

**ON BOUNDARY CORRESPONDENCE OF Q.C.  
HARMONIC MAPPINGS BETWEEN SMOOTH JORDAN  
DOMAINS**

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ABSTRACT. A quantitative version of an inequality obtained in [8, Theorem 2.1] is given. More precisely, for normalized  $K$  quasiconformal harmonic mappings of the unit disk onto a Jordan domain  $\Omega \in C^{1,\mu}$  ( $0 < \mu \leq 1$ ) we give an explicit Lipschitz constant depending on the structure of  $\Omega$  and on  $K$ . In addition we give a characterization of q.c. harmonic mappings of the unit disk onto an arbitrary Jordan domain with  $C^{2,\alpha}$  boundary in terms of boundary function using the Hilbert transformations. Moreover it is given a sharp explicit quasiconformal constant in terms of the boundary function.

1. INTRODUCTION AND AUXILIARY RESULTS

Let  $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ . We will consider the matrix norm:

$$|A| = \max\{|Az| : z \in \mathbf{R}^2, |z| = 1\}$$

and the matrix function

$$l(A) = \min\{|Az| : |z| = 1\}.$$

Let  $w = u + iv : D \rightarrow G$ ,  $D, G \subset \mathbf{C}$ , have partial derivative at  $z \in D$ . By  $\nabla w(z)$  we denote the matrix  $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ . For the matrix  $\nabla w$  we have

$$(1.1) \quad |\nabla w| = |w_z| + |w_{\bar{z}}|$$

and

$$l(\nabla w) = ||w_z| - |w_{\bar{z}}||,$$

where

$$w_z := \frac{1}{2} \left( w_x + \frac{1}{i} w_y \right) \quad \text{and} \quad w_{\bar{z}} := \frac{1}{2} \left( w_x - \frac{1}{i} w_y \right).$$

A sense-preserving homeomorphism  $w : D \rightarrow G$ , where  $D$  and  $G$  are subdomains of the complex plane  $\mathbf{C}$ , is said to be  $K$ -quasiconformal (K-q.c),

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$K \geq 1$ , if  $w$  is absolutely continuous on a.e. horizontal and a.e. vertical line and

$$(1.2) \quad |\nabla w| \leq Kl(\nabla w) \quad \text{a.e. on } D.$$

Notice that, condition (1.2) can be written as

$$|w_{\bar{z}}| \leq k|w_z| \quad \text{a.e. on } D \text{ where } k = \frac{K-1}{K+1} \text{ i.e. } K = \frac{1+k}{1-k},$$

or in its equivalent form

$$(1.3) \quad \left| \frac{\partial w}{\partial r} \right|^2 + \frac{1}{r^2} \left| \frac{\partial w}{\partial \varphi} \right|^2 \leq \frac{1}{2} \left( K + \frac{1}{K} \right) J_w \quad (z = re^{i\varphi}),$$

where  $J_w$  is the Jacobian of  $w$  (cf. [1], pp. 23-24). Finally the last is equivalent to:

$$\frac{1}{K} \leq \left| \frac{r\partial w}{\partial r} \right| \leq K.$$

This implies the inequality

$$(1.4) \quad \frac{1}{r^2} \left( 1 + \frac{1}{K^2} \right) \left| \frac{\partial w}{\partial \varphi} \right|^2 \leq K J_w \quad (z = re^{i\varphi}).$$

A function  $w$  is called *harmonic* in a region  $D$  if it has form  $w = u + iv$  where  $u$  and  $v$  are real-valued harmonic functions in  $D$ . If  $D$  is simply-connected, then there are two analytic functions  $g$  and  $h$  defined on  $D$  such that  $w$  has the representation

$$w = g + \bar{h}.$$

If  $w$  is a harmonic univalent function, then by Lewy's theorem (see [14]),  $w$  has a non-vanishing Jacobian and consequently, according to the inverse mapping theorem,  $w$  is a diffeomorphism. If  $k$  is an analytic function and  $w$  is a harmonic function then  $w \circ k$  is harmonic. However  $k \circ w$ , in general is not harmonic.

