INVERTIBLE HARMONIC MAPPINGS BEYOND KNESER THEOREM AND QUASICONFORMAL HARMONIC MAPPINGS

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ABSTRACT. In this paper we extend Rado-Choquet-Kneser theorem for the mappings with Lipschitz boundary data and essentially positive Jacobian at the boundary without restriction on the convexity of image domain. The proof is based on a recent extension of Rado-Choquet-Kneser theorem by Alessandrini and Nesi [2] and it is used an approximation schema. Some applications for the family of quasiconformal harmonic mappings between Jordan domains are given.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Harmonic mappings in the plane are univalent complex-valued harmonic functions of a complex variable. Conformal mappings are a special case where the real and imaginary parts are conjugate harmonic functions, satisfying the Cauchy-Riemann equations. Harmonic mappings were studied classically by differential geometers because they provide isothermal (or conformal) coordinates for minimal surfaces. More recently they have been actively investigated by complex analysts as generalizations of univalent analytic functions, or conformal mappings. For the background to this theory we refer to the book of Duren [6]. If w is a univalent complex-valued harmonic functions, then by Lewy's theorem (see [24]), w has a non-vanishing Jacobian and consequently, according to the inverse mapping theorem, w is a diffeomorphism. Moreover, if w is a harmonic mapping of the unit disk $\mathbf{U}$ onto a convex Jordan domain Ω, mapping the boundary $\mathbf{T} = \partial\mathbf{U}$ onto $\partial\Omega$ homeomorphically, then w is a diffeomorphism. This is celebrated theorem of Rado, Kneser and Choquet ([21]). This theorem has been extended in various directions (see for example [11], [3], [36] and [35]). One of the recent extensions is the following proposition, due to Nesi and Alessandrini, which is one of the main tools in proving our main result.

Proposition 1.1. [2] *Let $F : \mathbf{T} \to \gamma \subset \mathbf{C}$ be an orientation preserving diffeomorphism of class C^1 onto a simple closed curve of the complex plane $\mathbf{C}$. Let D be a bounded domain such that $\partial D = \gamma$. Let $w = P[F] \in C^1(\overline{\mathbf{U}}; \mathbf{C})$, where $P[f]$ is the Poisson extension of F. The mapping w is a diffeomorphism of $\overline{\mathbf{U}}$ onto $\overline{D}$ if and only if*

$$J_w(e^{it}) > 0 \text{ everywhere on } \mathbf{T}, \tag{1.1}$$

2000 *Mathematics Subject Classification.* Primary 30C55, Secondary 31C05.
Key words and phrases. Planar harmonic mappings, Quasiconformal mapping, Convex domains, Rado-Kneser-Choquet theorem.

where $J_w(e^{it}) := \lim_{r\to 1^-} J_w(re^{it})$, *and* $J_w(re^{it})$ *is the Jacobian of* w *at* re^{it}.

In this paper we generalize Rado-Kneser-Choquet theorem as follows.

Theorem 1.2 (The main result)**.** *Let* $F : \mathbf{T} \to \gamma \subset \mathbf{C}$ *be an orientation preserving Lipschitz weak homeomorphism of the unit circle* $\mathbf{T}$ *onto a* $C^{1,\alpha}$ *smooth Jordan curve. Let* D *be a bounded domain such that* $\partial D = \gamma$. *Then* $J_w(e^{it})/|F'(t)|$ *exists a.e. in* $\mathbf{T}$ *and has a continuous extension* $T_w(e^{it})$ *to* $\mathbf{T}$. *If*

$$T_w(e^{it}) > 0 \quad \text{everywhere on } \mathbf{T}, \tag{1.2}$$

then the mapping $w = P[F]$ *is a diffeomorphism of* $\mathbf{U}$ *onto* D.

In order to compare this statement with Kneser's Theorem, it is worth noticing that when D is convex, then by Remark 3.2 the condition (1.2) is automatically satisfied.

It follows from Theorem 1.2 that under its conditions, the Jacobian J_w of w has continuous extension to the boundary provided that $F \in C^1(\mathbf{T})$ and it should be noticed that this does not mean that the partial derivatives of w have necessarily a continuous extension to the boundary (see e.g. [27] for a counterexample).

Note that we do not have any restriction on convexity of image domain in Theorem 1.2 which is proved in section 3.

Using this theorem, in section 4 we characterize all quasiconformal harmonic mappings between the unit disk $\mathbf{U}$ and a smooth Jordan domain D, in terms of boundary data (see Theorem 4.1) which could be considered as a variation of Proposition 1.1.

2. Preliminaries

2.1. **Arc length parameterization of a Jordan curve.** Suppose that γ is a rectifiable Jordan curve in the complex plane $\mathbf{C}$. Denote by l the length of γ and let $g : [0, l] \mapsto \gamma$ be an arc length parameterization of γ, i.e. a parameterization satisfying the condition:

$$|g'(s)| = 1 \text{ for all } s \in [0, l].$$

We will say that γ is of class $C^{1,\alpha}$, $0 < \alpha \le 1$, if g is of class C^1 and

$$\sup_{t,s} \frac{|g'(t) - g'(s)|}{|t - s|^\alpha} < \infty.$$

Definition 2.1. Let $l = |\gamma|$. We will say that a surjective function $F = g \circ f : \mathbf{T} \to \gamma$ is a *weak homeomorphism*, if $f : [0, 2\pi] \to [0, l]$ is a nondecreasing surjective function.

Definition 2.2. Let $f : [a, b] \to \mathbf{C}$ be a continuous function. The modulus of continuity of f is

$$\omega(t) = \omega_f(t) = \sup_{|x-y|\le t} |f(x) - f(y)|.$$

The function f is called Dini continuous if

$$\int_{0+} \frac{\omega_f(t)}{t}\, dt < \infty. \tag{2.1}$$

Here $\int_{0+} := \int_0^k$ for some positive constant k. A smooth Jordan curve γ with the length $l = |\gamma|$, is said to be Dini smooth if g' is Dini continuous. Observe that every smooth $C^{1,\alpha}$ Jordan curve is Dini smooth.

