Complex Variables and Elliptic Equations, 2014
Vol. 59, No. 4, 539–552, http://dx.doi.org/10.1080/17476933.2012.759565

Radius of close-to-convexity and fully starlikeness of harmonic mappings

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Communicated by I. Sabadini

(*Received 22 July 2011; final version received 9 December 2012*)

Let $\mathcal{H}$ denote the class of all normalized complex-valued harmonic functions $f = h + \overline{g}$ in the unit disk $\mathbb{D}$, and let $\mathcal{S}_H^0$ denote the class of univalent and sense-preserving functions f in $\mathcal{H}$ such that $f_{\overline{z}}(0) = 0$. If $K = H + \overline{G}$ denotes the harmonic Koebe function whose dilation is $\omega(z) = z$, then $K \in \mathcal{S}_H^0$ and it is conjectured that $K(z)$ is extremal for the coefficient problem in $\mathcal{S}_H^0$. If the conjecture were true, then $\mathcal{F}$ contains the family $\mathcal{S}_H^0$, where

$$\mathcal{F} = \{f = h + \overline{g} \in \mathcal{H} : |a_n| \le A_n \text{ and } |b_n| \le B_n \text{ for } n \ge 1\}.$$

Here, a_n, b_n, A_n, and B_n denote the Maclaurin coefficients of h, g, H, and G. We show that the radius of univalence of the family $\mathcal{F}$ is $0.112903\ldots$. We also show that this number is also the radius of the fully starlikeness of $\mathcal{F}$. Analogous results are proved for a family which contains the class of harmonic convex functions in $\mathcal{H}$. We use the new coefficient estimate for bounded harmonic mappings and Lemma 1.6 to improve Bloch-Landau constant for bounded harmonic mappings.

Keywords: coefficient inequality; partial sums; radius of univalence; analytic, univalent, convex and starlike harmonic functions

AMS Subject Classifications: Primary: 30C45

1. Introduction and main results

Denote by $\mathcal{H}$ the class of all complex-valued harmonic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f_z(0) - 1$. Each f can be decomposed as $f = h + \overline{g}$, where g and h are analytic in $\mathbb{D}$ so that [1,2]

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1}$$

Let $\mathcal{S}_H$ denote the class of univalent and sense-preserving functions $f = h + \overline{g}$ in $\mathcal{H}$. Then, the Jacobian of f is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$. We note that if $f = h + \overline{g} \in \mathcal{S}_H$ and $g(z) \equiv 0$ in $\mathbb{D}$, then $f = h \in \mathcal{S}$, where $\mathcal{S}$ denotes the well-known

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class of normalized univalent analytic functions in $\mathbb{D}$. A necessary and sufficient condition (see [1] or Lewy [3]) for a harmonic function f to be locally univalent in $\mathbb{D}$ is that $J_f(z) \neq 0$ in $\mathbb{D}$. For a sense-preserving harmonic mappings, the function $\omega(z) = g'(z)/h'(z)$ denotes the complex dilatation of f. Thus, for $f = h + \overline{g} \in \mathcal{S}_H$ with $g'(0) = b_1$ and $|b_1| < 1$ (because $J_f(0) = 1 - |b_1|^2 > 0$), the function

$$F = \frac{f - \overline{b_1 f}}{1 - |b_1|^2}$$

is also in S_H with $F_{\overline{z}}(0) = 0$. Thus, it is customary to restrict our attention to the subclass

$$\mathcal{S}_H^0 = \{f \in \mathcal{S}_H : \ f_{\overline{z}}(0) = 0\}.$$

The family $\mathcal{S}_H^0$ is known to be compact. The uniqueness result of the Riemann mapping theorem does not extend to these classes of harmonic functions, [1,2]. Several authors have studied the subclass of functions that map $\mathbb{D}$ onto specific domains, e.g. starlike domains, convex and close-to-convex domains. Let $\mathcal{S}_H^*$ ($\mathcal{K}_H, \mathcal{C}_H$ resp.) consist of all sense-preserving harmonic mappings $f = h + \overline{g} \in \mathcal{H}$ of $\mathbb{D}$ onto starlike (convex, close-to-convex, resp.) domains. Denote by $\mathcal{S}_H^{*0}$ ($\mathcal{K}_H^0, \mathcal{C}_H^0$ resp.) the class consists of those functions f in $\mathcal{S}_H^*$ ($\mathcal{K}_H$, $\mathcal{C}_H$ resp.) for which $f_{\overline{z}}(0) = 0$.

Recall that a function $f \in \mathcal{H}$ is called close-to-convex in $\mathbb{D}$ if it is univalent and the range $f(\mathbb{D})$ is a close-to-convex domain, i.e. the complement of $f(\mathbb{D})$ can be written as the union of nonintersecting half-lines. A normalized analytic function f in $\mathbb{D}$ is close-to-convex in $\mathbb{D}$ if there exists an analytic function (not necessarily normalized) ϕ that is convex in $\mathbb{D}$ and such that

$$\mathrm{Re}\left(\frac{f'(z)}{\phi'(z)}\right) > 0 \quad (|z| < 1).$$

In [1, Lemma 5.15], Clunie and Sheil-Small proved the following result.