Let

$$P(r, x - \varphi) = \frac{1 - r^2}{2\pi(1 - 2r \cos(x - \varphi) + r^2)}$$

denote the Poisson kernel. Then every bounded harmonic function  $w$  defined on the unit disc  $\mathbf{U} := \{z : |z| < 1\}$  has the following representation

$$(1.5) \quad w(z) = P[w_b](z) = \int_0^{2\pi} P(r, x - \varphi) w_b(e^{ix}) dx,$$

where  $z = re^{i\varphi}$  and  $w_b$  is a bounded integrable function defined on the unit circle  $S^1 := \{z : |z| = 1\}$ .

In this paper we continue to establish Lipschitz and co-Lipschitz character of q.c. harmonic mappings between smooth domains. This class contains conformal mappings. The conformal case is well-known ([13], [23], [21], [3],

[18]) but it seems only here we yield an explicit constant even for conformal case.

The first result in the area of q.c. harmonic mappings was established by O. Martio ([16]). Recently there are several papers with deals with topic ([4]-[10], [19]-[20]). See also [22] for the similar problem of hyperbolic q.c. harmonic mappings of the unit disk.

It is worth to mention the following fact, q.c. harmonic mappings share with conformal mappings the following property (a result of M. Mateljevic and P. Pavlovic). This property do not satisfy hyperbolic q.c. harmonic mappings of the unit disk onto itself.

**Proposition 1.1.** *If  $w = P[f]$  is a q.c. harmonic mapping of the unit disk onto a Jordan domain  $\Omega$  with rectifiable boundary, then  $f$  is an absolutely continuous function.*

The proof can be found in [20], [19] or [11]. We will use Proposition 1.1 implicitly in our main Theorems 2.1 and 3.1.

Some of the notations are taken from [8]. Let  $\gamma \in C^{1,\mu}$ ,  $0 < \mu \leq 1$ , be a Jordan curve such that the interior of  $\gamma$  contains the origin and let  $g$  be the arc length parameterization of  $\gamma$ . Then  $|g'(s)| = 1$ . Let

$$(1.6) \quad K(s, t) = \operatorname{Re} [\overline{(g(t) - g(s))} \cdot ig'(s)]$$

be a function defined on  $[0, l] \times [0, l]$ . Denote by  $K$  its periodic extension to  $\mathbf{R}^2$  ( $K(s + nl, t + ml) = K(s, t)$ ,  $m, n \in \mathbf{Z}$ ).

Since  $K(s + nl, t + ml) = K(s, t)$ ,  $m, n \in \mathbf{Z}$ , it follows from [8, Lemma 1.1] that

$$(1.7) \quad |K(s, t)| \leq C_\gamma d_\gamma(g(s), g(t))^{1+\mu},$$

for

$$(1.8) \quad C_\gamma = \frac{1}{1 + \mu} \sup_{s \neq t} \frac{|g'(s) - g'(t)|}{(s - t)^\mu}$$

and  $d_\gamma$  is the distance between  $g(s)$  and  $g(t)$  along the curve  $\gamma$  i.e.

$$(1.9) \quad d_\gamma(g(s), g(t)) = \min\{|s - t|, (l - |s - t|)\}.$$

Using (1.7) and following the same lines as in the proof of [8, Lemma 2.7] we obtain the following lemma.

**Lemma 1.2.** *Let  $w = P[f](z)$  be a Lipschitz continuous harmonic function between the unit disk  $\mathbf{U}$  and a Jordan domain  $\Omega$ , such that  $f$  is injective, and  $\partial\Omega = f(S^1) \in C^{1,\mu}$ . Then for almost every  $e^{i\varphi} \in S^1$  one has*

$$(1.10) \quad \limsup_{r \rightarrow 1-0} J_w(re^{i\varphi}) \leq C_\gamma |f'(\varphi)| \int_{-\pi}^{\pi} \frac{d_\gamma(f(e^{i(\varphi+x)}), f(e^{i\varphi}))^{1+\mu}}{x^2} dx,$$

where  $J_w$  denotes the Jacobian of  $w$  at  $z$ , and  $f'(\varphi) := \frac{d}{d\varphi} f(e^{i\varphi})$ .

A closed rectifiable Jordan curve  $\gamma$  enjoys a  $B$ -chord-arc condition for some constant  $B > 1$  if for all  $z_1, z_2 \in \gamma$  there holds the inequality

$$(1.11) \quad d_\gamma(z_1, z_2) \leq B|z_1 - z_2|.$$

It is clear that if  $\gamma \in C^{1,\alpha}$  then  $\gamma$  enjoys a chord-arc condition for some  $B_\gamma > 1$ .