Let

$$K(s,t) = \text{Re}\,[\overline{(g(t)-g(s))} \cdot ig'(s)] \tag{2.2}$$

be a function defined on $[0,l] \times [0,l]$. By $K(s \pm l, t \pm l) = K(s,t)$ we extend it on $\mathbf{R} \times \mathbf{R}$. Note that $ig'(s)$ is the inner unit normal vector of γ at $g(s)$ and therefore, if γ is convex then

$$K(s,t) \geq 0 \text{ for every } s \text{ and } t. \tag{2.3}$$

Suppose now that $F : \mathbf{R} \mapsto \gamma$ is an arbitrary 2π periodic Lipschitz function such that $F|_{[0,2\pi)} : [0,2\pi) \mapsto \gamma$ is an orientation preserving bijective function. Then there exists an increasing continuous function $f : [0,2\pi] \mapsto [0,l]$ such that

$$F(\tau) = g(f(\tau)). \tag{2.4}$$

In the remainder of this paper we will identify $[0,2\pi)$ with the unit circle $\mathbf{T}$, and $F(s)$ with $F(e^{is})$. In view of the previous convention we have for a.e. $e^{i\tau} \in \mathbf{T}$ that

$$F'(\tau) = g'(f(\tau)) \cdot f'(\tau),$$

and therefore

$$|F'(\tau)| = |g'(f(\tau))| \cdot |f'(\tau)| = f'(\tau).$$

Along with the function K we will also consider the function K_F defined by

$$K_F(t,\tau) = \text{Re}\,[\overline{(F(t)-F(\tau))} \cdot iF'(\tau)].$$

It is easy to see that

$$K_F(t,\tau) = f'(\tau)K(f(t), f(\tau)). \tag{2.5}$$

Lemma 2.3. *If γ is Dini smooth, and ω is modulus of continuity of g', then*

$$|K(s,t)| \leq \int_0^{\min\{|s-t|, l-|s-t|\}} \omega(\tau) d\tau. \tag{2.6}$$

Proof. Note that

$$\begin{aligned} K(s,t) &= \text{Re}[\overline{(g(t)-g(s))} \cdot ig'(s)] \\ &= \text{Re}\left[\overline{(g(t)-g(s))} \cdot i\left(g'(s) - \frac{g(t)-g(s)}{t-s}\right)\right], \end{aligned}$$

and

$$g'(s) - \frac{g(t)-g(s)}{t-s} = \int_s^t \frac{g'(s)-g'(\tau)}{t-s} d\tau.$$

Therefore

$$\left|g'(s) - \frac{g(t)-g(s)}{t-s}\right| \le \int_s^t \frac{|g'(s)-g'(\tau)|}{t-s} d\tau \le \int_s^t \frac{\omega(\tau - s)}{t-s} d\tau = \frac{1}{t-s}\int_0^{t-s} \omega(\tau) d\tau.$$

On the other hand

$$|\overline{g(t)-g(s)}| \le \sup_{s\le x\le t} |g'(x)|(t-s) = (t-s).$$

It follows that

$$|K(s,t)| \le \int_0^{|s-t|} \omega(\tau) d\tau. \tag{2.7}$$

Since $K(s\pm l, t\pm l) = K(s,t)$ according to (2.7) we obtain (2.6). □

Lemma 2.4. *If $\omega : [0,l] \to [0,\infty)$, $\omega(0) = 0$, is a bounded function satisfying $\int_{0+} \omega(x)dx/x < \infty$, then for every constant a, we have $\int_{0+}\omega(ax)dx/x < \infty$. Next for every $0 < y \le l$ holds the following formula:*

$$\int_{0+}^{y} \frac{1}{x^2}\int_0^x \omega(at)dtdx = \int_{0+}^{y} \frac{\omega(ax)}{x} - \frac{\omega(ax)}{y} dx. \tag{2.8}$$

Proof. The first statement of the lemma is immediate. Taking the substitutions $u = \int_0^x \omega(at)dt$ and $dv = x^{-2}dx$, and using the fact that

$$\lim_{\alpha\to 0} \frac{\int_0^\alpha \omega(at)dt}{\alpha} = \omega(0) = 0$$

we obtain:

$$\begin{aligned}\int_{0+}^{y} \frac{1}{x^2}\int_0^x \omega(at)dtdx &= \lim_{\alpha\to 0+}\int_{\alpha}^{y} \frac{1}{x^2}\int_0^x \omega(at)dtdx \\ &= -\lim_{\alpha\to 0+} \left.\frac{\int_0^x \omega(at)dt}{x}\right|_\alpha^y + \lim_{\alpha\to 0+}\int_\alpha^y \frac{\omega(ax)}{x}dx \\ &= \int_{0+}^{y} \frac{\omega(ax)}{x} - \frac{\omega(ax)}{y}dx.\end{aligned}$$

□

A function $\varphi : A \to B$ is called $(\ell, \mathcal{L})$ bi-Lipschitz, where $0 < \ell < \mathcal{L} < \infty$, if $\ell|x-y| \le |\varphi(x) - \varphi(y)| \le \mathcal{L}|x-y|$ for $x, y \in A$.

Lemma 2.5. *If $\varphi : \mathbf{R} \to \mathbf{R}$ is a $(\ell, \mathcal{L})$ bi-Lipschitz mapping ($\mathcal{L}$ Lipschitz weak homeomorphism), such that $\varphi(x+a) = \varphi(x) + b$ for some a and b and every x, then there exist a sequence of $(\ell, \mathcal{L})$ bi-Lipschitz diffeomorphisms (respectively a sequence of diffeomorphisms) $\varphi_n : \mathbf{R} \to \mathbf{R}$ such that φ_n converges uniformly to φ, and $\varphi_n(x+a) = \varphi_n(x) + b$.*

Proof. We introduce appropriate mollifiers: fix a smooth function $\rho:\mathbb{R}\to[0,1]$ which is compactly supported on the interval $(-1,1)$ and satisfies $\int_{\mathbb{R}}\rho=1$. For $\varepsilon=1/n$ consider the mollifier

$$\rho_\varepsilon(t):=\frac{1}{\varepsilon}\,\rho\left(\frac{t}{\varepsilon}\right). \tag{2.9}$$

It is compactly supported in the interval $(-\varepsilon,\varepsilon)$ and satisfies $\int_{\mathbb{R}}\rho_\varepsilon=1$. Define

$$\varphi_\varepsilon(x)=\varphi*\rho_\varepsilon=\int_{\mathbf{R}}\varphi(y)\frac{1}{\varepsilon}\rho\left(\frac{x-y}{\varepsilon}\right)dy=\int_{\mathbf{R}}\varphi(x-\varepsilon z)\rho(z)dz.$$

Then

$$\varphi_\varepsilon'(x)=\int_{\mathbf{R}}\varphi'(x-\varepsilon z)\rho(z)dz.$$

It follows that

$$\ell\int_{\mathbf{R}}\rho(z)dz=\ell\le|\varphi_\varepsilon'(x)|\le\mathcal{L}\int_{\mathbf{R}}\rho(z)dz=\mathcal{L}.$$

The fact that $\varphi_\varepsilon(x)$ converges uniformly to φ follows by Arzela-Ascoli theorem.