LEMMA A *If h and g are analytic in $\mathbb{D}$ with $|h'(0)| > |g'(0)|$ and $h+\epsilon g$ is close-to-convex for each ϵ, $|\epsilon| = 1$, then $f = h + \overline{g}$ is close-to-convex in $\mathbb{D}$.*

This lemma has been used to obtain many important results. In the case of $\mathcal{S}_H^0$, we have the harmonic Koebe function $K = H + \overline{G}$ in $\mathcal{S}_H^0$, where

$$H(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} \ \text{ and } \ G(z) = \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}. \tag{2}$$

We see that the function K has the dilatation $\omega(z) = z$ and K maps the unit disk $\mathbb{D}$ onto the slit plane $\mathbb{C}\backslash\{u + iv : \ u \leq -1/6, \ v = 0\}$. Moreover,

$$H(z) = z + \sum_{n=2}^{\infty} A_n z^n \ \text{ and } \ G(z) = \sum_{n=2}^{\infty} B_n z^n,$$

where

$$A_n = \frac{1}{6}(2n+1)(n+1) \ \text{ and } \ B_n = \frac{1}{6}(2n-1)(n-1), \quad n \geq 1. \tag{3}$$

A well-known coefficient conjecture of Clunie and Sheil-Small [1] is that if $f = h + \overline{g} \in \mathcal{S}_H^0$, then the Taylor coefficients of the series of h and g satisfy the inequalities

$$|a_n| \le A_n \ \text{ and } \ |b_n| \le B_n \text{ for all } n \ge 1. \tag{4}$$

Although, the coefficient's conjecture remains an open problem for the full class $\mathcal{S}_H^0$, the same has been verified for certain subclasses: namely, the class $\mathcal{T}_H$ (see [2, Section 6.6]) of harmonic univalent typically real functions, the class of harmonic convex functions in one direction, harmonic starlike functions in $\mathcal{S}_H^0$ (see [2, Section 6.7]), and the class of harmonic close-to-convex functions (see [4]).

We remark that a normalized harmonic function $f = h + \overline{g}$ satisfying the inequalities (4) are not necessarily univalent in $\mathbb{D}$. For example,

$$f_\alpha(z) = z + \alpha \overline{z}^n \ \ (n \ge 2)$$

is not even sense-preserving if $|\alpha| > 1/n$. Therefore, it is natural to know to what extent do the conditions (4) influence the univalency of the normalized harmonic function $f(z)$ and of all of its partial sums, namely, $f_n(z)$ and $f_{\overline{m}}(z)$, where

$$f_n(z) = h_n(z) + \overline{g_m(z)} \text{ if } n \ge m; \ \ f_{\overline{m}}(z) = h_n(z) + \overline{g_m(z)} \text{ if } m \ge n.$$

Here, $h_n(z)$ and $g_m(z)$ represent the n-th section/partial sums of h and g given by

$$h_n(z) = z + \sum_{k=2}^{n} a_k z^k \ \text{ and } \ g_m(z) = \sum_{k=1}^{m} b_k z^k,$$

respectively. According to our notation, the degree of the polynomials $f_n(z)$ and $f_{\overline{m}}(z)$ is n if $n = m$.

Definition 1.1 A harmonic mapping $f \in \mathcal{H}$ is said to be *fully starlike* (resp. fully convex) if each $|z| < r$ is mapped onto a starlike (resp. convex) domain (see [5]).

Clearly, fully convex mappings are fully starlike but not the converse as the function

$$f(z) = z + \frac{1}{n}\overline{z}^n \ \ (n \ge 2)$$

shows. Furthermore, it can be easily seen that the Koebe function K is not fully starlike in $\mathbb{D}$. According to Radó–Kneser–Choquet theorem, a fully convex harmonic mapping is necessarily univalent in $\mathbb{D}$. However, a fully starlike mapping need not be univalent (see [5]). Finally, we remark that in the case of analytic functions, fully starlike (resp. fully convex) is same as starlike (resp. convex) and so, the sceneries in the harmonic case is different and difficult to handle sometimes, e.g. even a sharp estimate for the second coefficient $|a_2|$ of the analytic part of $f \in \mathcal{H}$ is still not known although Clunie and Sheil-Small conjectured that $|a_2| \le 5/2$ holds.