We will say that the q.c. mapping  $f : \mathbf{U} \rightarrow \Omega$  is normalized if  $f(1) = w_0$ ,  $f(e^{2\pi/3i}) = w_1$  and  $f(e^{-2\pi/3i}) = w_2$ , where  $w_0w_1$ ,  $w_1w_2$  and  $w_2w_0$  are arcs of  $\gamma = \partial\Omega$  having the same length  $|\gamma|/3$ .

The following lemma is a quasiconformal version of [23, Lemma 1]. Moreover, here we give an explicit Hölder constant  $L_\gamma(K)$ .

**Lemma 1.3.** *Assume that  $\gamma$  enjoys a chord-arc condition for some  $B$ . Then for every  $K$ -q.c. normalized mapping  $f$  between the unit disk  $\mathbf{U}$  and the Jordan domain  $\Omega = \text{int}\gamma$  there holds*

$$|f(z_1) - f(z_2)| \leq L_\gamma(K)|z_1 - z_2|^\alpha$$

for  $z_1, z_2 \in S^1$ ,  $\alpha = \frac{1}{K(1+2B)^2}$  and  $L_\gamma(K) = 4(1+2B)2^\alpha \sqrt{\frac{K|\Omega|}{\pi \log 2}}$ .

*Proof.* For  $a \in \mathbf{C}$  and  $r > 0$ ,  $D(a, r) := \{z : |z - a| < r\}$ . It is clear that if  $z_0 \in S^1 := \partial\mathbf{U}$ , then, because of normalization,  $f(S^1 \cap \overline{D(z_0, 1)})$  has common points with at most two of three arcs  $w_0w_1$ ,  $w_1w_2$  and  $w_2w_0$ . (Here  $w_0, w_1, w_2 \in \gamma$  divide  $\gamma$  into three arcs with the same length such that  $f(1) = w_0$ ,  $f(e^{2\pi i/3}) = w_1$ ,  $f(e^{4\pi i/3}) = w_2$ , and  $S^1 \cap \overline{D(z_0, 1)}$  do not intersect at least one of three arcs defined by  $1, e^{2\pi i/3}$  and  $e^{4\pi i/3}$ ).

Let  $k_\rho$  denotes the arc of the circle  $|z - z_0| = \rho < 1$  which lies in  $|z| \leq 1$  and let  $l_\rho = |f(k_\rho)|$ .

Let  $\gamma_\rho := f(S^1 \cap D(z_0, \rho))$  and let  $|\gamma_\rho|$  be its length. Assume  $w$  and  $w'$  are the endpoints of  $\gamma_\rho$  i.e. of  $f(k_\rho)$ . Then  $|\gamma_\rho| = d_\gamma(w, w')$  or  $|\gamma_\rho| = |\gamma| - d_\gamma(w, w')$ . If the first case hold, then since  $\gamma$  enjoys the  $B$ -chord-arc condition, it follows  $|\gamma_\rho| \leq B|w - w'| \leq Bl_\rho$ . Consider now the last case. Let  $\gamma'_\rho = \gamma \setminus \gamma_\rho$ . Then  $\gamma'_\rho$  contains one of the arcs  $w_0w_1, w_1w_2$  and  $w_2w_0$ . Thus  $|\gamma_\rho| \leq 2|\gamma'_\rho|$ , and therefore

$$|\gamma_\rho| \leq 2Bl_\rho.$$

On the other hand, by using (1.1), polar coordinates and the Cauchy-Schwartz inequality, we have

$$\begin{aligned} l_\rho^2 &= |f(k_\rho)|^2 = \left( \int_{k_\rho} |f_z dz + f_{\bar{z}} d\bar{z}| \right)^2 \\ &\leq \left( \int_{k_\rho} |\nabla f(z_0 + \rho e^{i\varphi})| \rho d\varphi \right)^2 \\ &\leq \int_{k_\rho} |\nabla f(z_0 + \rho e^{i\varphi})|^2 \rho d\varphi \cdot \int_{k_\rho} \rho d\varphi. \end{aligned}$$

