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Quasiconformal Harmonic Mappings and Generalizations

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Abstract. We present some new results concerning the class of K and (K, K') quasiconformal mappings in the plane. Among the other things, we consider the following results. A harmonic diffeomorphism w between two C^2 Jordan domains is a (K, K') quasiconformal mapping for some constants $K \geq 1$ and $K' \geq 0$ if and only if it is Lipschitz continuous. We also discuss a generalization of this result if $K' = 0$, which states that a harmonic diffeomorphism w between two $C^{1,\alpha}$ domains is quasiconformal if and only if it is bi-Lipschitz continuous.

Keywords. Harmonic mappigs, quasiconformal mappings, hyperbolic metric.

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1. Introduction

A function w is called *harmonic* in a region D if it has form $w = u + iv$ where u and v are real-valued harmonic functions in D. If D is simply-connected, then there are two analytic functions g and h defined on D such that w has the representation

$$w = g + \overline{h}.$$

If w is a harmonic univalent function, then by Lewy's theorem (see [38]), w has a non-vanishing Jacobian and consequently, according to the inverse mapping theorem, w is a diffeomorphism. If k is an analytic function and w is a harmonic function then $w \circ k$ is harmonic. However $k \circ w$, in general is not harmonic.

By $\mathbf{R}$ we denote the set of real numbers. Throughout this paper, we will use notation $z = re^{i\varphi}$, where $r = |z|$ and $\varphi \in \mathbf{R}$ are polar coordinates and by w_φ and w_r we denote partial derivatives of w with respect to φ and r, respectively. Let

$$P(r, x) = \frac{1 - r^2}{2\pi(1 - 2r\cos x + r^2)}$$

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denote the Poisson kernel. Note that every bounded harmonic function w defined on the unit disk $\mathbf{U} := \{z : |z| < 1\}$ has the following representation

$$w(z) = P[f](z) = \int_0^{2\pi} P(r, x - \varphi) f(e^{ix}) dx, \tag{1.1}$$

where $z = re^{i\varphi}$ and f is a bounded integrable function defined on the unit circle $\mathbf{T} := \{z : |z| = 1\}$.

The Hilbert transformation of a function χ is defined by the formula

$$H[\chi](\varphi) = -\frac{1}{\pi} \int_{0+}^{\pi} \frac{\chi(\varphi + t) - \chi(\varphi - t)}{2\tan(t/2)} \mathrm{d}t$$

for a.e. φ and $\chi \in L^1(\mathbf{T})$. The facts concerning the Hilbert transformation can be found in ([62], Chapter VII).

Here and in the remainder of this paper it is convenient to use the convention: if f is complex-valued function defined on $\mathbf{T}$ a.e. we consider also f as a periodic function defined on $\mathbf{R}$ by $f(t) = f(e^{it})$ and vise versa if the meaning of it is clear from the context; we also write $f'(t) = \frac{\partial f(e^{it})}{\partial t}$. Note that if γ is 2π-periodic absolutely continuous on $[0, 2\pi]$ (and therefore $\gamma' \in L^1[0, 2\pi]$) and $h = P[\gamma]$, then

$$(h'_r)^*(e^{i\theta}) = H(\gamma')(\theta) \text{ a.e.},$$

where H denotes the Hilbert transform.

Let Γ be a curve of $C^{1,\mu}$ class and $\gamma : \mathbf{R} \to \Gamma^*$ be arbitrary topological (homeomorphic) parameterization of Γ and $s(\varphi) = \int_0^{\varphi} |\gamma'(t)| dt$. It is convenient to abuse notation and to denote by $\Gamma(s)$ natural parameterization.

For $\Gamma(s) = \gamma(\varphi)$, we define $n_\gamma(\varphi) = i\Gamma'\big(s(\varphi)\big)$ and

$$R_\gamma(\varphi, t) = (\gamma(t) - \gamma(\varphi), n_\gamma(\varphi)).$$

For $\theta \in \mathbf{R}$ and $h = P[\gamma]$, define

$$E_\gamma(\theta) = \big((h'_r)^*(e^{i\theta}), n_\gamma(\theta)\big) = \big(H(\gamma')(\theta), n_\gamma(\theta)\big) \text{ a.e.} \quad \textit{and} \tag{1.2}$$

$$\upsilon(z, \theta) = \upsilon_\gamma(z, \theta) = \big(rh'_r(z), n_\gamma(\theta)\big), \quad z \in \mathbf{U}. \tag{1.3}$$

Note that $\upsilon_*(t, \theta) = \big(H(\gamma'_*)(t), n_\gamma(\theta)\big)$ a.e.

To get a filling about C^{1,μ_1} curve we give a basic example:

Example 1.4. For $c > 0$, $0 < \mu < 1$, and $x_0 > 0$ the curve

$$y = c|x|^{1+\mu}, \quad |x| < x_0 \tag{1.5}$$

is $C^{1,\mu}$ at origin but it is not C^{1,μ_1} for $\mu_1 > \mu$. It is convenient to write this equation using polar coordinates $z = re^{i\varphi}$: $r \sin\varphi = cr^{1+\mu}(\cos\varphi)^{1+\mu}$ and we find $\sin\varphi = cr^{\mu}(\cos\varphi)^{1+\mu}$, $0 \le r < r_0$, where r_0 is a positive number. Since $\sin\varphi = \varphi + o(\varphi)$ and $\cos\varphi = 1 = o(1)$, when $\varphi \to 0$, the curve $\gamma(c,\mu)$ defined by joining curves $\varphi = cr^{1+\mu}$ and $\pi - \varphi = cr^{1+\mu}$, $0 \le r < r_0$, which share the origin, has similar properties near the origin as the curve defined by (1). The reader can check that $\gamma(c,\mu)$ is $C^{1,\mu}$ at origin but it is not C^{1,μ_1} for $\mu_1 > \mu$. Note that if a curve satisfies $\varphi \le cr^{1+\mu}$, then it is is below the curve $\gamma(c,\mu)$.

Let
$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbf{R}^{2\times 2}.$$
We will consider the matrix norm:
$$\Lambda_A = |A| = \max\{|Az| : z \in \mathbf{R}^2, |z| = 1\}$$
and the matrix function
$$\lambda_A = \min\{|Az| : |z| = 1\}.$$
Let D and G be subdomains of the complex plane $\mathbf{C}$, and $w = u + iv : D \to G$ be a function that has both partial derivatives at a point $z \in D$. By $\nabla w(z)$ we denote the matrix $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$. For the matrix ∇w we have
$$\Lambda_w(z) = |\nabla w| = |w_z| + |w_{\bar z}| \tag{1.6}$$
and
$$\lambda_w(z) = ||w_z| - |w_{\bar z}||, \tag{1.7}$$
where
$$w_z := \frac{1}{2}\left(w_x + \frac{1}{i}w_y\right) \text{ and } w_{\bar z} := \frac{1}{2}\left(w_x - \frac{1}{i}w_y\right).$$

We say that a function $u : D \to \mathbf{R}$ is ACL (absolutely continuous on lines) in the region D, if for every closed rectangle $R \subset D$ with sides parallel to the x and y-axes, u is absolutely continuous on a.e. horizontal and a.e. vertical line in R. Such a function has of course, partial derivatives u_x, u_y a.e. in D.