LEMMA 1.2 *Let h and g have the form* (1) *and the coefficients of the series satisfy the conditions* (4). *Then $f = h + \overline{g}$ satisfies the inequality*

$$|h'(z) - 1| < 1 - |g'(z)| \tag{5}$$

in the disk $|z| < r_S$ and fully starlike in $|z| < r_S$, where

$$r_S = 1 + \frac{\sqrt{2}}{4} - \sqrt{\sqrt{2} + \frac{1}{8}} \approx 0.112903$$

is the root of the quadratic equation

$$\sqrt{2}\,r^2 - (1+2\sqrt{2})\,r + \sqrt{2} - 1 = 0$$

in the interval $(0, 1)$. *The result is sharp.*

We shall soon see that the harmonic function $f = h + \overline{g}$ satisfying the condition (5) in $\mathbb{D}$ is necessarily close-to-convex (and univalent), but not necessarily starlike in $\mathbb{D}$. In the other direction, we would like to mention that even a convex function f in $\mathbb{D}$ does not necessarily satisfy the condition $\operatorname{Re} h'(z) > |g'(z)|$ in $\mathbb{D}$ and hence, f does not necessarily satisfy the condition (5). So, the radius conclusion in Lemma 1.2 and related results below provide a stronger information than just the starlikeness.

The radii problems for various subclasses of univalent harmonic mappings are open [2, Problem 3.3] (see also [1,2,7,8]). However, Lemma 1.2 quickly yields.

THEOREM 1.3 *The radius of fully starlikeness for mappings in* $\mathcal{S}_H^{*0}$, $\mathcal{C}_H^0$, *and* $\mathcal{T}_H$ *is at least* 0.112903.

Under the hypotheses of Lemma 1.2, all the partial sums of f are close-to-convex (univalent), and fully starlike in $|z| < r_S$. Similar comments apply to the next two results.

Another well-known result due to Clunie and Sheil-Small [1] states that the coefficients of the series of h and g of every convex function $f = h + \overline{g} \in \mathcal{K}_H^0$ satisfy the inequalities

$$|a_n| \le \frac{n+1}{2} \quad \text{and} \quad |b_n| \le \frac{n-1}{2} \text{ for all } n \ge 1. \tag{6}$$

Equality occurs for the function $L = M + \overline{N} \in \mathcal{K}_H^0$, where

$$M(z) = \frac{1}{2}\left(\frac{z}{1-z} + \frac{z}{(1-z)^2}\right) \quad \text{and} \quad N(z) = \frac{1}{2}\left(\frac{z}{1-z} - \frac{z}{(1-z)^2}\right). \tag{7}$$

We observe that

$$L(z) = \operatorname{Re}\left(\frac{z}{1-z}\right) + \operatorname{Im}\left(\frac{z}{(1-z)^2}\right) = z + \sum_{n=2}^{\infty} \frac{n+1}{2} z^n - \overline{\sum_{n=2}^{\infty} \frac{n-1}{2} z^n}$$

and L maps $\mathbb{D}$ onto the half-plane $\operatorname{Re}(w) > -1/2$. At this place, it is worth recalling that the convexity (resp. starlikness) property is not a hereditary property in the harmonic case, unlike the analytic case. For instance, the convex function L maps the subdisk $|z| < r$ onto a convex domain for $r \le \sqrt{2} - 1$, but onto a nonconvex domain for $\sqrt{2} - 1 < r < 1$. Furthermore, it can be easily shown that the half-plane mapping L is not fully starlike in $\mathbb{D}$.

LEMMA 1.4 *Let h and g have the form* (1) *and the coefficients of the series satisfy the conditions* (6). *Then,* $f = h + \overline{g}$ *satisfies the inequality*

$$|h'(z) - 1| < 1 - |g'(z)|$$

in the disk $|z| < r_S$, *and* f *is fully starlike in* $|z| < r_S$, *where*

$$r_S = 1 + \frac{\sqrt[3]{-18+\sqrt{330}}}{6^{2/3}} - \frac{1}{\sqrt[3]{6(-18+\sqrt{330})}} \approx 0.164878$$

is the real root of the cubic equation

$$2r^3 - 6r^2 + 7r - 1 = 0$$

in the interval $(0, 1)$. *The result is sharp.*

Lemma 1.4 easily gives the following theorem although the conclusion of Lemma 1.4 is much more stronger.

THEOREM 1.5 *The radius of fully starlikeness for convex mappings in* $\mathcal{S}_H^0$ *is at least* 0.164878.

LEMMA 1.6 *Let h and g have the form* (1) *with* $|b_1| = |g'(0)| < 1$, *and the coefficients of the series satisfy the conditions*

$$|a_n| + |b_n| \le c \ \textit{ for all } n \ge 2.$$

Then $f = h + \overline{g}$ *satisfies the inequality* $|h'(z) - 1| < 1 - |g'(z)|$ *in the disk* $|z| < r_S$ *and is fully starlike in* $|z| < r_S$, *where*

$$r_S = 1 - \sqrt{\frac{c}{c + 1 - |b_1|}}.$$

The result is sharp.

