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QUASICONFORMAL AND HARMONIC MAPPINGS BETWEEN SMOOTH JORDAN DOMAINS

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Abstract. We present some recent results on the topic of quasiconformal harmonic maps. The main result is that every quasiconformal harmonic mapping w of $C^{1,\mu}$ Jordan domain Ω_1 onto $C^{1,\mu}$ Jordan domain Ω is Lipschitz continuous, which is the property shared with conformal mappings. In addition, if Ω has $C^{2,\mu}$ boundary, then w is bi-Lipschitz continuous. These results have been considered by the authors in various ways.

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1. Introduction

Let D and G be subdomains of the complex plane $\mathbf{C}$. A homeomorphism $f\colon D \mapsto G$, where is said to be K-quasiconformal (K-q.c), $K \geq 1$, if f is absolutely continuous on almost every horizontal and almost every vertical line and

$$\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2 \leq \left(K + \frac{1}{K}\right) J_f \quad \text{a.e. on } D, \tag{1.1}$$

where J_f is the Jacobian of f (cf. [1], pp. 23–24). Note that the condition (1.1) can be written as

$$|f_{\bar{z}}| \leq k|f_z| \quad \text{a.e. on } D \text{ where } k = \frac{K-1}{K+1} \text{ i.e. } K = \frac{1+k}{1-k}\ .$$

A function w is called *harmonic* in a region D if it is of the form $w = u + iv$ where u and v are real-valued harmonic functions in D. If D is simply-connected, there exist two analytic functions g and h defined on D such that w has the representation

$$w = g + \overline{h}.$$

If w is a harmonic univalent function, then by Lewy's theorem (see [14]), w has a non-vanishing Jacobian and consequently, according to the inverse mapping theorem, w is a diffeomorphism.

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Let

$$P(r, x-\varphi) = \frac{1-r^2}{2\pi(1-2r\cos(x-\varphi)+r^2)}$$

denote the Poisson kernel. Then every bounded harmonic function w defined on the unit disk $\mathbf{U} := \{z : |z| < 1\}$ has the representation

$$w(z) = P[f](z) = \int_0^{2\pi} P(r, x-\varphi) f(e^{ix}) dx, \tag{1.2}$$

where $z = re^{i\varphi}$ and f is a bounded integrable function defined on the unit circle S^1.

Suppose γ is a rectifiable, directed, differentiable curve given by its arc-length parametrization $g(s)$, $0 \le s \le l$, where l is the length of γ. Then $|g'(s)| = 1$ and $s = \int_0^s |g'(t)| dt$, for all $s \in [0, l]$.

If γ is a twice-differentiable curve, then the curvature of γ at a point $p = g(s)$ is given by $\kappa_\gamma(p) = |g''(s)|$. Let

$$K(s,t) = \mathrm{Re}\,[\overline{(g(t) - g(s))} \cdot ig'(s)] \tag{1.3}$$

be a function defined on $[0,l] \times [0,l]$. By $K(s \pm l, t \pm l) = K(s,t)$ we extend it on $\mathbb{R} \times \mathbb{R}$. Note that $ig'(s)$ is the unit normal vector of γ at $g(s)$ and therefore, if γ is convex then

$$K(s,t) \ge 0 \text{ for every } s \text{ and } t. \tag{1.4}$$

We say that $\gamma \in C^{1,\mu}$, $0 < \mu \le 1$, if $g \in C^1$ and

$$\sup_{t,s} \frac{|g'(t) - g'(s)|}{|t-s|^\mu} < \infty.$$

Let $\gamma \in C^{1,\mu}$ be a Jordan curve such that the interior of γ contains the origin. Let f be a $C^{1,\mu}$ function from the unit circle onto γ and let $F(x) = f(e^{ix})$, $x \in [0, 2\pi)$. Then the functions $\rho(x) = |F(x)|$ and $\theta(x) = \arg F(x) \mod 2\pi$ on $(0, 2\pi]$ have $C^{1,\mu}$ extension on $\mathbb{R}$. In the remainder of this paper we will use f and F interchangeably and will write $f'(x)$ instead of $F'(x)$.