A sense-preserving homeomorphism $w\colon D \to G$, where D and G are subdomains of the complex plane $\mathbf{C}$, is said to be (K,K')-quasiconformal (or shortly (K,K')-q.c. or q.c.) $(K \ge 1,\ K' \ge 0)$ if $w \in$ACL and

$$|\nabla w|^2 \le KJ_w + K' \quad (z = re^{i\varphi}), \tag{1.8}$$
where J_w is the Jacobian of w given by
$$J_w = |w_z|^2 - |w_{\overline{z}}|^2 = \Lambda_w(z)\lambda_w(z). \tag{1.9}$$

Mappings which satisfy (1.8) arise naturally in elliptic equations, where $w = u + iv$, and u and v are partial derivatives of solutions, cf [14, Chapter XII]. If $K' = 0$, and $k = \frac{K-1}{K+1}$ then instead of (K, K')-q.c. we write $K-$q.c. or $k-$q.c.

Let Ω be a Jordan domain with rectifiable boundary. We will say that a mapping $f : \overline{\mathbf{U}} \to \overline{\Omega}$ is *normalized* if $f(t_i) = \omega_i$, $i = 0, 1, 2$, where t_0t_1, t_1t_2, t_2t_0 and $\omega_0\omega_1, \omega_1\omega_2, \omega_2\omega_0$ are arcs of $\mathbf{T}$ and of $\gamma = \partial\Omega$ respectively, having the same length $2\pi/3$ and $|\gamma|/3$ respectively.

We will say that a mapping $f : U \to V$ is *Hölder (Lipschitz) continuous*, if there exists a constant L such that

$$|f(z) - f(w)| \leq L|z - w|^\alpha, \quad z, w \in U,$$

where $0 < \alpha < 1$ ($\alpha = 1$).

1.1. Background and new results. Let γ be a Jordan curve. By the Riemann mapping theorem there exists a Riemann conformal mapping of the unit disk onto a Jordan domain $\Omega = \text{int } \gamma$. By Caratheodory's theorem it has a continuous extension to the boundary. Moreover if $\gamma \in C^{n,\alpha}$, $n \in \mathbb{N}$, $0 \leq \alpha < 1$, then the Riemann conformal mapping has $C^{n,\alpha}$ extension to the boundary (this result is known as Kellogg's theorem), see [60]. Conformal mappings are quasiconformal and harmonic. Hence quasiconformal harmonic (shortly HQC) mappings are natural generalization of conformal mappings. The class of HQC automorphisms of the unit disk has been first considered by Martio in [43]. Hengartner and Schober have shown that, for a given second dilatation ($a = \overline{f_{\bar{z}}}/f_z$, with $\|a\| < 1$) there exist a q.c. harmonic mapping f between two Jordan domains with analytic boundary ([18, Theorem 4.1]).

Recently there has been a number of authors who are working on the topic. The situation in which the image domain is different from the unit disk firstly has been considered by the first author in [22]. There it is observed that if f is harmonic K- quasiconformal mapping of the upper half-plane onto itself normalized such that $f(\infty) = \infty$, then $\text{Im} f(z) = cy$, where $c > 0$; hence f is bi-Lipschitz. In [22] (see also [24]) also characterization of HQC automorphisms of the upper half-plane by means of integral representation of analytic functions is given.

Using the result of Heinz ([17]): If w is a harmonic diffeomorphism of the unit disk onto itself with $w(0) = 0$, then $|w_z|^2 + |w_{\bar{z}}|^2 \geq \frac{1}{\pi^2}$, it can be shown that, every quasiconformal harmonic mapping of the unit disk onto itself is co-Lipschitz.

Further, Pavlović [54], by using the Mori's theorem on the theory of quasiconformal mappings and by using an important approach, proved the following intrigue result: every quasiconformal selfmapping of the unit disk is Lipschitz continuous. Partyka and Sakan ([53]) yield explicit Lipschitz and co-Lipschitz constants depending on a constant of quasiconformality. Using the Hilbert transforms of the derivative of boundary function, the first characterizations of HQC automorphisms of the upper half-plane and of the unit disk have been given in [54, 24]; for further result cf. [46]. Among the other things Knežević and the second author in [35] showed that a q.c. harmonic mapping of the unit disk onto itself is a $(1/K, K)$ quasi-isometry with respect to Poincaré and Euclidean distance. See also the paper of Chen and Fang [8] for a generalization of the previous result to convex domains.

Since the composition of a harmonic mapping and of a conformal mapping is itself harmonic, using the case of the unit disk and Kellogg's theorem, these theorems can be generalized to the class of mappings from arbitrary Jordan domain with $C^{1,\alpha}$ boundary onto the unit disk. However the composition of a conformal and a harmonic mapping is not, in general, a harmonic mapping. This means in particular that the results of this kind for arbitrary image domain do not follow from the case of the unit disk or the upper half-plane and Kellogg's theorem.

Using some new methods the results concerning the unit disk and the half-plane have been extended properly in the papers [23]–[32], [41] and [46]. In particular, in [25] we show how to apply Kellogg's theorem and that simple proof in the case of the upper half-plane has an analogy for C^2 domain; namely, we prove a version of "inner estimate" for quasi-conformal diffeomorphisms, which satisfies a certain estimate concerning their laplacian. As an application of this estimate, we show that quasi-conformal harmonic mappings between smooth domains (with respect to the approximately analytic metric), have bounded partial derivatives; in particular, these mappings are Lipschitz.

For related results about quasiconformal harmonic mappings with respect to the hyperbolic metric we refer to the paper of Wan [58] and of Marković [42].

Very recently, Iwaniec, Kovalev and Onninen in [19] have shown that, the class of quasiconformal harmonic mappings is also interesting concerning the modulus of annuli in complex plane.

In [48] C. B. Morrey proved a local Hölder estimate for quasiconformal mappings in the plane. Such a Hölder estimate was a fundamental development in the theory of quasiconformal mappings, and had very important applications to partial differential equations. Nirenberg in [50] made significant simplifications and improvements to Morrey's work (in particular, the restriction that the mappings involved be $1-1$ was removed), and he was consequently able to develop a rather complete theory for second order elliptic equation with 2 independent variables. Simon [57, Theorem 2.2] (see also Finn-Serrin [12]) obtain a Hölder estimate for (K, K') quasiconformal mappings, which is analogous to that obtained by Nirenberg in [50], but which is applicable to quasiconformal mappings between surfaces in Euclidean space.

Global Hölder continuity of $(K, 0)$-quasiconformal mapping between domains satisfying certain boundary conditions has been extensively studied by many authors and the results of this kind can be considered as generalizations of Mori's theorem (see for example the papers of Gehring & Martio [13] and Koskela, Onninen & Tyson [36]).