xmlcommand

Lemma 1.6 helps to improve the Bloch-Landau's theorem for bounded harmonic functions. Consider the class $\mathcal{B}_H^M$ of a harmonic mapping f of the unit disk $\mathbb{D}$ with $f(0) = f_{\overline{z}}(0) = f_z(0) - 1 = 0$, and $|f(z)| < M$ for $z \in \mathbb{D}$. There are two important constants: one is relative to the domain of the function while the other one, namely the Bloch constant, is defined relative to the range. In [9], the authors proved that if $f \in \mathcal{B}_H^M$, then f is univalent in $|z| < \rho_0$ and $f(|z| < \rho_0)$ contains a disk $|w| < R_0$, where

$$\rho_0 \approx \frac{1}{11.105M} \ \text{ and } \ R_0 = \frac{\rho_0}{2} \approx \frac{1}{22.21M}.$$

Better estimates were given in [9,10,12,13] and later in [14]. See Table 1 for the best known results, where ϕ and ψ are explicitly given by

$$\phi(x) = \frac{x}{\sqrt{2}(x^2 + x - 1)} \ \text{ and } \ \psi(x) = \frac{1}{\sqrt{2}}\left[1 + \left(\frac{x^2 - 1}{x}\right)\log\left(\frac{x^2 - 1}{x^2 + x - 1}\right)\right].$$

This result is the best known but not sharp.

The purpose the next theorem is to give a new proof of one of these results. Indeed our method of proof is simple and improves the best known result. In fact, our distortion estimate for $f \in \mathcal{B}_H^M$ provides the following better estimate for the radius of close-to-convexity and the radius of fully starlikeness of $\mathcal{B}_H^M$.

THEOREM 1.7 *Let* $f \in \mathcal{B}_H^M$. *Then,* $f = h + \overline{g}$ *satisfies the inequality* $|h'(z) - 1| < 1 - |g'(z)|$ *in the disk* $|z| < r_0$ *and fully starlike in* $|z| < r_0$, *where*

$$r_0 = 1 - \sqrt{\frac{4M}{4M + \pi}}$$

Table 1. The left side columns refer to Theorem 4 in [14] and the right side columns refer to Theorem 1.7.

M	$r = \phi(8M/\pi)$	$R = \psi(8M/\pi)$	M	r_0	R_0
1	0.22421	0.12629	1	0.251602	0.143904
2	0.11992	0.06367	2	0.152633	0.082622
3	0.08311	0.04328	3	0.109765	0.0580693

and $f(\mathbb{D}_{r_0})$ contains a univalent disk of radius at least

$$R_0 = r_0 - \frac{4M}{\pi}\frac{r_0^2}{1-r_0}.$$

It would be interesting to know the improvement and sharpness of other versions of Bloch-Landau-type theorems for harmonic functions (see [9]).

2. Useful Lemmas and their Proofs

We need the following lemma to prove our main results.

LEMMA 2.1 *Let h and g have the form* (1) *with $|b_1| < 1$, $f = h + \overline{g}$, and satisfy the condition*

$$\sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1. \tag{8}$$

Then, $f \in \mathcal{C}_H^2$, where $\mathcal{C}_H^2 = \{f \in \mathcal{S}_H : |f_z(z) - 1| < 1 - |f_{\overline{z}}(z)| \text{ in } \mathbb{D}\}$. Moreover, $f \in \mathcal{S}_H^*$. The bound in (8) *is sharp as the harmonic function*

$$f(z) = z + \sum_{n=2}^{\infty} \frac{\epsilon_n}{n} z^n + \sum_{n=1}^{\infty} \frac{\epsilon_n'}{n} \overline{z^n},$$

for which $\sum_{n=2}^{\infty} |\epsilon_n| + \sum_{n=1}^{\infty} |\epsilon_n'| = 1$, shows.

Proof Note that the coefficient inequality (8) implies that both h and g are analytic in $\mathbb{D}$. Thus, $f = h + \overline{g} \in \mathcal{H}$. Without loss of generality, we may assume that f is not affine. Since $f_z = h'$, and $f_{\overline{z}} = \overline{g'}$, it follows from (8) that

$$|h'(z) - 1| \leq \sum_{n=2}^{\infty} n|a_n|\,|z|^{n-1} \leq \sum_{n=2}^{\infty} n|a_n| \leq 1 - \sum_{n=1}^{\infty} n|b_n| \leq 1 - |g'(z)|$$

implying that $f \in \mathcal{C}_H^2$ (since strict inequality occurs either at the second or fourth inequality). In particular, $\operatorname{Re} h'(z) > |g'(z)|$ in $\mathbb{D}$ and hence, f is sense-preserving, univalent, and close-to-convex in $\mathbb{D}$ (see also [1,14]).

Next, we show that f is fully starlike in $\mathbb{D}$. Indeed from the work of [15] (see also [17]), it follows by (8) with $b_1 = 0$ that

$$\frac{\partial}{\partial\theta} \arg f(re^{i\theta}) = \frac{\partial}{\partial\theta} \operatorname{Im}\, \log f(re^{i\theta}) = \operatorname{Re}\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} \geq 0$$

where, $z = re^{i\theta}$. That is, $\arg f(re^{i\theta})$ is a nondecreasing function of θ for each r, and so, f is fully starlike in $\mathbb{D}$ (see [6, Theorem 3]).