To prove the case, when φ is a $\mathcal{L}$ Lipschitz weak homeomorphism, we make use of the following simple fact. Since φ is $\mathcal{L}$ -Lipschitz, then

$$\varphi_m(x)=\frac{mb}{mb+a}(\varphi(x)+x/m)$$

is $(\ell_m,\mathcal{L}_m)$ bi-Lipschitz, for some $\ell_m,\mathcal{L}_m$, with $\varphi_m(x+a)=\varphi_m(x)+b$, and φ_m converges uniformly to φ. By the previous case, we can choose a diffeomorphism

$$\psi_m=\varphi_m*\rho_{\varepsilon_m}=\frac{mb}{mb+a}\left(\varphi*\rho_{\varepsilon_m}+\frac{x}{m}\right), \tag{2.10}$$

such that $\|\psi_m-\varphi_m\|_\infty\le 1/m$. Thus

$$\lim_{n\to\infty}\|\psi_n-\varphi\|_\infty=0.$$

The proof is completed. □

2.2. **Harmonic functions and Poisson integral.** The function

$$P(r,t)=\frac{1-r^2}{2\pi(1-2r\cos t+r^2)},\quad 0\le r<1,\ \ t\in[0,2\pi]$$

is called the Poisson kernel. The Poisson integral of a complex function $F\in L^1(\mathbf{T})$ is a complex harmonic function given by

$$w(z)=u(z)+iv(z)=P[F](z)=\int_0^{2\pi}P(r,t-\tau)F(e^{it})dt, \tag{2.11}$$

where $z=re^{i\tau}\in\mathbf{U}$. We refer to the book of Axler, Bourdon and Ramey [4] for good setting of harmonic functions.

The Hilbert transformation of a function $\chi\in\mathrm{L}^1(\mathbf{T})$ is defined by the formula

$$\tilde{\chi}(\tau)=H(\chi)(\tau)=-\frac{1}{\pi}\int_{0+}^{\pi}\frac{\chi(\tau+t)-\chi(\tau-t)}{2\tan(t/2)}\mathrm{d}t.$$

This integral is improper and converges for a.e. $\tau \in [0, 2\pi]$; this and other facts concerning the operator H used in this paper can be found in the book of Zygmund [41, Chapter VII]. If f is a harmonic function then a harmonic function $\tilde{f}$ is called the harmonic conjugate of f if $f + i\tilde{f}$ is an analytic function. Let $\chi, \tilde{\chi} \in L^1(\mathbf{T})$. Then

$$P[\tilde{\chi}] = \widetilde{P[\chi]}, \tag{2.12}$$

where $\tilde{k}(z)$ is the harmonic conjugate of $k(z)$ (see e.g. [33, Theorem 6.1.3]).

Assume that $z = x + iy = re^{i\tau} \in \mathbf{U}$. The complex derivatives of a differential mapping $w : \mathbf{U} \to \mathbf{C}$ are defined as follows:

$$w_z = \frac{1}{2}\left(w_x + \frac{1}{i}w_y\right)$$

and

$$w_{\bar{z}} = \frac{1}{2}\left(w_x - \frac{1}{i}w_y\right).$$

The derivatives of w in polar coordinates can be expressed as

$$w_\tau(z) := \frac{\partial w(z)}{\partial \tau} = i(zw_z - \overline{z}w_{\bar{z}})$$

and

$$w_r(z) := \frac{\partial w(z)}{\partial r} = (e^{i\tau}w_z + e^{-i\tau}w_{\bar{z}}).$$

The Jacobian determinant of w is expressed in polar coordinates as

$$J_w(z) = |w_z|^2 - |w_{\bar{z}}|^2 = \frac{1}{r}\mathrm{Im}(h_\tau \overline{h_r}) = \frac{1}{r}\mathrm{Re}(ih_r\overline{h}_\tau). \tag{2.13}$$

Assume that $w = P[F](z)$ is a harmonic function defined on the unit disk $\mathbf{U}$. Then there exist two analytic functions h and k defined in the unit disk such that $w = h + \overline{k}$. Moreover $w_\tau = i(zh'(z) - \bar{z}\overline{k'(z)})$ is a harmonic function and $rw_r = zh'(z) + \bar{z}\overline{k'(z)}$ is its harmonic conjugate.

It follows from (2.11) that w_τ equals the Poisson–Stieltjes integral of F':

$$\begin{aligned}
w_\tau(re^{i\tau}) &= \int_0^{2\pi} \partial_\tau P(r, \tau - t)F(t)dt \\
&= -\int_0^{2\pi} \partial_t P(r, \tau - t)F(t)dt \\
&= -\int_0^{2\pi} \partial_t P(r, \tau - t)F(t)dt \\
&= -P(r, \tau - t)F(t)|_{t=0}^{2\pi} + \int_0^{2\pi} P(r, \tau - t)dF(t) \\
&= \int_0^{2\pi} P(r, \tau - t)dF(t).
\end{aligned}$$

Hence, by Fatou's theorem, the radial limits of w_τ exist a.e. and

$$\lim_{r \to 1^-} w_\tau(re^{i\tau}) = F_0'(\tau), \quad a.e.,$$

where F_0 is the absolutely continuous part of F.

As rw_r is the harmonic conjugate of w_τ, it turns out that if F is absolutely continuous, then

$$\lim_{r\to 1^-} w_r(re^{i\tau}) = H(F')(\tau)\ (a.e.), \tag{2.14}$$

and

$$\lim_{r\to 1^-} w_\tau(re^{i\tau}) = F'(\tau)\quad (a.e). \tag{2.15}$$

3. The proof of the main theorem

The aim of this chapter is to prove Theorem 1.2. We will construct a suitable sequence w_n of univalent harmonic mappings, converging almost uniformly to $w = P[F]$. In order to do so, we will mollify the boundary function F, by a sequence of diffeomorphism F_n and take the Poisson extension $w_n = P[F_n]$. We will show that under the condition of Theorem 1.2 for large n, w_n satisfies the conditions of theorem of Alessandrini and Nesi. By a result of Hengartner and Schober [9], the limit function w of a locally uniformly convergent sequence of univalent harmonic mappings w_n is univalent, providing that F is a surjective mapping.

