omega$ we have

$$f(z_1) - f(z_2) = \int_{z_1}^{z_2} f'(z)\,dz = (z_1 - z_2)\int_0^1 f'(z_1 + t(z_2 - z_1))dt.$$

Therefore

$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = \int_0^1 f'(z_1 + t(z_2 - z_1))dt.$$

Since

$$\alpha a + (1-\alpha)b \le (\alpha^2 + (1-\alpha)^2)^{1/2}(a^2+b^2)^{1/2} \le \left(a^2+b^2\right)^{1/2},$$

it follows that

$$\begin{aligned}&|\frac{f(z_1) - f(z_2)}{z_1 - z_2}|\\ &\ge \int_0^1 \alpha \mathrm{Re}\, f'(z_1 + t(z_2 - z_1)) + (1-\alpha)\mathrm{Im}\, f'(z_1 + t(z_2 - z_1))dt > 0.\end{aligned}$$

This infer that f is univalent. □

Proof of Proposition 2.2. Since f is convex, then for every $\varepsilon = e^{i\varphi}$ the mapping $e^{-i\varphi/2}f = e^{-i\varphi/2}h + e^{i\varphi/2}\overline{g}$ is convex. On the other hand

$$G_\varepsilon := e^{-i\varphi/2}h - e^{i\varphi/2}g = e^{-i\varphi/2}f - 2\mathrm{Re}\, e^{-i\varphi/2}g.$$

It follows that $G_\varepsilon(\mathbb{U})$ is convex in direction of real axis. Prove that G_ε is injective. Since f is univalent, it follows that

$$G_\varepsilon \circ f^{-1}(w) = e^{-i\varphi}w + p(w),$$

where p is a real function. Let $w_1 = e^{i\varphi}\omega_1$ and $w_2 = e^{i\varphi}\omega_2$ be two points such that

$$w_1 + e^{i\varphi}p(w_1) = w_2 + e^{i\varphi}p(w_2),$$

i.e.

$$\omega_1 + q(\omega_1) = \omega_2 + q(\omega_2),$$

where $q(\omega) = p(e^{i\varphi}\omega)$. It follows that

$$\mathrm{Im}\,\omega_1 = \mathrm{Im}\,\omega_2 = v_0 \tag{2.3}$$

and

$$\mathrm{Re}\,\omega_1 + q(\omega_1) = \mathrm{Re}\,\omega_2 + q(\omega_2).$$

According to Lewy's theorem

$$G_\varepsilon := e^{-i\varphi/2}h' - e^{i\varphi/2}g' \ne 0.$$

Therefore $G_\varepsilon \circ f^{-1}$ is locally univalent. Write $\omega = u + iv$. Then the function $u \to u + q(u + iv_0)$ is locally univalent. Since it is a real function, it follows that it is univalent. As

$$u_1 + q(u_1 + iv_0) = u_2 + q(u_2 + iv_0),$$

it follows that $u_1 = u_2$ and consequently $w_1 = w_2$.

This means that G_ε is convex in direction of real axis and univalent. In particular $F_\varepsilon = h+\varepsilon g$ is close to convex for $|\varepsilon| = 1$. According to a Kaplan's theorem ([9, Eqs. (16′)]); this is equivalent to the fact that for $0 < r < 1$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$

$$\arg\left(h'(re^{i\theta_1}) + \varepsilon g'(re^{i\theta_1})\right) - \arg\left(h'(re^{i\theta_2}) + \varepsilon g'(re^{i\theta_2})\right) \le \pi + \theta_2 - \theta_1. \tag{2.4}$$

As the expression on the left-side of (2.4) is well-defined harmonic function in ε for $|\varepsilon| \le 1$ (because $h'(w) \neq 0$ and $\operatorname{Re}(1+\varepsilon \frac{g'(w)}{h'(w)}) > 0$), according to the maximum principle the inequality (2.4) continues to hold when $|\varepsilon| \le 1$. We proved that F_ε, $|\varepsilon| \le 1$, is close to convex. According to the introduction and Lemma 2.3 it is univalent.

To prove (ii) we argue by contradiction. Assume there exists an A: $|A| \ge 1$ and $z_1, z_2 \in \mathbb{U}$ such that

$$\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} = A.$$

Hence for $\varepsilon = -1/A$ we have

$$h(z_1) - h(z_2) + \varepsilon(g(z_1) - g(z_2)) = 0.$$

This contradicts (i).

□

Proof of Theorem 2.1. (a) Assume that f_a is not univalent. Then for some distinct points $z_1, z_2 \in \mathbb{U}$

$$f_a(z_1) = f_a(z_2).$$

It follows that,

$$\overline{h(z_1) - h(z_2)} = a(g(z_2) - g(z_1)).$$

This contradicts (2.2). The dilatation μ_a of f_a is equal to $\overline{a}\mu$. Thus f_a is $|a|$ quasiconformal.

To continue we need the following lemma.

Lemma 2.4. [4, Lemma 5.15] *Suppose G and H are harmonic in $\mathbb{U}$ with $|g'(0)| < |H'(0)|$ and that $H+\varepsilon G$ is close to convex for $|\varepsilon| = 1$. Then $H+\overline{G}$ is harmonic close to convex.*

First of all $|a||g'(0)| < |h'(0)|$. By Proposition 2.2 $h + \varepsilon\overline{a}g$ is close to convex for every $|\varepsilon| = 1$. Therefore $f_a = h + a\overline{g}$ is close to convex. □

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