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A Simple Proof of an Aomoto-Type Extension of Askey's Last Conjectured Selberg q-Integral

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We establish an Aomoto-type extension of Askey's last conjectured Selberg q-integral, which was recently proved by Evans. We follow the lines of our proofs of Aomoto-type extensions of the Morris constant term q-identity and Gustafson's Askey–Wilson Selberg q-integral. We require integral forms of the q-transportation theory and its alternative for the root system A_{n-1}, which are related to the simple reflections and minuscule weight of A_{n-1}. We use an elementary symmetry which is related to the minuscule weight of A_{n-1} to lift a proof of the one dimensional q-integral to the multivariable setting. © 2001 Academic Press

Key Words: Selberg's integral; Aomoto-type extension; q-transportation theory for the root system A_{n-1}.

1. INTRODUCTION AND SUMMARY

Throughout this paper, we let $n \geq 1$, $k \geq 0$, $N \geq 0$, $a \geq 0$, $b \geq 0$, m, and v be integers with $0 \leq m \leq n$ and $2 \leq v \leq n$, we let x and y be complex with positive real parts, and we let q be real with $0 < q < 1$. In 1944, Selberg [27] evaluated the multivariable beta integral

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^n t_i^{x-1}(1-t_i)^{y-1}\Delta_n^{2k}(t_1,\ldots,t_n)dt_1\cdots dt_n$$
$$= \prod_{i=1}^n \frac{\Gamma(x+(n-i)k)\Gamma(y+(n-i)k)\Gamma(1+ik)}{\Gamma(x+y+(2n-i-1)k) \qquad \Gamma(1+k)}, \tag{1.1}$$

where $\Delta_n(t_1,\ldots,t_n) = \prod_{1\leq i<j\leq n}(t_i - t_j)$ is the Vandermonde determinant.

Askey [5] conjectured a number of extensions of Selberg's integral based upon various beta integrals. The first three conjectures were based upon

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q-beta integrals given by Ramanujan [25, 26]; see also Askey [4] and Hardy [13]. Gustafson [10, 11] has proven many extensions of Selberg's integral including extensions based upon Mellin–Barnes type integrals and the Askey–Wilson integral with five parameters.

Let $(z;q)_a = \prod_{i=0}^{a-1}(1-q^i z)$ be the q-Pockhammer symbol. Following Askey [5], we set

$$ {}_q\Delta_n^{2k}(t_1,\ldots,t_n) = \prod_{1\le i<j\le n} t_i^{2k}(q^{1-k}t_j/t_i;q)_{2k}. \tag{1.2}$$

In the words of Askey [5], ${}_q\Delta_n^{2k}(t_1,\ldots,t_n)$ "vanishes when $t_i = t_j$ and on k lines on one side of this line and $k-1$ lines on the other side."

We associate

$$ {}_q a_{n-1}^{k}(t_1,\ldots,t_n) = \prod_{1\le i<j\le n} (t_i/t_j;q)_k(qt_j/t_i;q)_k \tag{1.3}$$

with the root system A_{n-1}. Observe that ${}_q a_{n-1}^k(t_1,\ldots,t_n)$ and ${}_q\Delta_n^{2k}(t_1,\ldots,t_n)$ have the same zeros. As in [19], the location of these zeros is the key to our proof.

We have the q-gamma function

$$\Gamma_q(x) = (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty}, \tag{1.4}$$

where $(z;q)_\infty = \prod_{i=0}^{\infty}(1-q^i z)$, and the Jackson q-integral

$$\int_0^a f(t)d_qt = a(1-q)\sum_{i=0}^{\infty} q^i f(aq^i), \tag{1.5}$$

which is the Riemann sum using the endpoints which are furthest from zero in the partition

$$(0,a]_q = \{aq^i \mid i \ge 0\}. \tag{1.6}$$

Askey's first conjecture [5] is given by

$$\begin{aligned} {}_qI_n^k(x,y) &= \int_0^1\cdots\int_0^1 \prod_{i=1}^n t_i^{x-1}\frac{(qt_i;q)_\infty}{(q^y t_i;q)_\infty}\, {}_q\Delta_n^{2k}(t_1,\ldots,t_n)d_qt_1\cdots d_qt_n \\ &= q^{2\binom{n}{3}k^2 + x\binom{n}{2}k}\prod_{i=1}^n \frac{\Gamma_q(x+(n-i)k)\Gamma_q(y+(n-i)k)\Gamma_q(1+ik)}{\Gamma_q(x+y+(2n-i-1)k)\Gamma_q(1+k)}. \end{aligned} \tag{1.7}$$