In order to prove the same for the case, $b_1 \neq 0$, it suffices to show that

$$|zf_z(z) - \overline{z}f_{\overline{z}}(z) - f(z)| < |zf_z(z) - \overline{z}f_{\overline{z}}(z) + f(z)|$$

for $|z| = r < 1$. Indeed for $|z| = r < 1$, we see that

$$\begin{aligned}
|zf_z(z) - \overline{z}f_{\overline{z}}(z) - f(z)| &= \left|\sum_{n=1}^{\infty}(n-1)a_n z^n - \overline{\sum_{n=1}^{\infty}(n+1)b_n z^n}\right| \\
&\leq \sum_{n=1}^{\infty}((n-1)|a_n| + (n+1)|b_n|)\,|z|^n \\
&\leq |z|\left(2 - \sum_{n=2}^{\infty}((n+1)|a_n| + (n-1)|b_n|)\,|z|^{n-1}\right) \text{ (by (8))} \\
&\leq \left|2z + \sum_{n=2}^{\infty}(n+1)a_n z^n - \sum_{n=1}^{\infty}(n-1)b_n z^n\right| \\
&= |zf_z(z) - \overline{z}f_{\overline{z}}(z) + f(z)|.
\end{aligned}$$

Here one of the inequalities could be strict. The proof is complete. □

For example, the functions

$$f_n(z) = z + \frac{n+1}{2n^2}z^n + \frac{n-1}{2n^2}\overline{z^n} \quad \text{for } n \geq 2$$

satisfy the condition (8) and hence, belong to the class $\mathcal{C}_H^2$. Moreover, the function

$$f(z) = z + \frac{1}{2}\overline{z}^2 e^{i\alpha}$$

satisfies the condition (8) and so, it belongs to $\mathcal{C}_H^2 \cap \mathcal{S}_H^{*0}$. This function is seen to be extremal for the area minimizing property of the family $\mathcal{S}_H^{*0}$ (see [2, p.89–90]).

In [16], under the hypotheses of Lemma 2.1, it was actually shown that $f \in \mathcal{C}_H^1$, where

$$\mathcal{C}_H^1 = \{f \in \mathcal{S}_H : \operatorname{Re} f_z(z) > |f_{\overline{z}}(z)| \text{ in } \mathbb{D}\}.$$

Clearly, Lemma 2.1 improves this result because of the strict inclusion $\mathcal{C}_H^2 \subsetneqq \mathcal{C}_H^1$ (see also [17]).

In the following, we show some necessary conditions for functions to be in $\mathcal{C}_H^2$.

Lemma 2.2 *Let h and g have the form* (1) *with $|b_1| < 1$, $f = h + \overline{g}$. Suppose $f \in \mathcal{C}_H^2$. Then, we have the following*

(a) $\big||a_n| - |b_n|\big| \leq 1/n$ *for $n \geq 2$ whenever $b_1 = 0$. The equality occurs, for example, for the function*

$$f(z) = z + \frac{e^{i\theta}}{n}z^n \;\; \textit{or} \;\; f(z) = z + \frac{e^{i\theta}}{n}\overline{z^n} \;\; \textit{for } n \geq 2 \textit{ and } \theta \textit{ real}.$$

(b) $\sum_{n=2}^{\infty} n^2(|a_n|^2 + |b_n|^2) \leq 1 - |b_1|^2.$

Proof First, we prove part **(a)**. Let $f = h + \overline{g} \in \mathcal{C}_H^2$ and $F = h + \epsilon g$, where $|\epsilon| = 1$.

Next, set $\omega(z) = F'(z) - 1$. Then, as $b_1 = g'(0) = 0$, we have $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. It is well-known property that the coefficients of such an analytic function ω satisfy the inequality $|\omega^{(n)}(0)| \leq n!$ for each $n \geq 1$. This gives the estimate

$$|na_n + \epsilon n b_n| \leq 1 \quad \text{for each } n \geq 2.$$

As $|\epsilon| = 1$, triangle inequality gives the proof for part **(a)**.

For the proof of part **(b)**, we observe that

$$|F'(z) - 1| = \left| \sum_{n=2}^{\infty} n a_n z^{n-1} + \epsilon \sum_{n=1}^{\infty} n b_n z^{n-1} \right| < 1, \quad z \in \mathbb{D}.$$

Therefore, with $z = re^{i\theta}$ for $r \in (0, 1)$ and $0 \leq \theta \leq 2\pi$, the last inequality gives

$$\sum_{n=2}^{\infty} n^2(|a_n|^2 + |b_n|^2) r^{2(n-1)} + |b_1|^2 = \frac{1}{2\pi} \int_0^{2\pi} |F'(re^{i\theta}) - 1|^2 \, d\theta \leq 1.$$