Suppose now that $f : \mathbb{R} \mapsto \gamma$ is an arbitrary 2π periodic C^1 function such that $f|_{[0,2\pi)} : [0, 2\pi) \mapsto \gamma$ is an orientation preserving bijective function.

Then there exists an increasing continuous function $s : [0, 2\pi] \mapsto [0, l]$ such that

$$f(\varphi) = g(s(\varphi)). \tag{1.5}$$

Hence

$$f'(\varphi) = g'(s(\varphi)) \cdot s'(\varphi),$$

and therefore

$$|f'(\varphi)| = |g'(s(\varphi))| \cdot |s'(\varphi)| = s'(\varphi).$$

Along with the function K we will also consider the function K_f defined by

$$K_f(\varphi, x) = \mathrm{Re}\,[\overline{(f(x) - f(\varphi))} \cdot if'(\varphi)].$$

It is easy to see that

$$K_f(\varphi, x) = s'(\varphi)\mathrm{Re}\,[\overline{(g(s(x)) - g(s(\varphi)))} \cdot ig'(s(\varphi))] = s'(\varphi)K(s(\varphi), s(x)). \tag{1.6}$$

2. The Lipschitz continuity of q.c. harmonic mapping

The following lemma is a slight modifications of the corresponding lemma in [8].

Lemma 2.1. *Let γ be a $C^{1,\mu}$ Jordan curve. Let $g : [0, l] \mapsto \gamma$ be a natural parametrization and $f : [0, 2\pi] \mapsto \gamma$, be arbitrary parametrization of γ. Then*

$$|K(s, t)| \le C_\gamma \min\{|s - t|^{1+\mu}, (l - |s - t|)^{1+\mu}\} \tag{2.1}$$

and

$$|K_f(\varphi, x)| \le C_\gamma s'(\varphi) \min\{|s(\varphi) - s(x)|^{1+\mu}, (l - |s(\varphi) - s(x)|)^{1+\mu}\}, \tag{2.2}$$

where

$$C_\gamma = \frac{1}{1+\mu} \sup_{0 \le t \ne s \le l} \frac{|g'(t) - g'(s)|}{|t - s|^\mu}.$$

Here $d_\gamma(f(e^{i\varphi}), f(e^{ix})) := \min\{|s(\varphi) - s(x)|, (l - |s(\varphi) - s(x)|)\}$ is the distance (shorter) between $f(e^{i\varphi})$ and $f(e^{ix})$ along γ which satisfies the relation

$$|f(e^{i\varphi}) - f(e^{ix}))| \le d_\gamma(f(e^{i\varphi}), f(e^{ix})) \le c_\gamma |(f(e^{i\varphi}) - f(e^{ix})|.$$

Moreover if γ has a bounded curvature then the relations (2.1) and (2.2) are true for

$$C_\gamma = \sup\{|\kappa_\gamma(g(s))|/2 : s \in [0, l]\}$$

and $\mu = 1$. In this case

$$\lim_{t \to s} \frac{K(s, t)}{(s - t)^2} = \frac{|\kappa_\gamma(g(s))|}{2} \quad \textit{and} \quad \lim_{x \to \varphi} \frac{K_f(\varphi, x)}{(s(x) - s(\varphi))^2} = \frac{|\kappa_\gamma(g(s))|}{2} s'(\varphi),$$

and the constant C_γ is the best possible.

Proof. Note that

$$\begin{aligned} K(s, t) &= \mathrm{Re}[\overline{(g(t) - g(s))} \cdot ig'(s)] \\ &= \mathrm{Re}\left[\overline{(g(t) - g(s))} \cdot i \left(g'(s) - \frac{g(t) - g(s)}{t - s}\right)\right], \end{aligned}$$

and

$$g'(s) - \frac{g(t) - g(s)}{t - s} = \int_s^t \frac{g'(s) - g'(\tau)}{t - s} d\tau.$$

If γ has a bounded curvature then g'' is bounded and

$$\left|g'(s)-\frac{g(t)-g(s)}{t-s}\right| \le \int_s^t \frac{|g'(s)-g'(\tau)|}{t-s}d\tau$$
$$\le \sup_{s\le x\le t}|g''(x)|\cdot\int_s^t\frac{\tau-s}{t-s}d\tau = \frac{1}{2}\sup_{s\le x\le t}|g''(x)|(t-s).$$