We begin by the following lemma.

Lemma 3.1. *Let γ be a Dini smooth Jordan curve, denote by g its arc-length parameterization and assume that $F(t) = g(f(t))$ is a Lipschitz weak homeomorphism from the unit circle onto γ. If $w(z) = u(z)+iv(z) = P[F](z)$ is the Poisson extension of F, then for almost every $\tau \in [0, 2\pi]$ exists the limit*

$$J_w(e^{i\tau}) := \lim_{r\to 1^-} J_w(re^{i\tau})$$

and there holds the formula

$$J_w(e^{i\tau}) = f'(\tau)\int_0^{2\pi} \frac{\mathrm{Re}\,[\overline{(g(f(t)) - g(f(\tau)))}\cdot ig'(f(\tau))]}{2\sin^2\frac{t-\tau}{2}}dt. \tag{3.1}$$

Proof. Let $z = re^{i\tau}$. Since F is Lipschitz it is absolutely continuous and by (2.15) and (2.14) we obtain that there exist radial derivatives of w_τ and w_r for a.e. $e^{i\tau} \in \mathbf{T}$. By Fatou's theorem (see e.g. [4, Theorem 6.39], c.f. (2.15))), we have

$$\lim_{r\to 1^-} w_\tau(re^{i\tau}) = F'(\tau) \tag{3.2}$$

for almost every $e^{i\tau} \in \mathbf{T}$.

Further for a.e. $\tau \in [0, 2\pi]$, by using Lagrange theorem we have

$$\frac{u(e^{i\tau}) - u(re^{i\tau})}{1-r} = u_r(pe^{i\tau}),\quad r < p < 1$$

and

$$\frac{v(e^{i\tau}) - v(re^{i\tau})}{1-r} = v_r(qe^{i\tau}),\quad r < q < 1.$$

It follows that for a.e. $\tau \in [0, 2\pi]$

$$\lim_{r\to 1^-} \frac{u(e^{i\tau}) - u(re^{i\tau})}{1-r} = \lim_{r\to 1^-} u_r(r(e^{i\tau})) \tag{3.3}$$

and

$$\lim_{r\to 1^-} \frac{v(e^{i\tau}) - v(re^{i\tau})}{1-r} = \lim_{r\to 1^-} v_r(r(e^{i\tau})) \tag{3.4}$$

and consequently for a.e. $\tau \in [0, 2\pi]$

$$\lim_{r\to 1^-} \frac{w(e^{i\tau}) - w(re^{i\tau})}{1-r} = \lim_{r\to 1^-} w_r(r(e^{i\tau})). \tag{3.5}$$

By using the previous facts and the formulas

$$w(e^{i\tau}) - w(re^{i\tau}) = \int_0^{2\pi} [F(\tau) - F(t)]P(r, \tau - t)dt$$

and (2.13) we obtain:

(3.6)

$$\begin{aligned}
\lim_{r\to 1^-} J_w(re^{i\tau}) &= \lim_{r\to 1^-} \frac{\mathrm{Re}[iw_r(re^{i\tau})\overline{w_\tau(re^{i\tau})}]}{r} \\
&= \lim_{r\to 1^-} \frac{\mathrm{Re}[i(w(e^{i\tau}) - w(re^{i\tau}))\overline{w_\tau(re^{i\tau})}]}{(1-r)r} \\
&= \lim_{r\to 1^-} \frac{1}{1-r} \int_0^{2\pi} P(r, \tau - t)\mathrm{Re}[i(F(\tau) - F(t))\overline{F'(\tau)}]dt \\
&= \lim_{r\to 1^-} \int_{-\pi}^{\pi} K_F(t + \tau, \tau) \frac{P(r,t)}{1-r} dt, \quad a.e.
\end{aligned}$$

where

$$K_F(t, \tau) = f'(\tau)\mathrm{Re}\,[\overline{(g(f(t)) - g(f(\tau)))} \cdot ig'(f(\tau)))], \tag{3.7}$$

and $P(r, t)$ is the Poisson kernel. We refer to [23, Eq. 5.6] for a similar approach, but for some other purpose.
To continue, observe first that

$$\frac{P(r,t)}{1-r} = \frac{1+r}{2\pi(1 + r^2 - 2r\cos t)} \le \frac{1}{\pi((1-r)^2 + 4r\sin^2 t/2)} \le \frac{\pi}{4rt^2}$$

for $0 < r < 1$ and $t \in [-\pi, \pi]$ because $|\sin(t/2)| \ge t/\pi$.
On the other hand by (2.6) and (3.7), for

$$\sigma = \min\{|f(t + \tau) - f(\tau)|, l - |f(t + \tau) - f(\tau)|\}$$

we obtain

$$|K_F(t + \tau, \tau)| \le \|F'\|_\infty \int_0^\sigma \omega(u)du,$$

where ω is the modulus of continuity of g'. Therefore for $r \geq 1/2$,

$$\begin{aligned}|K_F(t+\tau,\tau)\frac{P(r,t)}{1-r}| &\leq \frac{\|F'\|_\infty \pi}{4rt^2}\int_0^\sigma \omega(u)du \\ &\leq \frac{\sigma}{t}\frac{\|F'\|_\infty \pi}{4rt^2}\int_0^t \omega\left(\frac{\sigma}{t}u\right)du \\ &\leq \frac{\pi\|F'\|_\infty^2}{2}\frac{1}{t^2}\int_0^t \omega(\|F'\|_\infty u)du := Q(t).\end{aligned} \tag{3.8}$$

Thus $Q(t)$ is a dominant for the expression

$$|K_F(t+\tau,\tau)\frac{P(r,t)}{1-r}|,$$

for $r \geq 1/2$. Having in mind the equation (2.8), we obtain

$$\begin{aligned}\int_{-\pi}^{\pi}|Q(t)|dt &\leq \frac{2\pi\|F'\|_\infty^2}{2}\int_0^\pi \frac{1}{t^2}\int_0^t \omega(\|F'\|_\infty u)du \\ &= \pi\|F'\|_\infty^2\int_0^\pi \left(\frac{\omega(\|F'\|_\infty u)}{u} - \frac{\omega(\|F'\|_\infty u)}{\pi}\right)du \\ &< M < \infty.\end{aligned}$$

According to the Lebesgue Dominated Convergence Theorem, taking the limit under the integral sign in the last integral in (3.6) we obtain (3.1).

