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On quasiregular mappings between smooth Jordan domains

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ABSTRACT

It is proved that every proper quasiregular C^2 mapping w between two plane Jordan domains $\Omega \in C^{1,\alpha}$ and $G \in C^{2,\alpha}$, $0 < \alpha \leqslant 1$, satisfying the differential inequality $|\Delta w| \leqslant M|\nabla w|^2 + N$ is Lipschitz continuous. This extends the main result of the author and M. Mateljević (Kalaj and Mateljević, 2006 [7]).

1. Introduction

By $\mathbb{U}$ we denote the unit disk, and by S^1 its boundary. Also, by Ω and G we denote domains in $\mathbb{C} = \mathbb{R}^2$.

Assume a twice differentiable mapping

$$w(x, y) = \big(u(x, y), v(x, y)\big) : \Omega \mapsto \mathbb{R}^2$$

satisfies the following differential inequality:

$$|\Delta w| \leqslant a|\nabla w|^2 + b, \tag{1.1}$$

where $a, b > 0$, $\Delta w = D_{xx}w + D_{yy}w$, ∇w is the matrix defined by

$$\nabla w = \begin{pmatrix} D_x u & D_y u \\ D_x v & D_y v \end{pmatrix},$$

and $|\nabla w(z)| = \sup\{|\nabla w(z)\zeta|\colon \zeta \in S^1\}$, $z = (x, y) = x + iy$.

We are going to study some regularity behaviour of solutions of inequality (1.1), providing that they are quasiconformal. This inequality has been introduced by S. Bernstein in [2] and has been studied by E. Heinz in [5]. It has been motivated by the Dirichlet problem for the system

$$\Delta w = Q\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, w, x, y\right); \quad w = \big(u(x, y), v(x, y)\big), \tag{1.2}$$

where $Q = (Q_1, Q_2)$, and Q_j, $j = 1, 2$, are quadratic polynomials in the quantities $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial y}$ with the coefficients depending on w and $(x, y) \in \Omega$. An example of the system (1.2) is the system of differential equations that presents a regular surface S with fixed mean curvature H with respect of (x, y). Also it is important in the connection with the Monge–Ampére equation.

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It is easy to see that any solution $w = (u, v)$ of system (1.2) satisfies partial differential inequality (1.1).

A homeomorphism (continuous mapping) $w : \Omega \to G$ between two open subsets Ω and G of Euclidean space $\mathbb{R}^2$ will be called a K ($K \geqslant 1$) *quasiconformal* (*quasiregular*) mappings.

(i) w is absolutely continuous function on every segment parallel to some of the coordinate axis and there exist partial derivatives which are locally L^2 integrable functions on Ω. We will write $w \in ACL^2$ and
(ii) w satisfies the condition

$$\left|\nabla w(z)\right|^2 \leqslant K J_w(z) \tag{1.3}$$

for almost every z in Ω, where $J_w(z)$ is the Jacobian determinant of w.

We shall use "q.c." ("q.r.") as an abbreviation for "quasiconformal" ("quasiregular").

Notice that the condition $w \in ACL^2$ guarantees the existence of first derivatives of w almost everywhere in Ω (see [1]). Moreover, if w is a quasiregular mapping then

$$J_w(z) = \det\bigl(\nabla w(z)\bigr) \neq 0 \quad \text{for a.e. } z \in \Omega. \tag{1.4}$$

Notice that, for a continuous mapping w, the condition (i) is equivalent with the fact that w is continuous and belongs to the Sobolev space $W^1_{2,\mathrm{loc}}(\Omega)$.

Notice that, in complex notation (1.3) can be written as

$$\bigl(|w_z| + |w_{\bar z}|\bigr)^2 \leqslant K\bigl(|w_z|^2 - |w_{\bar z}|^2\bigr) \tag{1.5}$$

or in its equivalent form

$$\frac{|w_{\bar z}|}{|w_z|} \leqslant k := \frac{K-1}{K+1},$$

where w_z and $w_{\bar z}$ are complex partial derivatives of w:

$$w_z = \frac{1}{2}(w_x - i w_y) \quad \text{and} \quad w_{\bar z} = \frac{1}{2}(w_x + i w_y).$$

The above definition can be modified to the class of q.c. (q.r.) mappings in $\mathbb{R}^n$.

This paper has been motivated by the following theorem.

Theorem 1.1. *(See [7].) Let w be a quasiconformal C^2 diffeomorphism from a $C^{1,\alpha}$ Jordan domain Ω onto a $C^{2,\alpha}$ Jordan domain G. If there exists a constant M such that*

$$|\Delta w| \leqslant M|w_z \cdot w_{\bar z}|, \quad z \in \Omega, \tag{1.6}$$

then w has bounded partial derivatives. In particular, it is a Lipschitz mapping.