Letting $r \to 1^-$, we obtain the inequality

$$\sum_{n=2}^{\infty} n^2(|a_n|^2 + |b_n|^2) \leq 1 - |b_1|^2$$

and the proof is complete. □

3. Proofs of main results

Proof of Lemma 1.2 Let h and g have the form (1) satisfying the coefficient conditions (4). First, we observe that $b_1 = g'(0) = 0$. The conditions (4) imply that the series (1) are convergent in the unit disk $|z| < 1$, and hence, the sum h and g are analytic in $\mathbb{D}$. Thus, $f = h + \overline{g}$ is harmonic in $\mathbb{D}$. Let $0 < r < 1$, we let

$$f_r(z) := r^{-1} f(rz) = r^{-1} h(rz) + r^{-1} \overline{g(rz)}$$

so that $f_r(z) = h_r(z) + \overline{g_r(z)}$ and

$$f_r(z) = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \overline{\sum_{n=2}^{\infty} b_n r^{n-1} z^n}, \; z \in \mathbb{D}.$$

By hypotheses, $|a_n| \leq A_n$ and $|b_n| \leq B_n$ for $n \geq 2$, where A_n and B_n are given by (3). Using these coefficient estimates, we obtain

$$\begin{aligned} S &= \sum_{n=2}^{\infty} n|a_n| r^{n-1} + \sum_{n=2}^{\infty} n|b_n| r^{n-1} \\ &\leq \sum_{n=2}^{\infty} n A_n r^{n-1} + \sum_{n=2}^{\infty} n B_n r^{n-1}. \end{aligned}$$

We show that $f_r \in \mathcal{C}_H^2 \cap \mathcal{S}_H^{*0}$. According to Lemma 2.1, it suffices to show that $S \leq 1$. By the last inequality, $S \leq 1$ if r satisfies the inequality

$$\sum_{n=2}^{\infty} nA_n r^{n-1} \leq 1 - \sum_{n=2}^{\infty} nB_n r^{n-1},$$

or equivalently (as $A_n + B_n = (2n^2 + 1)/3$),

$$2\sum_{n=2}^{\infty} n^3 r^{n-1} + \sum_{n=2}^{\infty} nr^{n-1} \leq 3. \tag{9}$$

As

$$\frac{r}{(1-r)^2} = \sum_{n=1}^{\infty} nr^n \quad \text{and} \quad \frac{r(1+r)}{(1-r)^3} = \sum_{n=1}^{\infty} n^2 r^n,$$

it follows that

$$\frac{(1-r)(1+2r) + 3r(1+r)}{(1-r)^4} = \sum_{n=1}^{\infty} n^3 r^{n-1}$$

and (9) reduces to the inequality,

$$\frac{2(r^2+4r+1)}{(1-r)^4} + \frac{1}{(1-r)^2} \leq 6, \quad \text{i.e.} \;\; 2(1-r)^4 - (1+r)^2 \geq 0.$$

This gives

$$\sqrt{2}(1-r)^2 - (1+r) = \sqrt{2}\,r^2 - (1+2\sqrt{2})\,r + \sqrt{2} - 1 \geq 0.$$

Thus, from Lemma 2.1, f_r is close-to-convex (univalent) in $\mathbb{D}$ and fully starlike in $\mathbb{D}$ for all $0 < r \leq r_S$, where r_S is the root of the quadratic equation

$$\sqrt{2}\,r^2 - (1+2\sqrt{2})\,r + \sqrt{2} - 1 = 0$$

in the interval (0, 1). In particular, f is close-to-convex (univalent) and fully starlike in $|z| < r_S$.

Next, to prove the sharpness part of the statement of the theorem, we consider the function $F_0(z) = H_0(z) + \overline{G_0(z)}$ with

$$H_0(z) = 2z - H(z) \;\; \text{and} \;\; G_0(z) = -\overline{G(z)}.$$

Here, H and G are defined by (2). We note that

$$F_0(z) = z - \sum_{n=2}^{\infty} A_n z^n - \overline{\sum_{n=2}^{\infty} B_n z^n}.$$

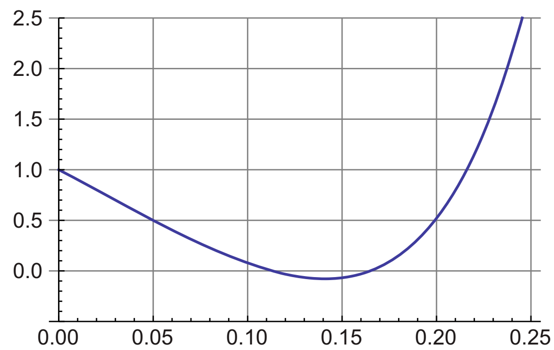

Figure 1. The graph of the Jacobian $J_{F_0}(r)$ for $r \in (0, 0.25)$.