□

For a Lipschitz non-decreasing function f and an arc-length parametrization g of the Dini's smooth curve γ we define the operator T as follows

$$T[f](\tau) = \int_0^{2\pi} \frac{\mathrm{Re}\left[\overline{(g(f(t)) - g(f(\tau)))}\cdot ig'(f(\tau)))\right]}{2\sin^2\frac{t-\tau}{2}}dt, \tau \in [0, 2\pi]. \tag{3.9}$$

According to Lemma 3.1, this integral converges. Notice that if γ is a convex Jordan curve then $\mathrm{Re}\left[\overline{(g(f(t)) - g(f(\tau)))}\cdot ig'(f(\tau))\right] \geq 0$, and therefore $T[f] > 0$. In the next proof, we will show that under the integral condition $T[f] > 0$ the harmonic extension of a bi-Lipschitz mapping is a diffeomorphism regardless of the condition of convexity.

Proof of Theorem 1.2. Assume for simplicity that $|\gamma| = 2\pi$. The general case follows by normalization. Let $g : [0, 2\pi] \to \gamma$ be an arc length parametrization of γ. Then $F(e^{it}) = g(f(t))$, where $f : \mathbf{R} \to \mathbf{R}$ is a Lipschitz weak homeomorphism such that $f(t + 2\pi) = f(t) + 2\pi$. From (3.9) we have

$$\begin{aligned}T[f](\tau) &= \lim_{\epsilon\to 0^+}\int_\epsilon^\pi \frac{\mathrm{Re}\left[\overline{(g(f(t+\tau)) - g(f(\tau)))}\cdot ig'(f(\tau)))\right]}{2\sin^2\frac{t}{2}}\frac{dt}{2\pi} \\ &+ \lim_{\epsilon\to 0^+}\int_{-\pi}^{-\epsilon} \frac{\mathrm{Re}\left[\overline{(g(f(t+\tau)) - g(f(\tau)))}\cdot ig'(f(\tau)))\right]}{2\sin^2\frac{t}{2}}\frac{dt}{2\pi}.\end{aligned}$$

Assume that $\beta : [0, 2\pi] \to \mathbf{R}$ is a continuous functions such that

$$g'(s) = e^{i\beta(s)}, \ \ \beta(0) = \beta(2\pi). \tag{3.10}$$

Then

$$|g'(s) - g'(t)| = 2|\sin\frac{\beta(t) - \beta(s)}{2}|. \tag{3.11}$$

Let ω_β be the modulus of continuity of g'. Then

$$\omega_\beta(\rho) = \max_{|t-s|\le\rho} 2|\sin\frac{\beta(t) - \beta(s)}{2}|. \tag{3.12}$$

Since $\gamma \in C^{1,\alpha}$,

$$\omega_\beta(\rho) \le c(\gamma)\rho^\alpha. \tag{3.13}$$

Further from (3.10), we have

$$\begin{aligned}\frac{\text{Re}\,[\overline{(g(f(t+\tau)) - g(f(\tau)))} \cdot ig'(f(\tau))]}{2\sin^2\frac{t}{2}} &= \frac{\text{Re}\,[\overline{\int_{f(\tau)}^{f(t+\tau)} g'(s)ds} \cdot ig'(f(\tau))]}{2\sin^2\frac{t}{2}} \\ &= \frac{\text{Re}\,[\overline{\int_{f(\tau)}^{f(t+\tau)} e^{i\beta(s)}ds} \cdot ie^{i\beta(\tau)}]}{2\sin^2\frac{t}{2}} \\ &= \frac{-\text{Im}\,[\overline{\int_{f(\tau)}^{f(t+\tau)} e^{i\beta(s)-\beta(\tau)}ds}]}{2\sin^2\frac{t}{2}} \\ &= \frac{\int_{f(\tau)}^{f(t+\tau)} \sin[\beta(s) - \beta(f(\tau))]ds}{2\sin^2\frac{t}{2}}.\end{aligned}$$

Taking

$$dU = \frac{1}{2\sin^2\frac{t}{2}}dt \ \text{ and } \ V = \int_{f(\tau)}^{f(t+\tau)} \sin[\beta(s) - \beta(f(\tau))]ds,$$

we obtain that

$$U = -\cot\frac{t}{2} \ \text{ and } \ dV = f'(t+\tau)\sin[\beta(f(t+\tau)) - \beta(f(\tau))]dt.$$

To continue recall that f is Lipschitz with a Lipschitz constant L. Thus

$$\begin{aligned}|\lim_{\epsilon\to0^+} U(t)V(t)|_\epsilon^\pi| &= \left|\lim_{\epsilon\to0^+} \cot\frac{\epsilon}{2}\int_{f(\tau)}^{f(\epsilon+\tau)} \sin[\beta(s) - \beta(f(\tau))]ds\right| \\ &\le \lim_{\epsilon\to0^+} \cot\frac{\epsilon}{2}|\sin[\beta(\epsilon+\tau) - \beta(f(\tau))]||f(\epsilon+\tau) - f(\tau)| \\ &\le \lim_{\epsilon\to0^+} L\epsilon\cot\frac{\epsilon}{2}\omega_\beta(\epsilon) = 0.\end{aligned}$$

Similarly we have

$$\lim_{\epsilon\to0^+} U(t)V(t)|_{-\pi}^{-\epsilon} = 0.$$

By a partial integration we obtain

$$\begin{aligned}T[f](\tau) &= \lim_{\epsilon\to 0^+}\left(UV|_{\epsilon}^{\pi} + \int_{\epsilon}^{\pi} f'(t+\tau)\cdot \sin[\beta(f(t+\tau))-\beta(f(\tau))]\cot\frac{t}{2}\frac{dt}{2\pi}\right)\\ &+ \lim_{\epsilon\to 0^+}\left(UV|_{-\pi}^{-\epsilon} + \int_{-\pi}^{-\epsilon} f'(t+\tau)\cdot \sin[\beta(f(t+\tau))-\beta(f(\tau))]\cot\frac{t}{2}\frac{dt}{2\pi}\right)\\ &= \int_{-\pi}^{\pi} f'(t+\tau)\cdot \sin[\beta(f(t+\tau))-\beta(f(\tau))]\cot\frac{t}{2}\frac{dt}{2\pi}.\end{aligned}$$

Hence

$$T[f](\tau) = \int_{-\pi}^{\pi} f'(t+\tau)\cdot \sin[\beta(f(t+\tau))-\beta(f(\tau))]\cot\frac{t}{2}\frac{dt}{2\pi}.$$

By using Lemma 2.5, we can choose a family of diffeomorphisms f_n converging uniformly to f. Then

$$T[f_n](\tau) = \int_{-\pi}^{\pi} f_n'(t+\tau)\cdot \sin[\beta(f_n(t+\tau))-\beta(f_n(\tau))]\cot\frac{t}{2}\frac{dt}{2\pi}.$$

We are going to show that $T[f_n]$ converges uniformly to $T[f]$. In order to do this, we apply Arzela-Ascoli theorem.