In this paper we overview some recent results concerning the class of harmonic quasiconformal mappings (HQC) and some new generalizations for the Hölder and Lipschitz continuity of the class of (K, K')-q.c. harmonic mappings between smooth domains.

In the section 2 we recall some results concerning the class of q.c. harmonic selfmappings of the unit disk and of the half-plane. The most important fact is that these classes are bi-Lipschitz mappings. Further in Section 3 we announce some new results concerning the class of HQC between Jordan domains with smooth $C^{1,\alpha}$ boundary, due to Božin and the second author, cf [6].

Section 4 contains Proposition 4.3 which can be considered as a Caratheodory theorem for $(K, K')-$ quasiconformal mappings and by using this proposition we extend Smirnov theorem for the class of (K, K')-quasiconformal harmonic mappings. By using Proposition 4.3, Heinz-Berenstein theorem (Lemma 5.8), and distance function with respect to image domain we first show that, (K, K') quasiconformal harmonic mappings are Lipschitz continuous, providing that the boundaries are twice differentiable Jordan curves (Theorem 5.9 the main result of [26] and of Section 5). Theorem 5.9 can be considered as extensions of Kellogg theorem and results of Martio, Pavlović, Partyka, Sakan and the authors. The method developed in [31], and Lemma 4.7 (which is a Mori's type theorem for the class of (K, K') quasiconformal mappings) has an important role on finding the quantitative Lipschitz constant, depending only on (K, K'), the domain and image domain, for normalized (K, K') quasiconformal harmonic mappings. By using Theorem 5.9, we prove Corollary 5.13, and this in turn implies that a harmonic diffeomorphism w between smooth Jordan domains is Lipschitz, if and only if w is (K, K') quasiconformal.

2. Characterizations of HQC

2.1. The half plane. In this section we present some results from [46]. By $\mathbf{H}$ we denote the upper-half plane and $\Pi^+ = \{z : \text{Re}z > 0\}$.

The first characterizations of the HQC conditions have been obtained by Kalaj in his thesis research.

In the case of the upper half plane, the following known fact plays an important role, cf for example [35]:

Lemma 2.1. *Let f be an euclidean harmonic $1-1$ mapping of the upper half-plane $\mathbf{H}$ onto itself, continuous on $\overline{\mathbf{H}}$, normalized by $f(\infty)=\infty$ and $v=\mathrm{Im} f$. Then $v(z)=c\,\mathrm{Im} z$, where c is a positive constant. In particular, v has bounded partial derivatives on $\mathbf{H}$.*

Lemma 2.1 is a corollary of the Herglotz representation of the positive harmonic function v (see for example [4]).

Theorem 2.2. *Let $h:\mathbf{H}\to\mathbf{H}$ be a harmonic function. Then h is orientation preserving harmonic diffeomorphism of $\mathbf{H}$ onto itself, continuous on $\mathbf{H}\cup\mathbf{R}$ such that $h(\infty)=\infty$ if and only if there are an analytic function $\phi:\mathbf{H}\to\Pi^+$ and constants $c>0$ and $c_1\in\mathbf{R}$ such that*

$\lim_{z\to\infty}\Phi_1(z)=\infty$, *where*

1. $\Phi(z)=\int_i^z\phi(\zeta)d\zeta$, $\Phi_1=\mathrm{Re}\,\Phi$, *and*
2. $h(z)=h^\phi(z)=\Phi_1(z)+icy+c_1$, $z\in\mathbf{H}$.

Let χ denote restriction of h on $\mathbf{R}$. In this setting, $h(z)=h^\phi(z)=P[\chi]+icy$, $z\in\mathbf{H}$, where $P=P_{\mathbf{H}}$ denotes the Poisson kernel for the upper half-plane $\mathbf{H}$.

A version of this result is proved in [22] and [24].

Let $h=u+iv$. By Lemma 2.1, $u=\mathrm{Re}\,\Phi$ and $v=cy$, where $c>0$ and Φ is analytic function in $\mathbf{H}$. Since $\Phi'_y=i\Phi'$ and

$$h(z)=h^\phi(z)=\frac{\Phi(z)+\overline{\Phi(z)}}{2}+icy+c_1,$$

we find

$$h'_y(z)=\frac{i\Phi'(z)+\overline{i\Phi'(z)}}{2}+ic=\frac{i\phi(z)-i\overline{\phi(z)}}{2}+ic=-\mathrm{Im}\,\phi(z)+ic$$

Hence $h'_x(z)=\mathrm{Re}\,\phi(z)$ and $h'_y(z)=-\mathrm{Im}\,\phi(z)+ic$. Since

$$h_z=\frac{h'_x-ih'_y}{2}=\frac{\phi}{2}+\frac{c}{2}$$

is analytic, $-h'_y$ is harmonic conjugate of h'_x and therefore $h'_y=H(h'_*)=H(\chi)=\mathrm{Im}\,\phi(z)-ic$, where $\chi=h_*$ denotes the restriction of h on $\mathbf{R}$.

By $HQC_0(\mathbf{H})$ (respectively $HQC_0^k(\mathbf{H})$) we denote the set of all qc (respectively k-qc) harmonic mappings h of $\mathbf{H}$ onto itself for which $h(\infty)=\infty$.

Recall by χ we denote restriction of h on $\mathbf{R}$. If $h\in HQC_0(\mathbf{H})$ it is well-known that $\chi:\mathbf{R}\to\mathbf{R}$ is a homeomorphism and $\mathrm{Re}\,h=P[\chi]$. Now we give characterizations of $h\in HQC_0(\mathbf{H})$ in terms of its boundary value χ.

Theorem 2.3 ([22], [46]). *The following condition are equivalent*

(A1) $h \in HQC_0(\mathbf{H})$

(A2) *there is an analytic function* $\phi : \mathbf{H} \to \Pi^+$ *such that* $\phi(\mathbf{H})$ *is relatively compact subset of* Π^+ *and* $h = h^\phi$.

Proof. Suppose (A1). We can suppose that h is K-qc and $c = 1$ in the representation (2). Since $v(z) = \operatorname{Im} h(z) = y$, we have $\lambda_h \geq 1/K$. Let $z_0 \in \mathbf{H}$ and define the curve $L = \{z : \Phi_1(z) = \Phi_1(z_0)\}$ and denote by l_0 the unit tangent vector to the curve L at z_0. Since $|dh_{z_0}(l_0)| \leq 1$, we have $\Lambda_h \leq K$ on $\mathbf{H}$. Hence absolute values of partial derivatives of h are bounded from above and below by two positive constants. Thus, by (3) and (4), ϕ is bounded on $\mathbf{H}$.

In particular, (A1) implies that h is bi-lipschitz.