On the other hand

$$|\overline{g(t)-g(s)}| \le \sup_{s\le x\le t}|g'(x)|(t-s) = (t-s),$$

and thus

$$|K(s,t)| \le \frac{1}{2}\sup_{s\le x\le t}|g''(x)|(s-t)^2.$$

It follows that the inequality (2.1) holds for $C_\gamma = \sup_p |\kappa_\gamma(p)|/2$ and $\mu = 1$. From (2.1) and (1.6) we obtain (2.2). Since

$$\frac{\partial}{\partial s}K(s,t) = \operatorname{Re}[\overline{(g(t)-g(s))}\cdot ig''(s)],$$

it follows that

$$\lim_{t\to s}\frac{K_g(s,t)}{(s-t)^2} = \lim_{t\to s}\frac{\operatorname{Re}[\overline{(g(t)-g(s))}\cdot ig''(s)]}{2(s-t)}$$
$$= \operatorname{Re}[\overline{-g'(s)}\cdot ig''(s)]/2 = \varepsilon|g''(s)|/2 = \kappa_\gamma(s)/2.$$

Here $\varepsilon = 1$ if $\kappa_\gamma > 0$ and $\varepsilon = -1$ if $\kappa_\gamma < 0$. Similarly we can prove the case $\gamma \in C^{1,\mu}$. □

Lemma 2.2. *[8] Let $w = u+iv$ be a differentiable function defined on* $\mathbf{U}$. *Then:*

$$J_w(re^{i\varphi}) = u_xv_y - u_yv_x = |w_z|^2 - |w_{\bar z}|^2 = \frac{1}{r}(u_rv_\varphi - u_\varphi v_r) \tag{2.3}$$

and

$$D(w)(re^{i\varphi}) := |w_z|^2 + |w_{\bar z}|^2 = \frac{|\partial_r w|^2}{2} + \frac{|\partial_\varphi w|^2}{2r^2}. \tag{2.4}$$

If in addition we suppose that $w = P[f](z)$, where $f \in C^{1,\mu}$, $f : S^1 \mapsto \gamma$, then there exist continuous functions J_w and $D(w)$ on the unit circle defined by:

$$J_w(e^{i\varphi}) = \lim_{r\to 1} J_w(re^{i\varphi}) \tag{2.5}$$

and

$$D(w)(e^{i\varphi}) = \lim_{r\to 1} D(w)(e^{i\varphi}) = \lim_{r\to 1}\frac{|\partial_r w(re^{i\varphi})|^2}{2} + \frac{|f'(\varphi)|^2}{2}. \tag{2.6}$$

Proposition 2.3 (Kellogg). *Let $\gamma \in C^{1,\mu}$ be a Jordan curve and let $\Omega = Int(\Gamma)$. If ω is a conformal mapping of $\mathbf{U}$ onto Ω, then ω' and $\ln \omega'$ are in Lip_μ. In particular, $|\omega'|$ is bounded from above and below by positive constants on $\mathbf{U}$.*

For the proof, see for example [12].

The following lemma is a generalization of Mori's Theorem, (cf. [1]).

Lemma 2.4. *If w is a K quasiconformal function between the unit disk and a Jordan domain Ω with $C^{1,\mu}$ boundary γ, then there exists a constant C_K depending only on γ and on $w(0)$ such that*

$$|w(z_1) - w(z_2)| \le C_K |z_1 - z_2|^\alpha, \quad \alpha = \frac{1-k}{1+k}, \quad z_1,\, z_2 \in \mathbf{U}\ .$$

Note that the constant α is the best possible (in general case).

In the following lemma, we give some estimates for the Jacobian of a harmonic univalent function. It is a slight improvement of [8, Lemma 2.7].

Lemma 2.5. *Let $w = P[f](z)$ be a harmonic function between the unit disk $\mathbf{U}$ and the Jordan domain Ω, such that f is injective, $f \in C^{1,\mu}$, and $\partial\Omega = f(S^1) \in C^{1,\mu}$. Then for*

$$C_1 = \frac{\pi}{4(1+\mu)} \sup_{s \ne t} \frac{|g'(s) - g'(t)|}{(s-t)^\mu}$$

one has

$$\lim_{z \to e^{i\varphi}} J_w(z) \le C_1 |f'(\varphi)| \int_{-\pi}^{\pi} \frac{d_\gamma(f(e^{i(\varphi+x)}), f(e^{i\varphi}))^{1+\mu}}{x^2} dx \tag{2.7}$$

for all $e^{i\varphi} \in S^1$.