First of all

$$\begin{aligned}|T[f_n](\tau)| &\le \frac{1}{\pi}\|f_n'\|_\infty \int_0^{\pi}\omega_\beta(\|f_n'\|_\infty t)\cot\frac{t}{2}dt\\ &\le \frac{1}{\pi}\|f'\|_\infty \int_0^{\pi}\omega_\beta(\|f'\|_\infty t)\cot\frac{t}{2}dt = C(f,\gamma) < \infty.\end{aligned}$$

We prove now that $T[f_n]$ is equicontinuous family of functions. We have to estimate the quantity:

$$|T[f_n](\tau) - T[f_n](\tau_0)|.$$

Assume without loss of generality that $\tau_0 = 0$. Then

$$\begin{aligned}|T[f_n](\tau) - T[f_n](0)| &= \Bigg|\int_{-\pi}^{\pi} f_n'(t+\tau)\cdot \sin[\beta(f_n(t+\tau))-\beta(f_n(\tau))]\cot\frac{t}{2}\frac{dt}{2\pi}\\ &\quad - \int_{-\pi}^{\pi} f_n'(t)\cdot \sin[\beta(f_n(t))-\beta(f_n(0))]\cot\frac{t}{2}\frac{dt}{2\pi}\Bigg| \le A+B,\end{aligned}$$

where

$$A = \left|\int_{-\pi}^{\pi}(f_n'(t+\tau)-f_n'(t))\cdot \sin[\beta(f_n(t+\tau))-\beta(f_n(\tau))]\cot\frac{t}{2}\frac{dt}{2\pi}\right|$$

and

$$B = \left|\int_{-\pi}^{\pi} f_n'(t)\cdot\{\sin[\beta(f_n(t))-\beta(f_n(0))] - \sin[\beta(f_n(t+\tau))-\beta(f_n(\tau))]\}\cot\frac{t}{2}\frac{dt}{2\pi}\right|.$$

Take $r\ge 1$, $p>1$, $q>1$ such that

$$\frac{1}{p}+\frac{1}{q} = 1,$$

and $\delta\in(0,1)$.

In what follows, for a function $g \in L^p(\mathbf{T})$ we have in mind the following $p-$norm:

$$\|g\|_p = \left(\int_0^{2\pi} |g(e^{it})|^p \frac{dt}{2\pi}\right)^{1/p}.$$

Define $f_\tau(x) := f(x+\tau)$. By (2.10) we have

$$f_n = \frac{n}{n+1}\left(f * \rho_{\varepsilon_n} + \frac{x}{n}\right).$$

Thus

$$|f'_{n,\tau} - f'_n| = \frac{n}{n+1}|(f'_\tau - f') * \rho_{\varepsilon_n}|. \tag{3.14}$$

According to Young's inequality for convolution ([40, pp. 54-55; 8, Theorem 20.18]), we obtain that

$$\|(f'_\tau - f') * \rho_{\varepsilon_n}\|_r \le \|f'_\tau - f'\|_r.$$

In view of (3.13) and (3.14), for $1 < q < \frac{1}{1-\alpha}$, according to Hölder inequality we have

$$\begin{aligned} A &\le \|f'_n(t+\tau) - f'_n(t)\|_p \cdot \|\sin[\beta(f_n(t+\tau)) - \beta(f_n(\tau))]\cot\frac{t}{2}\frac{1}{2\pi}\|_q \\ &\le \|f'(t+\tau) - f'(t)\|_p \cdot \|\omega_\beta(|f_n|_\infty t)\cot\frac{t}{2}\frac{1}{2\pi}\|_q \\ &\le C_1(\gamma)\|f'\|_\infty \|f'(t+\tau) - f'(t)\|_p, \end{aligned}$$

i.e.

$$A \le C_1(\gamma)\|f'\|_\infty \|f'(t+\tau) - f'(t)\|_p. \tag{3.15}$$

Let now estimate B. First of all

(3.16)

$$B \le \|f'\|_\infty \|\{\sin[\beta(f_n(t)) - \beta(f_n(0))] - \sin[\beta(f_n(t+\tau)) - \beta(f_n(\tau))]\}\cot\frac{t}{2}\|_1.$$

On the other hand, using again Hölder inequality we have

$$\begin{aligned} &\|\{\sin[\beta(f_n(t)) - \beta(f_n(0))] - \sin[\beta(f_n(t+\tau)) - \beta(f_n(\tau))]\}\cot\frac{t}{2}\|_1 \\ &\quad \le \|\{\sin[\beta(f_n(t)) - \beta(f_n(0))] - \sin[\beta(f_n(t+\tau)) - \beta(f_n(\tau))]\}^\delta\|_p \\ &\quad \times \|\{\sin[\beta(f_n(t)) - \beta(f_n(0))] - \sin[\beta(f_n(t+\tau)) - \beta(f_n(\tau))]\}^{1-\delta}\cot\frac{t}{2}\|_q. \end{aligned}$$

Further

$$\begin{aligned}
&\|\{\sin[\beta(f_n(t))-\beta(f_n(0))]-\sin[\beta(f_n(t+\tau))-\beta(f_n(\tau))]\}^\delta\|_p \\
&\quad\le \|\{|2\sin\frac{\beta(f_n(t))-\beta(f_n(0))-\beta(f_n(t+\tau))+\beta(f_n(\tau))}{2}|\}^\delta\|_p \\
&\quad\le \|\{|2\sin\frac{\beta(f_n(t+\tau))-\beta(f_n(t))}{2}|\}^\delta\|_p \\
&\quad+ \|\{|2\sin\frac{\beta(f_n(\tau))-\beta(f_n(0))}{2}|\}^\delta\|_p \\
&\quad\le \omega_\beta(|f_n'|_\infty\tau)^\delta+\omega_\beta(|f_n'|_\infty\tau)^\delta = 2\omega_\beta(|f_n'|_\infty\tau)^\delta \le 2\omega_\beta(|f'|_\infty\tau)^\delta,
\end{aligned}$$

and

$$\begin{aligned}
&\|\{\sin[\beta(f_n(t))-\beta(f_n(0))]-\sin[\beta(f_n(t+\tau))-\beta(f_n(\tau))]\}^{1-\delta}\cot\frac{t}{2}\|_q \\
&\qquad\qquad \le \|(2\omega_\beta(|f_n'|_\infty t)^{1-\delta}\cot\frac{t}{2}\|_q.
\end{aligned}$$