Hence there are two positive constants s_1 and s_2 such that $s_1 \leq \chi'(x) \leq s_2$, a.e. Since $\chi'(x) = \operatorname{Re}\phi^*(x)$ a.e. on $\mathbf{R}$ and ϕ is bounded on $\mathbf{H}$, we find $s_1 \leq \operatorname{Re}\phi(z) \leq s_2$, $z \in \mathbf{H}$; and (A2) follows.

We leave to the reader to prove that (A2) implies (A1) and using equation (2.5) below to prove (A1) $\Rightarrow$ (A2).

■

It is clear that the conditions (A1) and (A2) are equivalent to:

(A3) *There is analytic function* $\phi \in H^\infty(\mathbf{H})$ *and there two positive constants* s_1 *and* s_2 *such that* $s_1 \leq \operatorname{Re}\phi(z) \leq s_2$, $z \in \mathbf{H}$.

Since $\chi'(x) = \operatorname{Re}\phi^*(x)$ a.e. on $\mathbf{R}$ and $H\chi' = \operatorname{Im}\phi^*(x) - ic$ a.e. on $\mathbf{R}$, we get characterization in terms of Hilbert transform:

(A4) χ *is absolutely continuous, and there two positive constants* s_1 *and* s_2 *such that* $s_1 \leq \chi'(x) \leq s_2$, *a.e. and* $H\chi'$ *is bounded.*

A similar characterization holds for smooth domains and in particular in the case of the unit disk; see Theorems 2.10 and 2.8 below.

From the proof of Theorem 2.4 below, cf [35], it follows that if we set $c = 1$ in the representation (2), then $h = h^\phi \in HQC_0^k(\mathbf{H})$ if and only $\phi(\mathbf{H})$ is in a disk $B_k = B(a_k; R_k)$, where

$$a_k = \frac{1}{2}(K + 1/K) = \frac{1+k^2}{1-k^2}$$

and

$$R_k = \frac{1}{2}(K - 1/K) = \frac{2k}{1-k^2}.$$

First, we need to introduce some notation: For $a \in \mathbf{C}$ and $r > 0$ we define $B(a; r) = \{z : |z - a| < r\}$. In particular, we write $\mathbf{U}_r$ instead of $B(0; r)$.

Theorem 2.4 ([35], the half plane euclidean-qch version)**.** *Let f be a K-qc euclidean harmonic diffeomorphism from* **H** *onto itself. Then f is a $(1/K, K)$ quasi-isometry with respect to Poincaré distance and with respect to Euclidean distance.*

For higher dimension version of this result see [44, 47, 3].

Proof. We first show that, by pre composition with a linear fractional transformation, we can reduce the proof to the case $f(\infty) = \infty$. If $f(\infty) \neq \infty$, there is the real number a such that $f(a) = \infty$. On the other hand, there is a conformal automorphism A of **H** such that $A(\infty) = a$. Since A is an isometry of **H** onto itself and $f \circ A$ is a K-qc euclidean harmonic diffeomorphism from **H** onto itself, the proof is reduced to the case $f(\infty) = \infty$.

It is known that f has a continuous extension to $\overline{\mathbf{H}}$ (see for example [37]). Hence, by Lemma B, $f = u + ic\,\mathrm{Im}\,z$, where c is a positive constant. Using the linear mapping B, defined by $B(w) = w/c$, and a similar consideration as the above, we can reduce the proof to the case $c = 1$. Therefore we can write f in the form

$$f = u + i\mathrm{Im}\,z = \frac{1}{2}(F(z) + z + \overline{F(z) - z}),$$

where F is a holomorphic function in **H**. Hence,

$$\mu_f(z) = \frac{F'(z) - 1}{F'(z) + 1} \quad \text{and} \quad F'(z) = \frac{1 + \mu_f(z)}{1 - \mu_f(z)}, \quad z \in \mathbf{H}. \tag{2.5}$$

Define

$$k = \frac{K - 1}{K + 1}$$

and

$$w = S\zeta = \frac{1 + \zeta}{1 - \zeta}.$$

Then, $S(U_k) = B_k = B(a_k; R_k)$, where

$$a_k = \frac{1}{2}(K + 1/K) = \frac{1 + k^2}{1 - k^2}$$

and

$$R_k = \frac{1}{2}(K - 1/K) = \frac{2k}{1 - k^2}.$$

Since f is k-qc, then $\mu_f(z) \in U_k$ and therefore $F'(z) \in B_k$ for $z \in \mathbf{H}$. This yields, first,

$$K + 1 \geq |F'(z) + 1| \geq 1 + 1/K, \quad K - 1 \geq |F'(z) - 1| \geq 1 - 1/K,$$

and then,

$$1 \leq \Lambda_f(z) = \frac{1}{2}(|F'(z)+1| + |F'(z)-1|) \leq K.$$

So we have $\lambda_f(z) \geq \Lambda_f(z)/K \geq 1/K$.
Thus, we find

$$1/K \leq \lambda_f(z) \leq \Lambda_f(z) \leq K. \tag{2.6}$$

Let λ denote the hyperbolic density on $\mathbf{H}$.
Since $\lambda(f(z)) = \lambda(z)$, $z \in \mathbf{H}$, using (2.6) and (A3), we obtain

$$\frac{1-k}{1+k}\, d_h(z_1, z_2) \leq d_h(f(z_1), f(z_2)) \leq \frac{1+k}{1-k}\, d_h(z_1, z_2).$$

It also follows from (2.6) that

$$\frac{1}{K}\,|z_2 - z_1| \leq |f(z_2) - f(z_1)| \leq K\,|z_2 - z_1|, \quad z_1, z_2 \in \mathbf{H}.$$

We leave to the reader to prove this inequality as an exercise.
This estimate is sharp (see also [24] for an estimate with some constant $c(K)$). ■

Theorem 2.4 has its counterpart for the unit disk.

Theorem 2.7 ([35], the unit disk euclidean-qch version)**.** *Let f be a K-qc euclidean harmonic diffeomorphism from $\mathbf{U}$ onto itself. Then f is a $(1/K, K)$ quasi-isometry with respect to Poincaré distance.*

2.2. The unit disc. We give characterizations in the case of the unit disk and for smooth domains (see below) similar to that in Theorem 2.3, which is related to the half-plane case.

Theorem 2.8. *Let ψ be a continuous increasing function on $\mathbf{R}$ such that $\psi(t+2\pi) - \psi(t) = 2\pi$, $\gamma(t) = e^{i\psi(t)}$ and $h = P[\gamma]$. Then h is q.c. if and only if*

1. $\operatorname{ess\,inf} \psi' > 0$,
2. *there is analytic function $\phi : \mathbf{U} \to \Pi^+$ such that $\phi(\mathbf{U})$ is relatively compact subset of Π^+ and $\psi'(x) = \operatorname{Re} \phi^*(e^{i\,x})$ a.e.*

In the setting of this theorem we write $h = P[\gamma] = h_\phi$. The reader can use the above characterization and functions of the form $\phi(z) = 2 + M(z)$, where M is an inner function, to produce examples of HQC mappings $h = h_\phi$ of the unit disk onto itself so the partial derivatives of h have no continuous extension to certain points on the unit circle. In particular we can take $M(z) = \exp\frac{z+1}{z-1}$, cf [5].