See [7–11,13,14] for additional boundary regularity behaviour of quasiconformal harmonic mappings. Observe that inequality (1.6) implies inequality (1.1). Observe also that, every quasiconformal mapping is quasiregular.

Our idea is to extend Theorem 1.1 to the class of quasiregular mappings, which are subject to more general conditions (see below inequality (1.7)). However, to do this we must restrict to the class of proper mappings. A function $f : \Omega \to G$ is *proper* if the preimage of every compact set in G is compact in Ω.

Since analytic mappings are quasiregular, quasiregular mappings in $\mathbb{R}^2$ can be smooth without being locally invertible. In contrast, it has been known for some time that every quasiregular mapping $f : G \to \mathbb{R}^n$ that is C^3-smooth if $n = 3$, and C^2-smooth if $n \geqslant 4$, must be locally invertible (see e.g. [15] and [3] for this argument). If f is a proper locally homeomorphic mapping between Jordan domains (i.e. domains homeomorphic to the unit ball), it is a covering map and hence a homeomorphism since the target domain is simply connected. Thus, under C^3 smoothness, only in the plane there exist q.r. mappings that are not q.c.

Our main result is the following extension of Theorem 1.1:

Theorem 1.2 *(The main result). Let w be a proper quasiregular C^2 mapping of a Jordan domain Ω with $C^{1,\alpha}$ boundary onto a Jordan domain G with $C^{2,\alpha}$ boundary. If there exist constants M and N such that*

$$|\Delta w| \leqslant M|\nabla w|^2 + N, \quad z \in \Omega, \tag{1.7}$$

then w has bounded partial derivatives in Ω. In particular w is a Lipschitz mapping in Ω.

Its proof is given in the following section. It depends on the Heinz–Bernstein theorem (Proposition 2.6). The method of its proof, presented here, differs from the method of the proof of Theorem 1.1 presented in [7].

2. The proof of main results

Lemma 2.1. *Every real non-singular* 2×2 *matrix* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *satisfies the equality*

$$|\tilde{A}| = |A|, \tag{2.1}$$

where $\tilde{A}$ *is adjoint matrix of* A, *i.e.* $\tilde{A} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.

Proof. First of all

$$|A| = \max\bigl\{|Ah|\colon\ |h| = 1\bigr\} = |A_z| + |A_{\bar{z}}|,$$

where $A_z = \frac{1}{2}(Ae_1 - iAe_2)$ and $A_{\bar{z}} = \frac{1}{2}(Ae_1 + iAe_2)$. On the other hand

$$A^{-1} = \frac{1}{\det A}\tilde{A}$$

and

$$\det A = |A_z|^2 - |A_{\bar{z}}|^2.$$

From

$$\bigl|A^{-1}\bigr| = \frac{1}{\min\{|Ah|\colon\ |h| = 1\}} = \frac{1}{|A_z| - |A_{\bar{z}}|}$$

it follows (2.1). □

Lemma 2.2. *Every* K*-q.r. mapping* $w(z) = \rho(z)S(z) : \Omega \to G$, $\Omega, G \subset \mathbb{C}$, $\rho = |w|$, $S(z) = e^{is(z)}$, $s(z) \in [0, 2\pi)$, *satisfies the inequalities*

$$\rho|\nabla S| \leqslant K|\nabla\rho| \tag{2.2}$$

and

$$|\nabla\rho| \leqslant K\rho|\nabla S| \tag{2.3}$$

almost everywhere on Ω. *Inequalities* (2.2) *and* (2.3) *are sharp*: *the equality*

$$\rho|\nabla S| = |\nabla\rho| \tag{2.4}$$

holds if w *is a* 1*-quasiregular mapping. We also have*

$$K^{-1}|\nabla w| \leqslant |\nabla\rho| \leqslant |\nabla w|. \tag{2.5}$$

Proof. From $\rho = \langle w, w\rangle^{1/2}$ and $S = w \cdot \langle w, w\rangle^{-1/2}$ we obtain

$$\nabla\rho = (\nabla w)^t \frac{w}{|w|} \quad \text{and} \quad \nabla S = \frac{\nabla w}{|w|} - \bigl((\nabla w)^t \cdot w\bigr) \otimes \frac{w}{|w|^3},$$

i.e. for $h \in \mathbb{R}^2$:

$$\nabla\rho h = \frac{\langle \nabla wh, w\rangle}{|w|} \tag{2.6}$$

and

$$\nabla Sh = \frac{\nabla wh}{|w|} - \frac{w\langle \nabla wh, w\rangle}{|w|^3}. \tag{2.7}$$