Choose q and δ such that

$$(\alpha-\alpha\delta-1)q>-1.$$

Then the integral

$$\|2\omega_\beta(|f_n'|_\infty t)^{1-\delta}\cot\frac{t}{2}\|_q$$

converges and it is less or equal to

$$C(\gamma)\|f_n'\|_\infty^{1-\delta}\le C(\gamma)\|f'\|_\infty^{1-\delta}.$$

Therefore

$$B\le 2\|f'\|_\infty C(\gamma)\|f'\|_\infty^{1-\delta}\omega_\beta(\|f'\|_\infty\tau)^\delta. \tag{3.17}$$

Since a translation is continuous (see [37, Theorem 9.5]), (3.15) and (3.17) imply that the family $\{T[f_n]\}$ is equicontinuous. By Arzela-Ascoli theorem it follows that

$$\lim_{n\to\infty}\|T[f_n]-T[f]\|_\infty=0.$$

Thus $T[f]$ is continuous.
Moreover, since f_n is a diffeomorphism, for n sufficiently large there holds the following inequality

$$J_{w_n}(e^{i\tau})=f_n'(\tau)T[f_n](e^{i\tau})>0, e^{i\tau}\in\mathbf{T}.$$

Since $f_n\in C^\infty$, it follows that

$$w_n=P[F_n]\in C^1(\overline{\mathbf{U}}).$$

Therefore all the conditions of Proposition 1.1 are satisfied. This means that w_n is a harmonic diffeomorphism of the unit disk onto the domain D.

Since, by a result of Hengartner and Schober [9], the limit function w of a locally uniformly convergent sequence of univalent harmonic mappings w_n on $\mathbf{U}$ is either univalent on $\mathbf{U}$, is a constant, or its image lies on a straight-line, we obtain that $w=P[F]$ is univalent. The proof is completed. □

Remark 3.2. If γ is a $C^{1,\alpha}$ convex curve, then $\mathrm{Re}\,[\overline{(g(f(t))-g(f(\tau)))}\cdot ig'(f(\tau))]\geq 0$ and therefore $T[f](\tau)>0$. By the proof of Theorem 1.2, $\tau\to T[f](\tau)$ is continuous. Therefore $\min_{\tau\in[0,2\pi]}T[f](\tau)=\delta>0$.

4. QUASICONFORMAL HARMONIC MAPPINGS

An injective harmonic mapping $w=u+iv$, is called K-*quasiconformal* (K-q.c), $K\geq 1$, if

$$|w_{\bar z}|\leq k|w_z| \tag{4.1}$$

on D where $k=(K-1)/(K+1)$. Here

$$w_z:=\frac{1}{2}\left(w_x-iw_y\right) \text{ and } w_{\bar z}:=\frac{1}{2}\left(w_x+iw_y\right).$$

Notice that, since if $|\nabla w(z)|:=\max\{|\nabla w(z)h|:|h|=1\}=|w_z|+|w_{\bar z}|$ $l(\nabla w(z)):=\min\{|\nabla w(z)h|:|h|=1\}=||w_z|-|w_{\bar z}||$, the condition (4.1) is equivalent with

$$|\nabla w(z)|\leq Kl(\nabla w(z)). \tag{4.2}$$

For a general definition of quasiregular mappings and quasiconformal mappings we refer to the book of Ahlfors [1]. In this section we apply Theorem 1.2 to the class of q.c. harmonic mappings. The area of quasiconformal harmonic mappings is very active area of research. For a background on this theory we refer [10], [12]-[26], [27], [31], [32] and [38]. In this section we obtain some new results concerning a characterization of this class. We will restrict ourselves to the class of q.c. harmonic mappings w between the unit disk $\mathbf{U}$ and a Jordan domain D. The unit disk is taken because of simplicity. Namely, if $w:\Omega\to D$ is q.c. harmonic, and $a:\mathbf{U}\to\Omega$ is conformal, then $w\circ a$, is also q.c. harmonic. However the image domain D cannot be replaced by the unit disk.

The case when D is a convex domain is treated in detail by the author and others in above cited papers. In this section we will use our main result to yield a characterization of quasiconformal harmonic mappings onto a Jordan that is not necessarily convex in terms of boundary data.

To state the main result of this section, we make use of Hilbert transforms formalism. It provides a necessary and a sufficient condition for the harmonic extension of a homeomorphism from the unit circle to a C^2 Jordan curve γ to be a q.c mapping. It is an extension of the corresponding result [15, Theorem 3.1] related to convex Jordan domains.

Theorem 4.1. *Let $F:\mathbf{T}\to\gamma$ be a sense preserving homeomorphism of the unit circle onto the Jordan curve $\gamma=\partial D\in C^2$. Then $w=P[F]$ is a quasiconformal mapping of the unit disk onto D if and only if F is absolutely continuous and*

$$0<l(F):=\operatorname{ess\,inf} l(\nabla w(e^{i\tau})), \tag{4.3}$$

$$\|F'\|_\infty:=\operatorname{ess\,sup}|F'(\tau)|<\infty \tag{4.4}$$

and

$$\|H(F')\|_\infty:=\operatorname{ess\,sup}|H(F')(\tau)|<\infty. \tag{4.5}$$

If F satisfies the conditions (4.3), (4.4) *and* (4.5), *then $w = P[F]$ is K quasiconformal, where*

$$K := \frac{\sqrt{\|F'\|_\infty^2 + \|H(F')\|_\infty^2 - l(F)^2}}{l(F)}. \tag{4.6}$$

The constant K is approximately sharp for small values of K: if w is the identity or if it is a mapping close to the identity, then $K = 1$ or K is close to 1 (respectively).

The proof of necessity. Suppose that $w = P[F] = g + \overline{h}$ is a $K-$q.c. harmonic mapping that satisfies the conditions of the theorem. By [15, Theorem 2.1], we see that w is Lipschitz continuous,

$$L := \|F'\|_\infty < \infty \tag{4.7}$$

and

$$|\nabla w(z)| \le KL. \tag{4.8}$$

By [19, Theorem 1.4] we have for $b = w(0)$

$$|\partial w(z)| - |\bar\partial w(z)| \ge C(\Omega, K, b) > 0,\ z \in \mathbf{U}. \tag{4.9}$$

Because of (4.8), the analytic functions $\partial w(z)$ and $\bar\partial w(z)$ are bounded, and therefore by Fatou's theorem:

$$\lim_{r\to 1^-}(|\partial w(re^{i\tau})| - |\bar\partial w(re^{i\tau})|) = |\partial w(e^{i\tau})| - |\bar\partial w(e^{i\tau})| \ \ a.e. \tag{4.10}$$

Combining (4.7), (4.10) and (4.9), we get (4.3) and (4.4).