Proof. Since $f \in C^{1,\mu}$, by the proof of the Lemma 2.2 it follows that the partial derivatives of the function w have continuous extensions on the boundary. Since

$$F(x) = \rho(x) e^{i\theta(x)},$$

we obtain

$$u_r(e^{i\varphi}) = \lim_{z \to e^{i\varphi}} u_r(z), \qquad v_r(e^{i\varphi}) = \lim_{z \to e^{i\varphi}} v_r(z),$$

$$\lim_{z \to e^{i\varphi}} u_\varphi(z) = \mathrm{Re} \frac{\partial}{\partial\varphi} \left(\rho(\varphi) e^{i\theta(\varphi)}\right) = \rho'(\varphi)\cos\theta(\varphi) - \rho(\varphi)\theta'(\varphi)\sin\theta(\varphi)$$

and

$$\lim_{z \to e^{i\varphi}} v_\varphi(z) = \mathrm{Im} \frac{\partial}{\partial\varphi} \left(\rho(\varphi) e^{i\theta(\varphi)}\right) = \rho'(\varphi)\sin\theta(\varphi) + \rho(\varphi)\theta'(\varphi)\cos\theta(\varphi).$$

Observe that $u(e^{i\varphi}) = \rho(\varphi)\cos\theta(\varphi)$ and $v(e^{i\varphi}) = \rho(\varphi)\sin\theta(\varphi)$. Thus:

$$\begin{aligned}\lim_{z\to e^{i\varphi}} J_w(re^{i\varphi}) &= \lim_{r\to 1}\frac{1}{r}(u_r v_\varphi - u_\varphi v_r)\\ &= \lim_{r\to 1}\left(\frac{u(re^{i\varphi}) - u(e^{i\varphi})}{1-r}\right)(\rho'(\varphi)\sin\theta(\varphi) + \rho(\varphi)\theta'(\varphi)\cos\theta(\varphi))\\ &- \lim_{r\to 1}\left(\frac{v(re^{i\varphi}) - v(e^{i\varphi})}{1-r}\right)(\rho'(\varphi)\cos\theta(\varphi) - \rho(\varphi)\theta'(\varphi)\sin\theta(\varphi))\\ &= \lim_{r\to 1}\int_{-\pi}^{\pi} K_f(x,\varphi)\frac{P(r,\varphi - x)}{1-r}dx\\ &= \lim_{r\to 1}\int_{-\pi}^{\pi} K_f(x+\varphi,\varphi)\frac{P(r,x)}{1-r}dx.\end{aligned}$$

According to (2.2)

$$|K_f(x+\varphi,\varphi)| \le C_\gamma |f'(\varphi)| d_\gamma(f(e^{i(\varphi+x)}), f(e^{i\varphi}))^{1+\mu}.$$

On the other hand, using the inequality $|t| \le \pi/2|\sin t|$ for $-\pi/2 \le t \le \pi/2$, we obtain

$$\frac{P(r,x)}{1-r} = \frac{1+r}{2\pi(1+r^2-2r\cos x)} \le \frac{1}{\pi((1-r)^2 + 4r\sin^2 x/2)} \le \frac{\pi}{4rx^2}$$

for $0 < r < 1$ and $x \in [-\pi,\pi]$. Thus,

$$\lim_{r\to 1}\int_{-\pi}^{\pi} K(x,\varphi)\frac{P(r,\varphi - x)}{1-r}dx \le \frac{\pi C_\gamma}{4}|f'(\varphi)|\int_{-\pi}^{\pi}\frac{d_\gamma(f(e^{i(\varphi+x)}), f(e^{i\varphi}))^{1+\mu}}{x^2}dx.$$

The inequality now holds for

$$C_1 = \frac{\pi}{4(1+\mu)}\sup_{s\neq t}\frac{|g'(s) - g'(t)|}{(s-t)^\mu}.$$

□

Using Lemma 2.2, Proposition 2.3, Lemma 2.4 and Lemma 2.5 we obtain:

Theorem 2.6. *[8] Let $w = P[f](z)$ be a K q.c. harmonic function between the unit disk and a Jordan domain Ω, such that $w(0) = 0$. If $\gamma = \partial\Omega \in C^{1,\mu}$, then there exists a constant $C' = C'(\gamma, K)$ such that*

(2.8) $$|f'(\varphi)| \le C' \text{ for almost every } \varphi \in [0,2\pi],$$

and

(2.9) $$|w(z_1) - w(z_2)| \le KC'|z_1 - z_2| \text{ for } z_1, z_2 \in \mathbf{U}.$$

Notice that Theorem 2.6 is a generalization of the corresponding result for the harmonic q.c. of the unit disk onto itself, see [19]. Theorem 2.6 has its extension to the class of q.c. mappings satisfying the differential inequality $|\Delta w| \leq M|w_z||w_{\bar z}|$ (see [11]).

Example 2.7 ([3])**.** Let P_n be a regular $n-$polygon. Then the function

$$w(z) = \int_0^z (1-z^n)^{-2/n} dz$$

is a conformal mapping of the unit disk onto the polygon P_n. However $w'(z) = (1-z^n)^{-2/n}$ is an unbounded function on the unit disk and thus the condition $\gamma \in C^{1,\mu}$ in Theorem 2.6 is important.

Corollary 2.8. *[8] Let w be a quasiconformal harmonic mapping between Jordan domains Ω and Ω_1, such that $w(0) = 0$. If $\gamma = \partial\Omega \in C^{1,\mu}$ and $\gamma_1 = \partial\Omega_1 \in C^{1,\mu_1}$, $0 < \mu, \mu_1 \leq 1$, then there exist the constants C and C_1 depending on γ and γ_1 such that*

$$|w(z_1) - w(z_2)| \leq C|z_1 - z_2| \tag{2.10}$$

and

$$D(w)(z) = |w_z(z)|^2 + |w_{\bar z}(z)|^2 \leq C_1. \tag{2.11}$$

3. The bi-Lipschitz continuity of q.c. harmonic mappings

The following theorem provides a necessary and sufficient condition for the q.c. harmonic extension of a homeomorphism from the unit circle to a $C^{1,\mu}$ convex Jordan curve. It is an extension of the corresponding theorem of Pavlović ([19]):

Theorem 3.1. *[8] Let $f : S^1 \mapsto \gamma$ be an orientation preserving absolutely continuous homeomorphism of the unit circle onto a convex Jordan curve $\gamma \in C^{1,\mu}$. Then $w = P[f]$ is a quasiconformal mapping if and only if*

$$0 < \operatorname{ess\,inf} |f'(\varphi)|, \tag{3.1}$$

$$\operatorname{ess\,sup} |f'(\varphi)| < \infty \tag{3.2}$$

and

$$\operatorname*{ess\,sup}_{\varphi} \left| \int_0^\pi \frac{f'(\varphi+t) - f'(\varphi-t)}{\tan t/2} dt \right| < \infty. \tag{3.3}$$

Let us note that the hypothesis "absolutely continuous" in the previous theorem is needed, although this theorem appeared in [8] without this hypothesis.

Example 3.2 ([7]). Let

$$\theta(\varphi) = \frac{2 + b(\cos(\log|\varphi|) - \sin(\log|\varphi|))}{2 + b(\cos(\log\pi) - \sin(\log\pi))}\varphi, \ \varphi \in [-\pi, \pi],$$

where $0 < b < 1$. Then the function $w(z) = P[f](z) = P[e^{i\theta(\varphi)}](z)$ is a quasiconformal mapping of the unit disk onto itself such that $f'(\varphi)$ does not exist for $\varphi = 0$.

Hence a q.c. harmonic function does not have necessarily a C^1 extension to the boundary as in conformal case.