Let $\mathcal{D}_1$ (respectively $\mathcal{D}_2$) be the family of all Jordan domains in the plane which are of class $C^{1,\mu}$ (res $C^{2,\mu}$) for some $0 < \mu < 1$.

In the next subsection and Section 3 we extend the above theorem to smooth domains.

Note that the proof that a HQC mapping h between the unit disk and $\mathcal{D}_2$ (more generally C^2 and $C^{1,1}$) domain is bi-Lipshitz is more delicate than in the case of the upper half plane, see Theorem 3.1 below. For example, one can use Lemma 5.8 (which we call Heinz-Berenstein inner estimate) instead of Lemma 2.1 to prove that h is Lipshitz.

Note that the proof that a HQC mapping between the unit disk and $\mathcal{D}_1$ domain is bi-Lipshitz is much more delicate than in the case of the upper half plane, cf Theorem 3.5 below.

2.3. HQC and convex smooth codomains. We need the following result related to convex codomains.

Theorem 2.9. *Suppose that h is a euclidean harmonic mapping from $\mathbf{U}$ onto a bounded convex domain $D = h(\mathbf{U})$, which contains the disc $B(h(0); R_0)$. Then*

1. $d(h(z), \partial D) \geq (1 - |z|)R_0/2, \quad z \in \mathbf{U}$.
2. *Suppose that $\omega = h^*(e^{i\theta})$ and $h_r^* = h_r'(e^{i\theta})$ exist at a point $e^{i\theta} \in \mathbf{T}$, and there exists the unit inner normal $n = n_\omega$ at $\omega = h^*(e^{i\theta})$ with respect to ∂D. Then $E := (h_r^*, n_{h_*}) \geq c_0$, where $c_0 = \frac{R_0}{2}$.*
3. *In addition to the hypothesis stated in the item (2), suppose that h_*' exists at the point $e^{i\theta}$. Then $|J_h| = |(h_r^*, N)| = \big|(h_r^*, n)\big||N| \geq c_0|N|$, where $N = i\, h_*'$ and the Jacobian is computed at the point $e^{i\theta}$ with respect to the polar coordinates.*
4. *If D is of $C^{1,\mu}$ class and h is quasiconformal, then the function E is continuous on $\mathbf{T}$.*

For the proof of first three items we refer to [44] *(see also* [22] *for related results). For the proof of (4) see* [6] *or* [21].

Theorem 2.10. *Suppose that $C^{1,\alpha}$ domain D is convex and denote by γ positively oriented boundary of D. Let $h_0 : \mathbf{T} \to \gamma$ be an orientation preserving homeomorphism and $h = P[h_0]$. The following conditions are then equivalent*

a) h is K-qc mapping
b) h is bi-Lipschitz in the Euclidean metric
c) the boundary function h_* is bi-Lipschitz in the Euclidean metric and Hilbert transform $H[h_*']$ of its derivative is in L^∞.
d) the boundary function h_* is absolutely continuous, $\operatorname{ess\,sup}|h_*'| < +\infty$, $\operatorname{ess\,inf}|h_*'| > 0$ and Hilbert transform $H[h_*']$ of its derivative is in L^∞.

For an extension of this theorem concerning Cauchy transform $C[h'_*]$ see [46]. For the proof of this result for the case of the unit disk we refer to [54]. Theorem 2.10 has been proved in [31].

3. HQC are bi-Lipschitz

In this section we discus the co-Lipschitz character of the class HQC for non-convex smooth Jordan domains. Recall first that Theorem 2.10, which assert that a harmonic q.c. mapping of the unit disk of a convex domain is bi-Lipschitz, provided that the image domain is $C^{1,\alpha}$. For the proof of co-Lipschitz part of Theorem 2.10 the first author used a version of Heinz lemma for convex domains ([20]) refereed in this approach as *convexity type argument*. On the other hand, the first author proved the following result:

Theorem 3.1 ([32])**.** *Let $w = f(z)$ be a K quasiconformal harmonic mapping between a Jordan domain Ω_1 with $C^{1,\alpha}$ boundary and a Jordan domain Ω with $C^{2,\alpha}$ boundary. Let in addition $a \in \Omega_1$ and $b = f(a)$. Then w is bi-Lipschitz. Moreover there exists a positive constant $c = c(K, \Omega_1, \Omega, a, b) \geq 1$ such that*

$$\frac{1}{c}|z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq c|z_1 - z_2|, \quad z_1, z_2 \in \Omega_1. \tag{3.2}$$

Theorem 3.1 has been extended by the first author for $C^{1,1}$ image domains in [29].

In the following approach we announce some generalizations of results Theorem 3.1 for the class of HQC between $C^{1,\alpha}$ domains ($0 < \alpha < 1$).

3.1. The Gehring-Osgood inequality. For a domain $G \subset \mathbf{R}^n$ let $\rho : G \to (0, \infty)$ be a function. We say that ρ is a weight function or a metric density if for every locally rectifiable curve γ in G, the integral

$$l_\rho(\gamma) = \int_\gamma \rho(x) ds$$

exists.

In this case we call $l_\rho(\gamma)$ the ρ-length of γ. A metric density defines a metric $d_\rho : G \times G \to (0, \infty)$ as follows. For $a, b \in G$, let

$$d_\rho(a, b) = \inf_\gamma l_\rho(\gamma)$$

where the infimum is taken over all locally rectifiable curves in G joining a and b. It is an easy exercise to check that d_ρ satisfies the axioms of a metric. For instance, the hyperbolic (or Poincaré) metric of $\mathbf{D}$ is defined in terms of the density $\rho(x) = c/(1 - |x|^2)$ where $c > 0$ is a constant.

The quasi-hyperbolic metric $k = k_G$ of G is a particular case of the metric d_ρ when $\rho(x) = \frac{1}{d(x,\partial G)}$.

Suppose that $G \subset \mathbf{R}^n$, $f : G \to \mathbf{R}^n$ is K-qr and $G' = f(G)$. Let $\partial G'$ be a continuum containing at least two distinct points. By Gehring-Osgood inequality [11], there exists a constant $c > 0$ depending only on n and K such that

$$k_{G'}(fy, fx) \leq c \max\{k_G(y,x)^\alpha, k_G(y,x)\}, \alpha = K^{1/(1-n)}, \quad x, y \in G\,.$$

Using Gehring-Osgood inequality, the following result is proved:

Theorem 3.3. [47] *Under the above condition, if f, in addition, is a harmonic mapping, then $f : (G, k_G) \to (G', k_{G'})$ is Lipschitz.*

We can compute the quasihyperbolic metric k on $\mathbf{C}^*$ by using the covering $\exp : \mathbf{C} \to \mathbf{C}^*$, where exp is exponential function. Let $z_1, z_2 \in \mathbf{C}^*$, $z_1 = r_1 e^{it_1}, z_2 = r_2 e^{it_2}$ and $\theta = \theta(z_1, z_2) \in [0, \pi]$ the measure of convex angle between z_1, z_2. We will prove

$$k(z_1, z_2) = \sqrt{|\log \frac{r_2}{r_1}|^2 + \theta^2}\,.$$

Let $l = l(z_1)$ be line defined by 0 and z_1. Then z_2 belongs to one half-plane, say M, on which $l = l(z_1)$ divides $\mathbf{C}$.