Next we prove (4.5). Observe first that $w_r = e^{i\tau}w_z + e^{-i\tau}w_{\overline{z}}$. Thus

$$|w_r| \le |\nabla w| \le KL. \tag{4.11}$$

Therefore $rw_r = P[H(F')]$ is a bounded harmonic function which implies that $H(F') \in L^\infty(\mathbf{T})$. Therefore (4.5) holds and the necessity proof is completed.

The proof of sufficiency. We have to prove that under the conditions (4.3), (4.4) and (4.5) w is quasiconformal. From

$$0 < l(F) := \operatorname{ess\,inf} l(\nabla w(e^{i\tau}))$$

we obtain that

$$J_w(e^{i\tau}) = (|w_z| + |w_{\bar z}|)l(\nabla w(e^{i\tau})) \ge l(F)^2 \ \ (a.e.)$$

As rw_r is a harmonic conjugate of w_τ, it turns out that if F is absolutely continuous, then

$$\lim_{r\to 1^-} w_r(re^{i\tau}) = H(F')(\tau) \ (a.e.), \tag{4.12}$$

and

$$\lim_{r\to 1^-} w_\tau(re^{i\tau}) = F'(\tau) \ \ (a.e.). \tag{4.13}$$

As

$$|w_z|^2 + |w_{\bar z}|^2 = \frac{1}{2}\left(|w_r|^2 + \frac{|w_\tau|^2}{r^2}\right),$$

it follows that for a.e. $\tau \in [0, 2\pi)$
(4.14)
$$\lim_{r\to 1^-} |w_z(re^{i\tau})|^2+|w_{\bar z}(re^{i\tau})|^2 = |w_z(e^{i\tau})|^2+|w_{\bar z}(e^{i\tau})|^2 \le \frac{1}{2}(\|F'\|^2_\infty+\|H(F')\|^2_\infty).$$
To continue we make use of (4.3). From (4.14), (4.3) and (4.2), for a.e. $\tau \in [0, 2\pi)$,
$$\frac{|w_z(e^{i\tau})|^2 + |w_{\bar z}(e^{i\tau})|^2}{(|w_z(e^{i\tau})| - |w_{\bar z}(e^{i\tau})|)^2} \le \frac{\|F'\|^2_\infty + \|H(F')\|^2_\infty}{2l(F)^2}. \tag{4.15}$$
Hence
$$|w_z(e^{i\tau})|^2 + |w_{\bar z}(e^{i\tau})|^2 \le S(|w_z(e^{i\tau})| - |w_{\bar z}(e^{i\tau})|)^2 \quad (a.e.), \tag{4.16}$$
where
$$S := \frac{\|F'\|^2_\infty + \|H(F')\|^2_\infty}{2l(F)^2}. \tag{4.17}$$
According to (4.15), $S \ge 1$. Let
$$\mu(e^{i\tau}) := \left|\frac{w_{\bar z}(e^{i\tau})}{w_z(e^{i\tau})}\right|.$$
Since every C^2 curve is $C^{1,\alpha}$ curve, Theorem 1.2 shows that $w = g+\overline{k}$ is univalent and according to Lewy's theorem $J_w(z) = |g'(z)|^2 - |h'(z)|^2 > 0$. Thus $a(z) = \overline{w_{\bar z}}/w_z = h'/g'$ is an analytic function bounded by 1. As $\mu(e^{i\tau}) = |a(e^{i\tau})|$, we have $\mu(e^{i\tau}) \le 1$. Then (4.16) can be written as
$$1 + \mu^2(e^{i\tau}) \le S(1 - \mu(e^{i\tau}))^2,$$
i.e. if $S = 1$, then $\mu(e^{i\tau}) = 0$ a.e. and if $S > 1$, then
$$\mu^2(S - 1) - 2\mu S + S - 1 = (S - 1)(\mu - \mu_1)(\mu - \mu_2) \ge 0, \tag{4.18}$$
where
$$\mu_1 = \frac{S + \sqrt{2S - 1}}{S - 1}$$
and
$$\mu_2 = \frac{S - 1}{S + \sqrt{2S - 1}}.$$
If $S > 1$, then from (4.18) it follows that $\mu(e^{i\tau}) \le \mu_2$ or $\mu(e^{i\tau}) \ge \mu_1$. But $\mu(e^{i\tau}) \le 1$ and therefore
$$\mu(e^{i\tau}) \le \frac{S - 1}{S + \sqrt{2S - 1}} \quad (a.e.). \tag{4.19}$$
If $S = 1$, then (4.19) clearly holds. Define $\mu(z) = |a(z)|$. Since a is a bounded analytic function, by the maximum principle it follows that
$$\mu(z) \le k := \mu_2,$$
for $z \in \mathbf{U}$. This yields that
$$K(z) \le K := \frac{1 + k}{1 - k} = \frac{2S - 1 + \sqrt{2S - 1}}{\sqrt{2S - 1} + 1} = \sqrt{2S - 1},$$

i.e.

$$K(z) \leq \frac{\sqrt{\|F'\|_\infty^2 + \|H(F')\|_\infty^2 - l(F)^2}}{l(F)}$$

which means that w is $K = \frac{\sqrt{\|F'\|_\infty^2 + \|H(F')\|_\infty^2 - l(F)^2}}{l(F)}-$quasiconfomal. The result is asymptotically sharp because $K = 1$ for w being the identity. This finishes the proof of Theorem 4.6. □

A conjecture. Let $F : \mathbf{T} \to \gamma \subset \mathbf{C}$ be a homeomorphism of bounded variation, where γ is Dini smooth. Let D be the bounded domain such that $\partial D = \gamma$. The mapping $w = P[F]$ is a diffeomorphism of $\mathbf{U}$ onto D if and only if

$$\operatorname{ess\,inf}\{J_w(e^{it}) : t \in [0, 2\pi]\} \geq 0. \tag{4.20}$$

Acknowledgment. I am thankful to the referee for providing very constructive comments and help in improving the contents of this paper.

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