Since  $l(k_\rho) \leq 2\rho\pi/2$ , for  $r \leq 1$ , denoting  $\Delta_r = \mathbf{U} \cap D(r, z_0)$ , we have

$$(1.12) \quad \begin{aligned} \int_0^r \frac{l_\rho^2}{\rho} d\rho &\leq \int_0^r \int_{k_\rho} |\nabla f(z_0 + \rho e^{i\varphi})|^2 \rho d\varphi d\rho \\ &\leq K \int_0^r \int_{k_\rho} J_f(z_0 + \rho e^{i\varphi}) \rho d\varphi d\rho = \pi A(r)K, \end{aligned}$$

where  $A(r)$  is the area of  $f(\Delta_r)$ . Using the first part of the proof it follows that, the length of boundary arc  $\gamma_r$  of  $f(\Delta_r)$  does not exceed  $2Bl_r$  which, according to the fact  $\partial f(\Delta_r) = \gamma_r \cup f(k_r)$ , implies  $|\partial f(\Delta_r)| \leq l_r + 2Bl_r$ . Therefore by the isoperimetric inequality

$$A(r) \leq \frac{|\partial f(\Delta_r)|^2}{4\pi} \leq \frac{(l_r + 2Bl_r)^2}{4\pi} = l_r^2 \frac{(1 + 2B)^2}{4\pi}.$$

Employing now (1.12) we obtain

$$F(r) := \int_0^r \frac{l_\rho^2}{\rho} d\rho \leq Kl_r^2 \frac{(1 + 2B)^2}{4}.$$

Observe that for  $0 < r \leq 1$  there hold the relation  $rF'(r) = l_r^2$ . Thus

$$F(r) \leq KrF'(r) \frac{(1 + 2B)^2}{4}.$$

It follows that, for

$$\alpha = \frac{2}{K(1 + 2B)^2}$$

there holds

$$\frac{d}{dr} \log(F(r) \cdot r^{-2\alpha}) \geq 0$$

i.e. the function  $F(r) \cdot r^{-2\alpha}$  is increasing. This yields

$$F(r) \leq F(1)r^{2\alpha} \leq K \frac{|\Omega|}{2\pi} r^{2\alpha}.$$

Now there exists for every  $r \leq 1$  an  $r_1 \in [r/\sqrt{2}, r]$  such that

$$F(r) = \int_0^r \frac{l_\rho^2}{\rho} d\rho \geq \int_{r/\sqrt{2}}^r \frac{l_\rho^2}{\rho} d\rho = l_{r_1}^2 \log \sqrt{2}.$$

Hence

$$l_{r_1}^2 \leq K \frac{|\Omega|}{\pi \log 2} r^{2\alpha}.$$

Thus if  $z$  is a point of  $|z| \leq 1$  with  $|z - z_0| = r/\sqrt{2}$ , then

$$|f(z) - f(z_0)| \leq (1 + 2B)l_r \leq (1 + 2B)l_{r_1}.$$

Therefore

$$|f(z) - f(z_0)| \leq H|z - z_0|^\alpha,$$

where

$$H = (1 + 2B)2^\alpha \sqrt{\frac{K|\Omega|}{\pi \log 2}}.$$

Thus we have for  $z_1, z_2 \in S^1$  the inequality

$$(1.13) \quad |f(z_1) - f(z_2)| \leq 4H|z_1 - z_2|^\alpha.$$

□

**Remark 1.4.** By applying Lemma 1.3, and by using the Möbius transformations, it follows that, if  $f$  is arbitrary conformal mapping between the unit disk  $\mathbf{U}$  and  $\Omega$ , where  $\Omega$  satisfies the conditions of Lemma 1.3, then  $|f(z_1) - f(z_2)| \leq C(f, \gamma)K|z_1 - z_2|^\alpha$  on  $S^1$ .

## 2. QUANTITATIVE BOUND FOR LIPSCHITZ CONSTANT

The aim of this section is to prove Theorem 2.1. This is a quantitative version of [8, Theorem 2.1]. Notice that, the proof presented here is direct (it does not depend on Kellogg's nor on Lindelöf theorem on the theory of conformal mappings (see [3] for this topic)).