Locally denote by log a branch of Log on M. Note that log maps M conformally onto horizontal strip of width π. Since $w = \log z$, we find the quasi-hyperbolic metric

$$|dw| = \frac{|dz|}{|z|}.$$

Note that $\rho(z) = \frac{1}{|z|}$ is the quasi-hyperbolic density for $z \in \mathbf{C}^*$ and therefore $k(z_1, z_2) = |w_1 - w_2| = |\log z_1 - \log z_2|$. Let $z_1, z_2 \in \mathbf{C}^*$, $w_1 = \log z_1 = \log r_1 + it_1$. Then $z_1 = r_1 e^{it_1}$; there is $t_2 \in [t_1, t_1 + \pi)$ or $t_2 \in [t_1 - \pi, t_1)$ and $w_2 = \log z_2 = \log r_2 + it_2$. Hence

$$k(z_1, z_2) = \sqrt{|\log \frac{r_2}{r_1}|^2 + (t_2 - t_1)^2}\,,$$

and therefore as a corollary of Gehring-Osgood inequality, we have

Proposition 3.4. *Let f be a K-qc mapping of the plane such that $f(0) = 0$, $f(\infty) = \infty$ and $\alpha = K^{-1}$. If $z_1, z_2 \in \mathbf{C}^*$, $|z_1| = |z_2|$ and $\theta \in [0, \pi]$* (respectively $\theta^* \in [0, \pi]$) is the measure of convex angle between z_1, z_2 (respectively $f(z_1), f(z_2)$), then

$$\theta^* \le c \max\{\theta^\alpha, \theta\},$$

where $c = c(K)$. In particular, if $\theta \le 1$, then $\theta^* \le c\theta^\alpha$.

We announce some results obtained in [6]. The results make use of Proposition 3.4, which is a corollary of the Gehring-Osgood inequality [11], as we are going to explain.

Let Ω be Jordan domain in $\mathcal{D}_1$, γ curve defined by $\partial\Omega$ and h K-qch from $\mathbf{U}$ onto Ω and $h(0) = a_0$. Then h is L-Lipschitz, where L depends only on $K, \mathrm{dist}(a_0, \partial\Omega)$ and $\mathcal{D}_1$ constant C_γ. In [6] we give an explicit bound for the Lipschitz constant.

Let h be a harmonic quasiconformal map from the unit disk onto D in class $\mathcal{D}_1$. Examples show that a q.c. harmonic function does not have necessarily a C^1 extension to the boundary as in conformal case. In [6] it is proved that the corresponding functions E_{h_*} is continuous on the boundary and for fixed θ_0, $\upsilon_{h_*}(z, \theta_0)$ is continuous in z at $e^{i\theta_0}$ on $\mathbf{U}$.

The main result in [6] is

Theorem 3.5. *Let Ω and Ω_1 be Jordan domains in $\mathcal{D}_1$, and let $h : \Omega \mapsto \Omega_1$ be a harmonic q.c. homeomorphism. Then h is bi-Lipschitz.*

It seems that we use a new idea here. Let Ω_1 be $C^{1,\mu}$ curve. We reduce proof to the case when $\Omega = H$. Suppose that $h(0) = 0 \in \Omega_1$. We show that there is a convex domain $D \subset \Omega_1$ in $\mathcal{D}_1$ such that $\gamma_0 = \partial D$ touch the boundary of Ω_1 at 0 and that the part of γ_0 near 0 is a curve $\gamma(c) = \gamma(c, \mu)$. Since there is qc extension h_1 of h to $\mathbf{C}$, we can apply Proposition 3.4 to $h_1 : \mathbf{C}^* \to \mathbf{C}^*$. This gives estimate for $\arg \gamma_1(z)$ for z near 0, where $\gamma_1 = h^{-1}(\gamma(c))$, and we show that there constants $c_1 > 0$ and μ_1 such that the graph of the curve $h^{-1}(\gamma(c))$ is below of the graph of the curve $\gamma(c_1) = \gamma(c_1, \mu_1)$. Therefore there is a domain $D_0 \subset \mathbf{H}$ in $\mathcal{D}_1$ such that $h(D_0) \subset D$. Finally, we combine the convexity type argument and noted continuity of functions E and υ to finish the proof.

4. Global and Hölder continuity of (K, K')-q.c. mappings

For $a \in \mathbf{C}$ and $r > 0$, put $D(a, r) := \{z : |z-a| < r\}$ and define $\Delta_r = \Delta_r(z_0) = \mathbf{U} \cap D(z_0, r)$. Denote by k_ρ the circular arc whose trace is $\{\zeta \in \mathbf{U} : |\zeta - \zeta_0| = \rho\}$.

Lemma 4.1 (The length-area principle, [26]). *Assume that f is a $(K, K')-$ q.c. on Δ_r, $0 < r < 1$, $z_0 \in \mathbf{T}$. Then*

$$F(r) := \int_0^r \frac{l_\rho^2}{\rho} d\rho \le \pi K A(r) + \frac{\pi}{2} K' r^2 , \tag{4.2}$$

where $l_\rho = |f(k_\rho)|$ *denote the length of* $f(k_\rho)$ *and* $A(r)$ *is the area of* $f(\Delta_r)$.

A topological space X is said to be locally connected at a point x if every neighborhood of x contains a connected open neighborhood. It is easy to verify that the set A is locally connected in $z_0 \in A$ if for every sequence $\{z_n\} \subset A$, which converges to z_0 there exists, for big enough n, connected set $L_n \subset A$ which contains z_0 and z_n such that $\mathrm{diam}(L_n) \to 0$. The set is locally connected if it is locally connected at every point. By using Lemma 4.1 we obtain

Proposition 4.3 (Caratheodory theorem for (K, K') mappings, [26]). *Let* D *be a simply connected domain in* $\overline{\mathbf{C}}$ *whose boundary has at least two boundary points such that* $\infty \notin \partial D$. *Let* $f : \mathbf{U} \to D$ *be a continuous mapping of the unit disk* $\mathbf{U}$ *onto* D *and* (K, K') *quasiconformal near the boundary* $\mathbf{T}$.

Then f *has a continuous extension to the boundary if and only if* ∂D *is locally connected.*

Remark 4.4. If we replace the hypothesis that f is (K, K') in Proposition 4.3 with $f \in W^{1,2}(\mathbf{U})$, for some $0 < r < 1$, then f has also continuous extension to $\overline{\mathbf{U}}$. After we wrote a version of this paper, Vuorinen informed us that results of this type related to Proposition 4.3 has been announced in [40].

