CHAPTER 12

ENTIRE FUNCTIONS

1.12. THE INVERSE PROBLEM OF THE BEST APPROXIMATION OF BOUNDED, UNIFORMLY CONTINUOUS FUNCTIONS BY ENTIRE FUNCTIONS OF EXPONENTIAL TYPE AND RELATED PROBLEMS*

Let E be a separable, infinite-dimensional Banach space; $E_0 \subset E_1 \subset E_2 \subset \ldots$ be a chain of finite-dimensional subspaces such that $\dim E_n = n$ and $\bigcup_{n\geq 1} E_n$ is dense in E. For each x, $x \in$ E, we define the sequence of "deviations" from the subspace E_n:

$$d(x,n) = \inf\{\|x-y\| : y \in E_n\}, \quad n = 0, 1, 2, \ldots .$$

Bernshtein [1] (see also [2]) has proved that for any numerical sequence $\{d_n\}_{n\geq 0}$, $d_n \downarrow 0$, there exists an element x, $x \in E$, such that

$$d(x,n) = d_n, \quad (n = 0, 1, 2, \ldots).$$

This is the (positive) solution of the inverse problem of the best approximation in the case of a separable space and finite-dimensional approximating subspaces. Strictly speaking, Bernshtein has considered the case when $E = C[a, b]$ and E_n is the subspace of algebraic polynomials of degree $\leqslant n - 1$; however, his solution can be reproduced without change in the general case.

Assume now that $B(\mathbb{R})$ is the Banach space of all bounded, uniformly continuous functions defined on the axis, with the usual norm; B_σ is its closed subspace formed by the entire functions (more precisely, by their restrictions to the axis) of exponential type $\leqslant\sigma$. As shown by Bernshtein [3], many facts from the theory of the best approximation of continuous functions by polynomials can be carried over to the case of the approximation of functions from $B(\mathbb{R})$ by functions from B_σ ($0 \leqslant \sigma < \infty$).

We define the deviation of the function f from B_σ:

$$A(f,\sigma) = \inf\{\|f-g\| : g \in B_\sigma\}.$$

For a fixed f the function $A(f, \sigma)$ satisfies the following conditions: 1. $A(f, \rho) \geqslant A(f, \sigma)$ for $\rho < \sigma$; 2. $A(f, \sigma) = A(f, \sigma + 0)$; 3. $\lim_{\sigma\to\infty} A(f,\sigma) = 0$.

Problem 1. Assume that the bounded function $\sigma \mapsto F(\sigma)$ $(0 \leqslant \sigma < \infty)$ satisfies the conditions 1-3. Is there a function f, $f \in B(\mathbb{R})$, for which $A(f,\sigma) \equiv F(\sigma)$?

Problem 2. If in the uniform metric we close the union of all the subspaces B_ρ ($\rho < \sigma$) we obtain a proper subspace $B_{\sigma-0} \subset B_\sigma$. What is its codimension in B_σ?

Problem 3. Let f be an almost periodic function according to Bohr. Will the deviation $A(f, \sigma)$ be a function of pure jumps?

Problem 4. Let $A(f, \sigma)$ be a function of pure jumps. Will f be almost periodic?

LITERATURE CITED

1. S. N. Bernshtein, "On the inverse problem of the theory of best approximation of continuous functions," in: Collected Works, Vol. 2, Akad. Nauk SSSR, Moscow (1954), pp. 292-294.
2. I. P. Natanson, Constructive Function Theory, Vols. I-III, Ungar, New York (1964-5).
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