**Theorem 2.1.** *Let  $w = P[f](z)$  be a harmonic normalized  $K$  quasiconformal mapping between the unit disk and the Jordan domain  $\Omega$ . If  $\gamma = \partial\Omega \in C^{1,\mu}$ , then there exists a constant  $L = L(\gamma, K)$  (which satisfies the inequality (2.8) below) such that*

$$(2.1) \quad |f'(\varphi)| \leq L \text{ for almost every } \varphi \in [0, 2\pi],$$

and

$$(2.2) \quad |w(z_1) - w(z_2)| \leq KL|z_1 - z_2| \text{ for } z_1, z_2 \in \mathbf{U}.$$

*Proof.* Assume first that  $w = P[f]$  is Lipschitz and thus

$$\operatorname{ess\,sup}_{0 \leq \theta \leq 2\pi} |f'(\theta)| < \infty.$$

It follows that

$$(2.3) \quad \frac{\partial w}{\partial \varphi}(z) = P[f'](z).$$

Therefore for  $\varepsilon > 0$  there exists  $\varphi$  such that

$$(2.4) \quad \left| \frac{\partial w}{\partial \varphi}(z) \right| \leq \operatorname{ess\,sup}_{0 \leq \theta \leq 2\pi} |f'(t)| =: L \leq |f'(\varphi)| + \varepsilon.$$

According to (1.4) and (1.10) we obtain:

$$\left(1 + \frac{1}{K^2}\right) |f'(\varphi)|^2 \leq \frac{\pi}{4} C_\gamma K |f'(\varphi)| \int_{-\pi}^{\pi} \frac{d_\gamma(f(e^{i(\varphi+x)}), f(e^{i\varphi}))^{1+\mu}}{x^2} dx.$$

If

$$C_2 = \frac{\pi}{4} C_\gamma \frac{K^3}{1 + K^2}$$

then

$$(2.5) \quad \begin{aligned} L - \varepsilon &\leq C_2 \int_{-\pi}^{\pi} \frac{d_{\gamma}(f(e^{i(\varphi+x)}), f(e^{i\varphi}))^{1+\mu}}{x^{1+\mu}} \frac{dx}{x^{1-\mu}} \\ &\leq C_2 \int_{-\pi}^{\pi} \frac{d_{\gamma}(f(e^{i(\varphi+x)}), f(e^{i\varphi}))^{1+\mu-\beta}}{x^{1+\mu-\beta}} L^{\beta} \frac{dx}{x^{1-\mu}}. \end{aligned}$$

Thus

$$(L - \varepsilon)/L^{\beta} \leq C_2 \int_{-\pi}^{\pi} \frac{d_{\gamma}(f(e^{i(\varphi+x)}), f(e^{i\varphi}))^{1+\mu-\beta}}{x^{1+\mu-\beta}} \frac{dx}{x^{1-\mu}}.$$

Choose  $\beta$ :  $0 < \beta < 1$  sufficiently close to 1 so that  $\sigma = (\alpha - 1)(1 + \mu - \beta) + \mu - 1 > -1$ . For example

$$\beta = 1 - \frac{\mu\alpha}{2 - \alpha},$$

and consequently

$$\sigma = \frac{\mu\alpha}{2 - \alpha} - 1.$$

From Lemma 1.3 and (1.11), letting  $\varepsilon \rightarrow 0$ , we get

$$L^{1-\beta} \leq C_2 \cdot (B_{\gamma} L_{\gamma})^{1+\mu-\beta} \int_{-\pi}^{\pi} x^{\sigma} dx = C_3,$$

and hence

$$(2.6) \quad L \leq C_3^{1/(1-\beta)} = C_3^{\frac{2-\alpha}{\mu\alpha}}.$$

By (2.3) it follows that

$$|zg'(z) - \overline{zh'(z)}| \leq L.$$

On the other hand,

$$|\nabla w| = |g'| + |h'|$$

is subharmonic. This follows that

$$(2.7) \quad |\nabla w(z)| \leq \max_{|z|=1} \{|g'(z)| + |h'(z)|\} \leq K \max_{|z|=1} \{|g'(z)| - |h'(z)|\} = KL.$$

This implies (2.2).