From (2.6) we obtain

$$|\nabla\rho| = \max_{|h|=1}|\nabla\rho h| = \max_{|h|=1}\frac{\langle \nabla wh, w\rangle}{|w|}.$$

Choose h_1 such that

$$\nabla wh_1 = \frac{w}{|w|}.$$

Then

$$\nabla\rho h_1 = \frac{\langle \frac{w}{|w|}, w\rangle}{|w|} = 1.$$

Thus for $h = \frac{h_1}{|h_1|}$ we obtain $\nabla\rho h = \frac{1}{|h_1|}$. Hence, according to (1.4)

$$|\nabla\rho| \geqslant \left|(\nabla w)^{-1}\frac{w}{|w|}\right|^{-1}$$

almost everywhere. Employing now (2.1) we have

$$|\nabla\rho| \geqslant \left|(\nabla w)^{-1}\right|^{-1} = |\det\nabla w||\widetilde{\nabla w}|^{-1} = |\det\nabla w|\cdot|\nabla w|^{-1}.$$

Thus

$$\frac{1}{|\nabla\rho|} \leqslant \frac{|\nabla w|}{|\det\nabla w|}. \tag{2.8}$$

On the other hand according to (2.7) we obtain

$$\rho^2|\nabla S h|^2 = |\nabla w h|^2 - \left\langle \nabla w h, \frac{w}{|w|}\right\rangle^2 \leqslant |\nabla w|^2|h|^2. \tag{2.9}$$

Thus

$$\rho|\nabla S| \leqslant |\nabla w|. \tag{2.10}$$

Combining (2.8) and (2.10) we obtain

$$\frac{\rho|\nabla S|}{|\nabla\rho|} \leqslant \frac{|\nabla w|^2}{|\det\nabla w|} \leqslant K. \tag{2.11}$$

This yields (2.2). From (2.6) we obtain

$$|\nabla\rho| \leqslant |\nabla w|. \tag{2.12}$$

Choose h_1: $|h_1| = 1$ so that $\langle\nabla w h_1, \frac{w}{|w|}\rangle = 0$. Then by (2.9) we obtain

$$\rho^2|\nabla S h_1|^2 = |\nabla w h_1|^2. \tag{2.13}$$

In view of the definition (see (1.5)), it follows that

$$\rho|\nabla S| \geqslant \min\bigl\{|\nabla w h|:\ |h| = 1\bigr\} = |w_z| - |w_{\bar z}| \geqslant (J_w/K)^{1/2}. \tag{2.14}$$

Combining (2.12) and (2.14), we obtain

$$\frac{|\nabla\rho|}{\rho|\nabla S|} \leqslant K^{1/2}\left(\frac{|\nabla w|^2}{J_w}\right)^{1/2} \leqslant K.$$

This gives (2.3). To continue, observe that

$$\nabla w = (\nabla\rho)^t S + \rho\nabla S$$

and thus

$$|\nabla w h|^2 = |\rho\nabla S h|^2 + |\nabla\rho h\cdot S|^2 + 2\rho\nabla\rho h\langle\nabla S h, S\rangle.$$

Hence

$$|\nabla w h|^2 = \rho^2|\nabla S h|^2 + |\nabla\rho h|^2. \tag{2.15}$$

Choose h_1: $|h_1| = 1$ so that $\nabla S h_1 = 0$. Then, by (2.15), we infer

$$|\nabla w h_1| \leqslant |\nabla\rho h_1|.$$

According to the definition of quasiregular mappings we obtain

$$K^{-1}|\nabla w| \leqslant |\nabla\rho|. \qquad \square$$

Remark 2.3. The previous proof applies as well to the class of q.r. mappings in n-dimensional space, and instead of (2.2) and (2.3) there hold

$$\rho|\nabla S| \leqslant \frac{\sqrt{n}K(K+\sqrt{K^2-1})}{\sqrt{n-1+(K+\sqrt{K^2-1})^2}}|\nabla\rho| \tag{2.16}$$

and

$$|\nabla\rho| \leqslant K^{2/n}\rho|\nabla S|. \tag{2.17}$$

Lemma 2.4. *(See [6].) For every twice differentiable mapping $w = \rho S : \Omega \to G$, $\Omega, G \subset \mathbb{C}$, $\rho = |w|$, $S = w/|w|$, we have*

$$\Delta\rho = 2\rho|\nabla S|^2 + \frac{1}{2}\langle \Delta w, S\rangle. \tag{2.18}$$

To prove the main theorem we need the following propositions:

Proposition 2.5. *(Kellogg and Warschawski, see [4,12,16,17].) If Ω and G are Jordan domains having $C^{l,\alpha}$ $(l \geqslant 1)$ boundary and if ω is a conformal mapping of Ω onto G, then*:

(a) $|\omega'(z)| \geqslant \inf\{|\omega'(\zeta)|:\ \zeta \in \Omega\} > 0$ *for* $z \in \Omega$,
(b) $\omega^{(l)} \in C^\alpha(\overline{\Omega})$. *In particular* $|\omega^{(l)}|_\infty := \sup\{|\omega^{(l)}(z)|:\ z \in \Omega\} < \infty$.