Let $\gamma \in C^{1,\mu}$, $0 < \mu \le 1$, be a Jordan curve and let g be the arc length parameterization of γ and let $l = |\gamma|$ be the length of γ. Let d_γ be the distance between $g(s)$ and $g(t)$ along the curve γ, i.e.

$$d_\gamma(g(s), g(t)) = \min\{|s-t|, (l - |s-t|)\}. \tag{4.5}$$

A closed rectifiable Jordan curve γ enjoys a $b-$ chord-arc condition for some constant $b > 1$ if for all $z_1, z_2 \in \gamma$ there holds the inequality

$$d_\gamma(z_1, z_2) \le b|z_1 - z_2|. \tag{4.6}$$

It is clear that if $\gamma \in C^{1,\alpha}$ then γ enjoys a chord-arc condition for some $b_\gamma > 1$.

The following lemma is a (K, K')-quasiconformal version of [59, Lemma 1]. Moreover, here we give an explicit Hölder constant $L_\gamma(K, K')$. It is one of the main tools in proving Theorem 5.9.

Lemma 4.7. [26] *Assume that* γ *enjoys a chord-arc condition for some* $b > 1$. *Then for every* $(K, K')-$ *q.c. normalized mapping* f *between the unit disk* $\mathbf{U}$ *and the Jordan domain* $\Omega = \mathrm{int}\gamma$ *there holds*

$$|f(z_1) - f(z_2)| \le L_\gamma(K, K')|z_1 - z_2|^\alpha$$

for $z_1, z_2 \in \mathbf{T}$, $\alpha = \frac{1}{K(1+2b)^2}$ *and*

$$L_\gamma(K, K') = 4(1+2b)2^\alpha \sqrt{\max\left\{\frac{2\pi K|\Omega|}{\log 2}, \frac{2\pi K'}{K(1+2b)^2+4}\right\}}. \tag{4.8}$$

Remark 4.9. By applying Lemma 4.7, and by using the Möbius transforms, it follows that, if f is an arbitrary $(K, K')-$q.c. mapping between the unit disk $\mathbf{U}$ and Ω, where Ω satisfies the conditions of Lemma 4.7, then $|f(z_1) - f(z_2)| \leq C(f, \gamma, K, K')|z_1 - z_2|^\alpha$ on $\mathbf{T}$.

4.1. A question. Lemma 4.7 states that, every (K, K') quasiconformal mapping of the unit disk onto a Jordan domain with rectifiable boundary satisfying chord-arc condition is Hölder on the boundary. This can be extended a little bit, for example the lemma remains true if we put $z_1 \in \mathbf{T}$ and $z_2 \in \mathbf{U}$ instead of $z_1, z_2 \in \mathbf{T}$. On the other hand the results of Nirenberg, Finn, Serrin and Simon state that f is Holder continuous in every compact set of the unit disk. It remains an interesting and important open question, does every (K, K') quasiconformal mapping f between the unit disk and a Jordan domain with smooth boundary enjoy Hölder continuity.

The following theorem is an extension of the Smirnov theorem on the theory of conformal mappings to the class of (K, K') quasiconformal harmonic mappings. Let $h^1 = h^1(\mathbf{U})$ and $H^1 = H^1(\mathbf{U})$ be Hardy spaces of harmonic respectively analytic functions defined on the unit disk.

Theorem 4.10 (Smirnov theorem for (K, K') q.c. harmonic mappings,[26]). *Let w be a (K, K') quasiconformal harmonic mapping of the unit disk $\mathbf{U}$ onto a Jordan domain D. Then $\nabla w \in h^1$ if and only if ∂D is a rectifiable Jordan curve. Moreover, $\nabla w \in h^1$ implies that w is absolutely continuous on $\mathbf{T}$.*

5. Lipschitz continuity of (K, K')-q.c. harmonic mappings

In this section we formulate Theorem 5.9 which is the main result of the paper [26]. The proof is based on a result of Heinz and Berenstein (Lemma 5.8) and on auxiliary results (Lemmas 5.1- 5.6):

Lemma 5.1 ([30])**.** *Let Ω be a Jordan C^2 domain, $f : \mathbf{T} \to \partial\Omega$ injective continuous parameterization of $\partial\Omega$ and $w = P[f]$. Suppose that $w = P[f]$ is a Lipschitz continuous harmonic function between the unit disk $\mathbf{U}$ and Ω. Then for almost every $e^{i\varphi} \in \mathbf{T}$ we have*

$$\limsup_{r\to 1-0} J_w(re^{i\varphi}) \leq \frac{\pi}{4}\frac{\kappa_0}{2}|f'(\varphi)| \int_{-\pi}^{\pi} \frac{d_\gamma(f(e^{i(\varphi+x)}), f(e^{i\varphi}))^2}{x^2} dx, \tag{5.2}$$

where $J_w(z)$ *denotes the Jacobian of* w *at* z*,* $f'(\varphi) := \frac{d}{d\varphi} f(e^{i\varphi})$ *and*

$$\kappa_0 = \sup_s |\kappa_s|, \tag{5.3}$$

and κ_s *is the curvature of* γ *at the point* $g(s)$*.*

If we denote $\partial\Omega$ by γ, then under the conditions of the above lemma we have first $|R_f(\varphi, x)| \leq \frac{\kappa_0}{2} d_\gamma(f(e^{i\varphi}), f(e^{ix}))^2$ and therefore

$$E_\gamma(\varphi) \leq \frac{\pi}{4}\frac{\kappa_0}{2}\int_{-\pi}^{\pi} \frac{d_\gamma(f(e^{i(\varphi+x)}), f(e^{i\varphi}))^2}{x^2} dx\,,$$

for almost every $e^{i\varphi} \in \mathbf{T}$.

Let d be the distance function with respect to the boundary of the domain Ω: $d(w) = \operatorname{dist}(w, \partial\Omega)$. Let $\Gamma_\mu := \{z \in \Omega : d(z) \leq \mu\}$. For basic properties of distance function we refer to [14]. For example $\nabla d(w)$ is a unit vector for $w \in \Gamma_\mu$, and $d \in C^2(\overline{\Gamma_\mu})$, provided that $\partial\Omega \in C^2$ and $\mu \leq 1/\sup\{|\kappa_z| : z \in \partial\Omega\}$. We now have.