As F_0 has real coefficients we obtain.

$$\begin{aligned} J_{F_0}(r) &= \left(H_0'(r) + G_0'(r)\right)\left(H_0'(r) - G_0'(r)\right) \\ &= \left(1 - \sum_{n=2}^{\infty} nA_n r^{n-1} - \sum_{n=2}^{\infty} nB_n r^{n-1}\right)\left(1 - \sum_{n=2}^{\infty} n(A_n - B_n) r^{n-1}\right) \\ &= \left(1 - \sum_{n=2}^{\infty} \frac{n(2n^2+1)}{3} r^{n-1}\right)\left(1 - \sum_{n=2}^{\infty} n^2 r^{n-1}\right) \\ &= \left(1 - \frac{-4r^2 + 3r^3 - r^4}{(-1+r)^3 r}\right)\left(1 + \frac{-6r^2 + 5r^3 - 4r^4 + r^5}{(-1+r)^4 r}\right) \\ &= \frac{(-1 + 7r - 6r^2 + 2r^3)(1 - 10r + 11r^2 - 8r^3 + 2r^4)}{(-1+r)^7}. \end{aligned}$$

Thus, $J_{F_0}(r) = 0$, $0 < r < 1$ if and only if

$$r = r_S = \frac{1}{4}\left(4 + \sqrt{2} - \sqrt{2 + 16\sqrt{2}}\right) \approx 0.112903$$

or

$$r = r_S' = 1 + \left(-18 + \sqrt{330}\right)^{1/3} 6^{-2/3} - \left(6\left(-18 + \sqrt{330}\right)\right)^{-1/3} \approx 0.164878.$$

Moreover, for $r_S < r < r_S'$, we have $J_{F_0}(r) < 0$. The graph of the function $J_{F_0}(r)$ for $r \in (0, 0.25)$ is shown in Figure 1.

This observation together with Lewy's theorem gives that (as the Jacobian changes sign) the function $F_0(z)$ is not univalent in $|z| < r$ if $r > r_{\mathcal{S}}$ and thus, $r_{\mathcal{S}}$ cannot be replaced by a larger number. □

Proof of Lemma 1.4 Following the notation and the method of the proof of Lemma 1.2, it suffices to show that $f_r \in \mathcal{C}_H^2 \cap \mathcal{S}_H^{*0}$. According to Lemma 2.1, $f_r \in \mathcal{C}_H^2 \cap \mathcal{S}_H^{*0}$ whenever $S \le 1$, where

$$S = \sum_{n=2}^{\infty} n|a_n| r^{n-1} + \sum_{n=2}^{\infty} n|b_n| r^{n-1}$$

when a_n and b_n satisfy the coefficient inequalities given by (6). Finally, using (6), we see that $S \leq 1$ if r satisfies the inequality

$$\sum_{n=2}^{\infty} \frac{n(n+1)}{2} r^{n-1} \leq 1 - \sum_{n=2}^{\infty} \frac{n(n-1)}{2} r^{n-1}.$$

The last inequality is easily seen to be equivalent to

$$\frac{1}{2}\left[\frac{1}{(1-r)^2} + \frac{1+r}{(1-r)^3} - 1\right] \leq 1 + \frac{1}{2}\left[\frac{1}{(1-r)^2} - \frac{1+r}{(1-r)^3} - 1\right]$$

which upon simplification reduces to

$$2(1-r)^3 - 1 - r = -(2r^3 - 6r^2 + 7r - 1) \geq 0.$$

The first part of the conclusion easily follows as in the proof of Lemma 1.2.

The sharpness part of the statement of Lemma 1.4 follows if we consider the function

$$L_0(z) = 2z - M(z) - \overline{N(z)},$$

where M and N are defined by (7). We note that

$$L_0(z) = z - \sum_{n=2}^{\infty} \frac{n+1}{2} z^n + \overline{\sum_{n=2}^{\infty} \frac{n-1}{2} z^n}.$$

Again, as L_0 has real coefficients, we can easily obtain that for $r \in (0, 1)$

$$\begin{aligned} J_{L_0}(r) &= \left(2 - M'(r)\right)^2 - \left(N'(r)\right)^2 \\ &= \left(2 - M'(r) + N'(r)\right)\left((2 - M'(r) - N'(r)\right) \\ &= \left(2 - \frac{1+r}{(1-r)^3}\right)\left(2 - \frac{1}{(1-r)^2}\right) \\ &= \frac{2}{(1-r)^5}\left(2(1-r)^3 - (1+r)\right)\left(r - 1 - \frac{\sqrt{2}}{2}\right)\left(r - 1 + \frac{\sqrt{2}}{2}\right). \end{aligned}$$

We see that $J_{L_0}(r_S) = 0$, $0 < r < 1$ if and only if

$$r = r_S \approx 0.16487$$

or

$$r = r'_S = \frac{2-\sqrt{2}}{2} \approx 0.292893.$$

Moreover, for $r_S < r < r'_S$, we have $J_{L_0}(r) < 0$. The graph of the function $J_{L_0}(r)$ for $r \in (0, 0.35)$ is shown in Figure 2.