Using the previous case and making the same approach as in the second part of theorem [8, Theorem 2.1] it follows that  $w$  is a Lipschitz mapping. Now applying again the previous case we obtain the desired conclusion.  $\square$

**Remark 2.2.** The previous proof yields the following estimate of a Lipschitz constant  $L$  for a normalized  $K$ -quasiconformal mapping between the unit disk and a Jordan domain  $\Omega$  bounded by a Jordan curve  $\gamma \in C^{1,\mu}$  satisfying a  $B$ -chord-arc condition.

$$(2.8) \quad L \leq 4\pi \left( \frac{\pi}{2} \frac{K^3}{1 + K^2} C_{\gamma} \frac{2 - \alpha}{\mu\alpha} \right)^{\frac{2-\alpha}{\mu\alpha}} \left\{ 4B(1 + 2B) \sqrt{\frac{K|\Omega|}{\pi \log 2}} \right\}^{\frac{2}{\alpha}},$$

where

$$\alpha = \frac{1}{K(1+2B)^2}$$

and  $C_\gamma$  is defined in (1.8). See [20], [19], [4] and [5] for more explicit (more precise) constants, in the special case where  $\gamma$  is the unit circle.

### 3. BOUNDARY CORRESPONDENCE UNDER Q.C. HARMONIC MAPPINGS

If  $w = g + \bar{h}$  is a harmonic function then

$$w_\varphi = i(zg'(z) - \overline{zh'(z)})$$

is also harmonic. On the other hand

$$rw_r = zg'(z) + \overline{zh'(z)}.$$

Hence the function  $rw_r$  is the harmonic conjugate of  $w_\varphi$  (this means that  $w_\varphi + irw_r$  is analytic). The Hilbert transformation of  $f'$  is defined by the formula

$$H(f')(\varphi) = -\frac{1}{\pi} \int_{0+}^{\pi} \frac{f'(\varphi+t) - f'(\varphi-t)}{2 \tan(t/2)} dt$$

for a.e.  $\varphi$  and  $f' \in L^1(S^1)$ . The facts concerning the Hilbert transformation can be found in ([24], Chapter VII).

There holds

$$(3.1) \quad w_\varphi = P[f'] \text{ and } rw_r = P[H(f')],$$

if  $w_\varphi$  and  $rw_r$  are bounded harmonic.

The following theorem provides a necessary and a sufficient condition for the harmonic extension of a homeomorphism from the unit circle to a  $C^{2,\mu}$  Jordan curve  $\gamma$  to be a q.c mapping, once we know that its image is  $\Omega = \text{int } \gamma$ . It is an extension of the corresponding result [8, Theorem 3.1] from convex domains to arbitrarily smooth domains.

**Theorem 3.1.** *Let  $f : S^1 \rightarrow \gamma$  be an orientation preserving absolutely continuous homeomorphism of the unit circle onto the Jordan curve  $\gamma = \partial\Omega \in C^{2,\mu}$ . If  $P[f](\mathbf{U}) = \Omega$ , then  $w = P[f]$  is a quasiconformal mapping if and only if*

$$(3.2) \quad 0 < l(f) := \text{ess inf } l(\nabla w(e^{i\varphi})),$$

$$(3.3) \quad \|f'\|_\infty := \text{ess sup } |f'(\varphi)| < \infty$$

and

$$(3.4) \quad \|H(f')\|_\infty := \text{ess sup}_\varphi |H(f')(\varphi)| < \infty.$$

If  $f$  satisfies the conditions (3.2), (3.3) and (3.4), then  $w = P[f]$  is  $K$  quasiconformal, where

$$(3.5) \quad K := \frac{\sqrt{\|f'\|_\infty^2 + \|H(f')\|_\infty^2 - l(f)^2}}{l(f)}.$$



The constant  $K$  is the best possible in the following sense, if  $w$  is the identity or it is a mapping close to the identity, then  $K = 1$  or  $K$  is close to 1 (respectively).

*Proof.* Under the above conditions the harmonic mapping  $w$ , by a result of Kneser, is univalent (see for example [2, p. 31]). Therefore  $w = g + \bar{h}$ , where  $g$  and  $h$  are analytic and  $J_u = |g'|^2 - |h'|^2 > 0$ . This infers that the second dilatation  $\mu = h'/g'$  is well defined analytic function bounded by 1.

**3.1. The proof of necessity.** Suppose  $w = P[f] = g + \bar{h}$  is a  $K$ -q.c. harmonic mapping that satisfies the conditions of the theorem. By [10, Theorem 2.1]) we have

$$(3.6) \quad |\partial w(z)| - |\bar{\partial} w(z)| \geq \frac{C(\Omega, K, a)}{K} > 0, z \in \mathbf{U}.$$

By [8, Thoerem 2.1] or Theorem 3.1 we get

$$(3.7) \quad |f'(\varphi)| \leq L \text{ a.e.}$$

and

$$(3.8) \quad \lim_{r \rightarrow 1} |\partial w(re^{i\varphi})| - |\bar{\partial} w(re^{i\varphi})| = |\partial w(e^{i\varphi})| - |\bar{\partial} w(e^{i\varphi})| \text{ a.e..}$$

Combining (3.7), (3.8) and (3.6) we get (3.2) and (3.3).