Corollary 3.3. *[8] Let w be a K quasiconformal harmonic function between a Jordan domain Ω and a convex Jordan domain Ω_1, such that $w(0) = 0$ and $\partial\Omega,\ \partial\Omega_1 \in C^{1,\mu}$. Then w is bi-Lipschitz, i.e. there exists a constant $L \geq 1$ such that*

$$L^{-1}|z_1 - z_2| < |w(z_1) - w(z_2)| < L|z_1 - z_2|, \ z_1, z_2 \in \Omega. \tag{3.4}$$

Moreover, there exists $C_D = C(K, \Omega, \Omega_1) \geq 1$ such that

$$1/C_D \leq |D(w)(z)| \leq C_D, \ \text{for } z \in \Omega. \tag{3.5}$$

One of the recent results of the first author is the following theorem. It is an extension of Corollary 3.3 for a nonconvex case.

Theorem 3.4. *[9] Let $w = f(z)$ be a K quasiconformal harmonic mapping between a Jordan domain Ω with $C^{1,\mu}$ boundary and a Jordan domain Ω_1 with $C^{2,\mu}$ boundary. Let in addition $a \in \Omega$ and $b = f(a)$. Then w is bi-Lipschitz. Moreover there exists a positive constant $c = c(K, \Omega, \Omega_1, a, b) \geq 1$ such that*

$$\frac{1}{c}|z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq c|z_1 - z_2|, \quad z_1, z_2 \in \Omega. \tag{3.6}$$

First, we need to introduce some notations:
We write $L_f = L_f(z) = |\partial f(z)| + |\bar\partial f(z)|$ and $l_f = l_f(z) = |\partial f(z)| - |\bar\partial f(z)|$, if $\partial f(z)$ and $\bar\partial f(z)$ exist.

In [13], the following results have been obtained (see also [15]) :

Theorem 3.5. *Let f be a k-qc euclidean harmonic diffeomorphism from the upper half-plane $\mathbb{H}$ onto itself and $K = \frac{1+k}{1-k}$. Then f is a $(1/K, K)$ quasi-isometry with respect to the Poincaré distance d_h.*

Outline of the proof: Precomposing f with a linear fractional transformation, we can suppose that $f(\infty) = \infty$ and therefore we can write f in the form $f = u + iy = \frac{1}{2}(F(z) + z + \overline{F(z) - z})$, where F is a holomorphic function in $\mathbb{H}$. Hence the complex dilatation $\mu_f = \frac{F'(z)-1}{F'(z)+1}$, $L_f(z) = \frac{1}{2}(|F'(z)+1| + |F'(z)-1|)$ and $l_f(z) = \frac{1}{2}(|F'(z)+1| - |F'(z)-1|)$; which yields

$$1 + 1/K \leqslant |F'(z) + 1| \leqslant K + 1, \quad 1 - 1/K \leqslant |F'(z) - 1| \leqslant K - 1$$

and therefore it follows

$$1 \leqslant L_f(z) = \frac{1}{2}(|F'(z)+1| + |F'(z)-1|) \leqslant K,$$

and consequently

$$l_f(z) \geq L_f(z)/K \geq 1/K.$$

Now using a known procedure, we obtain

$$\frac{1}{K}|z_2 - z_1| \leq |f(z_2) - f(z_1)| \leq K|z_2 - z_1| \quad z_1, z_2 \in H, \tag{3.7}$$

$$\frac{1-k}{1+k} d_h(z_1, z_2) \leqslant d_h(f(z_1), f(z_2)) \leqslant \frac{1+k}{1-k} d_h(z_1, z_2) \quad z_1, z_2 \in H. \tag{3.8}$$

Both estimates are sharp (see also [4], [6] for an estimate with some constant $c(K)$ in (3.7)). □

The following generalization of Theorem 3.4 will appear in [18].

Ii is partially based on the results obtained in [9] and on Bochner formula for harmonic maps.

Theorem 3.6. *[18] Let w be a C^2 K quasiconformal mapping of the unit disk onto a $C^{2,\alpha}$ Jordan domain. Let ρ be a C^1 metric on $\overline{\Omega}$ of non-negative curvature and w ρ-harmonic, that is*

$$w_{z\bar{z}} + (\log \rho)_w w_z w_{\bar{z}} = 0.$$

Then $J_w \neq 0$ and w is bi-Lipschitz.

Finally, notice that the proof of Theorem 3.1, which was published in [8], can be also based on the results presented in [16] and [17].

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