Thus, according to Lewy's theorem, $L_0(z)$ is not univalent in $|z| < r$ if $r > r_{\mathcal{S}}$ and this observation shows that $r_{\mathcal{S}}$ cannot be replaced by a larger number. □

Proof of Lemma 1.6 This time, we apply Lemma 2.1 and show that f_r defined by $f_r(z) := r^{-1} f(rz) = r^{-1}h(rz) + r^{-1}\overline{g(rz)}$ belongs to $\mathcal{C}_H^2$.

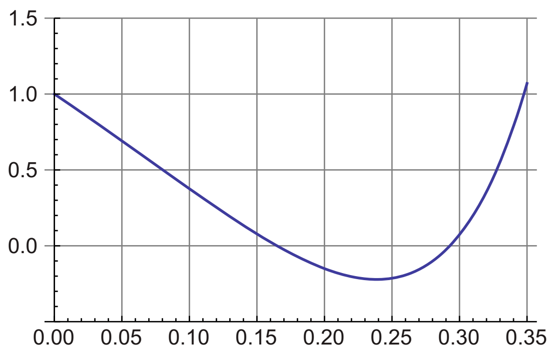

Figure 2. The graph of the Jacobian $J_{L_0}(r)$ for $r \in (0, 0.35)$.

As in the proof of previous two theorems, it suffices to show the corresponding coefficient inequality (8), namely,

$$S = \sum_{n=2}^{\infty} n(|a_n| + |b_n|)r^{n-1} + |b_1| \leq 1.$$

By the hypothesis, $|a_n| + |b_n| \leq c$ for all $n \geq 2$ and so, the last inequality $S \leq 1$ clearly holds if r satisfies the inequality

$$c\left(\frac{1}{(1-r)^2} - 1\right) \leq 1 - |b_1|, \quad \text{i.e. } r \leq r_S = 1 - \sqrt{\frac{c}{c+1-|b_1|}}.$$

Thus, by Lemma 2.1,

$$|h_r'(z) - 1| < 1 - |g_r'(z)|$$

holds for all $z \in \mathbb{D}$ whenever $r \leq r_S$. Thus, $f_r \in \mathcal{C}_H^2$ for $r \leq r_S$.

The function $f_0(z) = h_0(z) + \overline{g_0(z)}$, where

$$h_0(z) = z - \frac{c}{2}\left(\frac{z^2}{1-z}\right) \text{ and } g_0(z) = -|b_1|z - \frac{c}{2}\left(\frac{z^2}{1-z}\right),$$

shows that the result is sharp. Indeed, it is easy to compute that

$$J_{f_0}(r) = |h_0'(r)|^2 - |g_0'(r)|^2 = (1 + |b_1|)\left(1 + c - |b_1| - \frac{c}{(1-r)^2}\right)$$

which shows that $J_{f_0}(r_S) = 0$ and $J_{f_0}(r) < 0$ for $r > r_S$. The proof of the theorem is complete. □

Proof of Theorem 1.7 Let $f = h + \overline{g}$ be a harmonic mapping defined on the unit disk $\mathbb{D}$ with $f(0) = f_{\overline{z}}(0) = f_z(0) - 1 = 0$, and $|f(z)| < M$ for $z \in \mathbb{D}$, where h and g have the form (1) with $b_1 = 0$. According to [4, Lemma 1] (see also [14]), we obtain the sharp estimate

$$|a_n| + |b_n| \leq \frac{4M}{\pi} \quad \text{for any } n \geq 1. \tag{10}$$

As $b_1 = 0$ and $a_1 = 1$, it follows that $M \geq \pi/4 \approx 0.785398$. By Lemma 1.6 with $c = 4M/\pi$, we conclude that f is close-to-convex and fully starlike (because $b_1 = 0$) for $|z| < 1 - \sqrt{c/(c+1)} = r_0$.

In particular, f is univalent for $|z| < r_0$ and furthermore, we have for $|z| = r_0$,

$$\begin{aligned} |f(z)| &= \left| z + \sum_{n=2}^{\infty} \left(a_n z^n + \overline{b_n z^n} \right) \right| \\ &\geq |z| - \left| \sum_{n=2}^{\infty} \left(a_n z^n + \overline{b_n z^n} \right) \right| \\ &\geq r_0 - \sum_{n=2}^{\infty} (|a_n| + |b_n|) r_0^n \\ &\geq r_0 - \frac{4M}{\pi} \sum_{n=2}^{\infty} r_0^n \\ &= r_0 - \frac{4M}{\pi} \frac{r_0^2}{1 - r_0} = R_0 \end{aligned}$$

and the proof is complete. □

Acknowledgements

The authors thank the referee for many helpful comments. The research of the second author was supported by the Academy of Finland, Project No. 2600066611, coordinated by the third author. The original article of the authors from `http://arxiv.org/abs/1107.0610` does contain some general results. The second author is currently at Indian Statistical Institute (ISI), Chennai Centre, SETS (Society for Electronic Transactions and security), MGR Knowledge City, CIT Campus, Taramani, Chennai 600 113, India.

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