Next we prove (3.4). Observe first that

$$w_r = e^{i\varphi} w_z + e^{-i\varphi} w_{\bar{z}}.$$

Thus

$$(3.9) \quad |w_r| \leq |\nabla w|.$$

By using (3.9) and (2.7) it follows that

$$(3.10) \quad |w_r(z)| \leq KL.$$

The last inequality implies that there exist the radial limits of the harmonic conjugate  $rw_r$  a.e. and

$$(3.11) \quad \lim_{r \rightarrow 1} rw_r(re^{i\varphi}) = \lim_{r \rightarrow 1} w_r(re^{i\varphi}) = H(f')(\varphi) \text{ a.e.,}$$

where  $H(f')$  is the Hilbert transform of  $f'$ . Since  $rw_r$  is a bounded harmonic function it follows that  $rw_r = P[H(f')]$ , and therefore

$$\|H(f')\|_\infty = \text{ess sup } |H(f')(\varphi)| < \infty.$$

Thus we obtain (3.4).

**3.2. The proof of sufficiency.** We have to prove that under the conditions (3.2), (3.3) and (3.4)  $w$  is quasiconformal. This means that we need to prove the function

$$(3.12) \quad K(z) = \frac{|w_z| + |w_{\bar{z}}|}{|w_z| - |w_{\bar{z}}|} = \frac{1 + |\mu|}{1 - |\mu|}$$

is bounded.

Since  $\mu = \overline{w_{\bar{z}}}/w_z$  is an analytic function it follows that  $|\mu|$  is subharmonic. (Notice that, as  $\phi(t) = \frac{1+t}{1-t}$  is convex this yields that  $K(z) = \phi(|\mu(z)|)$  is subharmonic).

It follows from (1.1) that  $w_\varphi$  is equals the Poisson-Stieltjes integral of  $f'$ :

$$w_\varphi(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \varphi - t) df(t).$$

Hence, by Fatou's theorem, the radial limits of  $f_\varphi$  exist almost everywhere and  $\lim_{r \rightarrow 1^-} f_\varphi(re^{i\varphi}) = f'_0(\theta)$  a.e., where  $f_0$  is the absolutely continuous part of  $f$ .

As  $rw_r$  is harmonic conjugate of  $w_\varphi$ , it turns out that if  $f$  is absolutely continuous, then

$$\lim_{r \rightarrow 1^-} f_r(re^{i\varphi}) = H(f')(\theta) \text{ (a.e.)},$$

and

$$\lim_{r \rightarrow 1^-} f_\varphi(re^{i\varphi}) = f'(\theta).$$

As

$$|w_z|^2 + |w_{\bar{z}}|^2 = \frac{1}{2} \left( |w_r|^2 + \frac{|f_\varphi|^2}{r^2} \right)$$

it follows that

$$(3.13) \quad \lim_{r \rightarrow 1^-} |w_z|^2 + |w_{\bar{z}}|^2 \leq \frac{1}{2} (\|f'\|_\infty^2 + \|H(f')\|_\infty^2).$$

To continue we make use of (3.2). From (3.13) and (3.2) we obtain that

$$(3.14) \quad \operatorname{ess\,sup}_{\varphi \in [0, 2\pi)} \frac{|w_z(e^{i\varphi})|^2 + |w_{\bar{z}}(e^{i\varphi})|^2}{(|w_z(e^{i\varphi})| - |w_{\bar{z}}(e^{i\varphi})|)^2} \leq \frac{\|f'\|_\infty^2 + \|H(f')\|_\infty^2}{2l(f)^2}.$$