Proposition 2.6 *(Interior estimate). (Heinz and Bernstein, see [5] and [2].) Let $s : \overline{\mathbb{U}} \to \mathbb{R}$ be a continuous function from the closed unit disc $\overline{\mathbb{U}}$ into the real line satisfying the conditions*:

(1) *s is C^2 on $\mathbb{U}$,*
(2) *$s_b(\theta) = s(e^{i\theta})$ is C^2, and*
(3) *$|\Delta s| \leqslant M_0|\nabla s|^2 + N_0$, on $\mathbb{U}$ for some constants M_0 and N_0.*

Then the function $|\nabla s|$ is bounded on $\mathbb{U}$.

Proof of Theorem 1.2. Let ω be a conformal mapping of the unit disc onto Ω and let g be a conformal mapping of G onto the unit disk. Let $\tau = g \circ w \circ \omega$. Then τ is a K quasiregular self-mapping of the unit disk and

$$\Delta\tau = (4g'' \cdot w_z w_{\bar z} + g'\Delta w)|\omega'|^2 = \left(\frac{4g'' \cdot \tau_z\tau_{\bar z}}{g'^2} + g'|\omega'|^2\Delta w\right). \tag{2.19}$$

Next, we have

$$|4\tau_z\tau_{\bar z}| \leqslant |\nabla\tau|^2. \tag{2.20}$$

By (1.7) it follows that

$$|\Delta w| \leqslant M|\nabla w|^2 + N \leqslant M\frac{|\nabla\tau|^2}{|g'|^2|\omega'|^2} + N. \tag{2.21}$$

From (2.19)–(2.21) we have

$$|\Delta\tau| \leqslant \left(\frac{|g''|_\infty}{(\inf\{|g'(z)|,\ z \in G\})^2} + \frac{M}{\inf\{|g'(z)|,\ z \in G\}}\right)|\nabla\tau|^2 + N|g'|_\infty|\omega'|^2_\infty. \tag{2.22}$$

Let $\rho = |\tau|$. Combining (2.18), (2.2), (2.22), (2.5) and Proposition 2.5 we obtain

$$|\Delta\rho| \leqslant \frac{N_1}{\rho}|\nabla\rho|^2 + M_1,$$

where

$$N_1 = 2K^2 + \frac{K^2|g''|_\infty}{2(\inf\{|g'(z)|,\ z \in G\})^2} + \frac{K^2 M}{2\inf\{|g'(z)|,\ z \in G\}} < \infty,$$

and

$$M_1 = \frac{|g'|_\infty|\omega'|^2_\infty N}{2} < \infty.$$

Since w is a proper quasiregular mapping, it follows that τ is a proper self-mapping of the unit disk. Thus $\lim_{|z|\to 1}\rho(z)=1$ uniformly on S^1, which means that ρ has a continuous (constant) extension up to the boundary. Therefore, there exists an $r>0$ such that $r\leqslant |z|\leqslant 1$ implies $\rho(z)>1/2$. Let $\tilde{\rho}$ be an extension of the function $\rho|_{r<|z|<1}$ in $\mathbb{U}$ (by Whitney theorem it exists, see e.g. [18]). Let

$$M_0=\max\{|\Delta\tilde{\rho}(z)|:\ |z|\leqslant (r+1)/2\}.$$

Then

$$|\Delta\tilde{\rho}|\leqslant 2N_1|\nabla\tilde{\rho}|^2+M_1+M_0.$$

Thus the conditions of Proposition 2.6 are satisfied. The conclusion is that $\nabla\tilde{\rho}$ is bounded. According to inequality (2.5) $\nabla\tau$ is bounded in $r\leqslant|z|<1$ and hence in $\mathbb{U}$ as well. Since $w=g^{-1}\circ\tau\circ\omega^{-1}$ it follows that $|\nabla w|=|\nabla\tau||g'\omega'|^{-1}$. Applying Proposition 2.5 again, we obtain that ∇w is bounded on Ω. $\square$

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