Lemma 5.4. [26] *Let* Ω *be a* C^2 *Jordan domain,* $w : \Omega_1 \mapsto \Omega$ *be a* C^1*,* (K, K') *q.c.,* $\chi = -d(w(z))$ *and* $\mu > 0$ *such that* $1/\mu > \kappa_0 = \operatorname{ess\,sup}\{|\kappa_z| : z \in \partial\Omega\}$*.*

Then:

$$|\nabla\chi| \leq |\nabla w| \leq K|\nabla\chi| + \sqrt{K'} \tag{5.5}$$

in $w^{-1}(\Gamma_\mu)$*.*

Lemma 5.6. [29] *Let* $\{e_1, e_2\}$ *be the natural basis in the space* $\mathbf{R}^2$ *and* Ω, Ω_1 *be two* C^2 *domains. Let* $w : \Omega_1 \mapsto \Omega$ *be a harmonic mapping and let* $\chi = -d(w(z))$ *and* $\mu > 0$ *such that* $1/\mu > \kappa_0 = \operatorname{ess\,sup}\{|\kappa_z| : z \in \partial\Omega\}$*. Then*

$$\Delta\chi(z_0) = \frac{\kappa_{\omega_0}}{1 - \kappa_{\omega_0} d(w(z_0))} |(O_{z_0}\nabla w(z_0))^T e_1|^2, \tag{5.7}$$

where $e_1 \in T_{z_0}$ *and* T_{z_0} *denotes the tangent space at* z_0*,* $z_0 \in w^{-1}(\Gamma_\mu)$*,* $\omega_0 \in \partial\Omega$ *with* $|w(z_0) - \omega_0| = \operatorname{dist}(w(z_0), \partial\Omega)$*, and* O_{z_0} *is an orthogonal transformation.*

Lemma 5.8 (Heinz-Berenstein)**.** [16]*. Let* $\chi : \overline{\mathbf{U}} \mapsto \mathbf{R}$ *be a continuous function between the unit disc* $\overline{\mathbf{U}}$ *and the real line satisfying the conditions:*

1. χ *is* C^2 *on* $\mathbf{U}$*,*
2. $\chi(\theta) = \chi(e^{i\theta})$ *is* C^2 *and*
3. $|\Delta\chi| \leq a|\nabla\chi|^2 + b$ *on* $\mathbf{U}$ *for some constant* c_0 *(natural growth condition).*

Then the function $|\nabla\chi| = |\operatorname{grad}\chi|$ *is bounded on* $\mathbf{U}$*.*

Theorem 5.9 (The main theorem)**.** [26] *Suppose that*

(a1) Ω *is a Jordan domain with* C^2 *boundary and*
(a2) w *is* (K, K') *-q.c. harmonic mapping between the unit disk and* Ω.

Then

(c1) w *has a continuous extension to* $\overline{\mathbf{U}}$, *whose restriction to* $\mathbf{T}$ *we denote by* f.
(c2) *Furthermore,* w *is Lipschitz continuous on* $\mathbf{U}$.
(c3) *If* f *is normalized, there exists a constant* $L = L(K, K', \partial\Omega)$ *(which satisfies the inequality* (5.17) *below) such that*

$$|f'(t)| \le L \text{ for almost every } t \in [0, 2\pi], \tag{5.10}$$

and

$$|w(z_1) - w(z_2)| \le (KL + \sqrt{K'})|z_1 - z_2| \quad \text{for } z_1, z_2 \in \mathbf{U}. \tag{5.11}$$

Remark 5.12. Note that a C^2 curve satisfies $b-$chord-arc condition for some b and has bounded curvature and that the constant L in the previous theorem depends only on K, K', κ_0 and b, where κ_0 is its maximal curvature.

By using Theorem 5.9 we deduce

Corollary 5.13 ([26]). *Let* h *be a harmonic orientation preserving diffeomorphism between two plane Jordan domains* Ω *and* D *with* C^2 *boundaries. Let in addition* $\phi : \mathbf{U} \to \Omega$ *be a conformal transformation and take* $w = h \circ \phi = P[f]$. *Then the following conditions are equivalent*

1. h *is a* (K, K')*-qc mapping.*
2. h *is Lipschitz w.r. to the Euclidean metric.*
3. f *is absolutely continuous on* $\mathbf{T}$, $f' \in L^\infty(\mathbf{T})$ *and* $H[f'] \in L^\infty(\mathbf{T})$.

The following proposition makes clear difference between (K, K')-q.c. harmonic mappings and K-q.c. ($(K, 0)$-q.c.) harmonic mappings.

Proposition 5.14 ([29, 21]). *Under conditions of Corollary 5.13, the following conditions are equivalent:*

1. h *is a* K*-qc mapping.*
2. h *is bi-Lipschitz w.r. to the Euclidean metric.*
3. f *is absolutely continuous on* $\mathbf{T}$ *and* $f', 1/l(\nabla h), H[f'] \in L^\infty(\mathbf{T})$.

By using Corollary 5.13 and Proposition 5.14, we obtain that the function given in the following example is a (K, K')-quasiconformal harmonic mapping which is not $(K, 0)$ quasiconformal.

Example 5.15. Let $f(e^{ix}) = e^{i(x+\sin x)}$. Then the mapping $w = P[f]$ is a Lipschitz mapping of the unit disk $\mathbf{U}$ onto itself, because $f \in C^\infty(\mathbf{T})$ and therefore w is (K, K')-quasiconformal for some K and K' (Corollary 5.13) but it is not $(K, 0)$-quasiconformal for any K, because f is not bi-Lipschitz.

Remark 5.16. a) The proof of Theorem 5.9 yields the following estimate of a Lipschitz constant L for a normalized $(K, K')-$quasiconformal harmonic mapping between the unit disk and a Jordan domain Ω bounded by a Jordan curve $\gamma \in C^2$ satisfying a $b-$chord-arc condition.

$$L \leq \Big(K\lambda\kappa_0 b\,(L_\gamma(K, K'))^{1+1/\lambda}\pi^{1/\lambda} + \sqrt{K'}\Big)^\lambda, \tag{5.17}$$

where

$$\alpha = \frac{1}{K(1+2b)^2},\ \lambda = \frac{2-\alpha}{\alpha},$$

κ_0 is defined by (5.3) and $L_\gamma(K, K')$ in (4.8). Thus L depends only on K, K', κ_0 and $b-$chord-arc condition.

See [54], [53], [35] and [23] for estimates, in the special case where γ is the unit circle, and w is $K-$q.c. $(K' = 0)$.

b) Notice that, the proof of Theorem 5.9 did not depend on Kellogg's and Warschawski theorem (that implies that a conformal mapping of the unit disk onto a Jordan domain Ω with $C^{1,\alpha}$ boundary is bi-Lipschitz) nor on Lindelöf theorem in the theory of conformal mappings (see [15] for this topic). For a generalization of Kellogg's theorem we refer to the paper of Lesley and Warschawski [39], where they gave an example of C^1 Jordan domain D, such that the Riemann conformal mapping of the unit disk $\mathbf{U}$ onto D is not Lipschitz. We expect that, the conclusion of Theorem 5.9 remains true, assuming only that the boundary of Ω is $C^{1,\alpha}$. This problem has been overcome for the class of K-q.c. mappings in [31] by composing by conformal mappings and by using "approximation argument". However, the composition of a (K, K') q.c. mapping and a conformal mapping is not necessarily a (K_1, K_1') q.c. mapping, and it causes further difficulties because the method used in [31] does not work for (K, K') q.c. mappings in general.

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