Hence

$$(3.15) \quad |w_z(e^{i\varphi})|^2 + |w_{\bar{z}}(e^{i\varphi})|^2 \leq S (|w_z(e^{i\varphi})| - |w_{\bar{z}}(e^{i\varphi})|)^2 \text{ (a.e.)},$$

where

$$(3.16) \quad S := \frac{\|f'\|_\infty^2 + \|H(f')\|_\infty^2}{2l(f)^2}.$$

According to (3.14),  $S \geq 1$ . Let

$$\mu(e^{i\varphi}) := \left| \frac{w_{\bar{z}}(e^{i\varphi})}{w_z(e^{i\varphi})} \right|.$$

As  $w$  is a diffeomorphism,  $|\mu(e^{i\varphi})| \leq 1$ . Then (3.15) can be written as follows:

$$1 + \mu^2(e^{i\varphi}) \leq S(1 - \mu(e^{i\varphi}))^2,$$

i.e.  $\mu = \mu(e^{i\varphi})$  satisfies the inequality

$$(3.17) \quad \mu^2(S - 1) - 2\mu S + S - 1 = (S - 1)(\mu - \mu_1)(\mu - \mu_2) \geq 0,$$

where

$$\mu_1 = \frac{S + \sqrt{2S - 1}}{S - 1}$$

and

$$\mu_2 = \frac{S - 1}{S + \sqrt{2S - 1}}.$$

From (3.17) it follows that  $\mu(e^{i\varphi}) \leq \mu_2$  or  $\mu(e^{i\varphi}) \geq \mu_1$ . But  $\mu(e^{i\varphi}) \leq 1$  and therefore

$$(3.18) \quad \mu(e^{i\varphi}) \leq \frac{S - 1}{S + \sqrt{2S - 1}} \quad (a.e.).$$

As  $\mu(z) = |a(z)|$ , where  $a$  is an analytic function, it follows that

$$\mu(z) \leq k := \mu_2,$$

for  $z \in \mathbf{U}$ .

This yields that

$$K(z) \leq K := \frac{1 + k}{1 - k} = \frac{2S - 1 + \sqrt{2S - 1}}{\sqrt{2S - 1} + 1} = \sqrt{2S - 1},$$

i.e.

$$K(z) \leq \frac{\sqrt{\|f'\|_\infty^2 + \|H(f')\|_\infty^2 - l(f)^2}}{l(f)}$$

which means that  $w$  is  $K = \frac{\sqrt{\|f'\|_\infty^2 + \|H(f')\|_\infty^2 - l(f)^2}}{l(f)}$  quasiconformal. The sharpness of the last results follows from the fact that  $K = 1$  for  $w$  being the identity.  $\square$

**3.3. Two examples.** The following example shows that, a  $K$  (with  $K$  arbitrary close to 1) q.c. harmonic selfmapping of the unit disk exists, having non-smooth extension to the boundary, contrary to the conformal case.

**Example 3.2.** ([9]). Let

$$\theta(\varphi) = \varphi \frac{1 + b \sin(\log |\varphi| - \pi/4)}{1 + b \sin(\log \pi - \pi/4)}, \quad \varphi \in [-\pi, \pi],$$

where  $0 < b < \sqrt{2}/2$ , and let  $w(z) = P[f](z) = P[e^{i\theta(\varphi)}](z)$ . Then  $w$  is a quasiconformal mapping of the unit disk onto itself such that  $f'(\varphi)$  does not exist for  $\varphi = 0$ . Using a similar approach as in Theorem 3.1 it can be shown that

$$K_w := \sup_{|z| < 1} \frac{|w_z| + |w_{\bar{z}}|}{|w_z| - |w_{\bar{z}}|} \rightarrow 1$$

as  $b \rightarrow 0$  and this means that, there exists a q.c. harmonic mapping close enough to the identity, but its boundary function is not differentiable at 1. Details we will discuss elsewhere.

The next example shows that, the condition (3.2) of the main theorem is important even for harmonic polynomials.

**Example 3.3.** Let  $w$  be the harmonic polynomial defined in the unit disk by:

$$w(z) = z - 1 - (z - 1)^2 + \overline{z - 1} = 3z - 3 - z^2 + \bar{z}.$$

Then  $w$  is a univalent harmonic mapping of the unit disk onto the domain bounded by the  $C^\infty$  convex curve  $\gamma = \{(4 \cos t - \cos(2t) - 3, \sin(2t) - 2 \sin(t)), t \in [0, 2\pi)\}$ . But  $w_z(1) = w_{\bar{z}}(1) = 1$ , and therefore  $w$  is not quasiconformal.

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