This conjecture was proved independently by Habsieger [12] and Kadell [17]. Habsieger extended Selberg's original proof [27], replacing the symmetry

$$q^{-x\binom{n}{2}k}\,{}_qI_n^k(x,y) = q^{-y\binom{n}{2}k}\,{}_qI_n^k(y,x) \tag{1.8}$$

by the asymptotic relation

$$q^{-x\binom{n}{2}k}\,{}_qI_n^k(x,-(n-1)k+\epsilon)\sim_{\epsilon\to 0} C(1-q^{\epsilon})^{-n}, \tag{1.9}$$

where the constant C is independent of x and y. Kadell's proof generalized Aomoto's proof [3] of an extension of Selberg's integral. See Evans [7] for a proof using Anderson's argument [1].

Let $[w]f$ denote the coefficient of the monomial w in the Laurent expansion of f. Habsieger [12] and Kadell [17] independently observed that the Selberg q-integral (1.7) is equivalent to the Morris constant term q-identity

$$\begin{aligned}[1]\prod_{i=1}^{n}(t_i;q)_a(q/t_i;q)_b\,{}_qa_{n-1}^k(t_1,\ldots,t_n)\\ =\prod_{i=1}^{n}\frac{(q;q)_{a+b+(n-i)k}(q;q)_{ik}}{(q;q)_{a+(n-i)k}(q;q)_{b+(n-i)k}(q;q)_k}\end{aligned} \tag{1.10}$$

A natural extension of the Jackson q-integral (1.5) is given by

$$\begin{aligned}\int_{-c}^{d}f(t)d_qt&=-\int_0^{-c}f(t)d_qt+\int_0^{d}f(t)d_qt\\ &=c(1-q)\sum_{i=0}^{\infty}q^if(-q^ic)+d(1-q)\sum_{i=0}^{\infty}q^if(q^id),\end{aligned} \tag{1.11}$$

which is the Riemann sum using the endpoints which are furthest from zero in the partition

$$[-c,d]_q=\{-q^ic\mid i\geq 0\}\cup\{q^id\mid i\geq 0\}. \tag{1.12}$$

Andrews and Askey [2] have given the q-beta integral

$$\begin{aligned}{}_qI(x,y;c,d)&=\int_{-c}^{d}\frac{(-qt/c;q)_\infty(qt/d;q)_\infty}{(-q^xt/c;q)_\infty(q^yt/d;q)_\infty}d_qt\\ &=\frac{cd}{(c+d)}\frac{(-d/c;q)_\infty(-c/d;q)_\infty}{(-q^xd/c;q)_\infty(-q^yc/d;q)_\infty}\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}.\end{aligned} \tag{1.13}$$

Observe that the integrand has no poles with $t\in[-c,d]_q$.

Omitting m as a subscript when $m=0$, we set

$$\begin{aligned}&{}_qas_{n,m}^k(x,y;c,d;t_1,\ldots,t_n)\\ &=\prod_{i=1}^{n}\frac{(-qt_i/c;q)_\infty(qt_i/d;q)_\infty}{(-q^{x+\chi(n-i+1\leq m)}t_i/c;q)_\infty(q^yt_i/d;q)_\infty}\,{}_q\Delta_n^{2k}(t_1,\ldots,t_n),\end{aligned} \tag{1.14}$$

where $\chi(A)$ is one or zero according to whether A is true or false, respectively, and we use capital letters to denote the q-integral

$$ {}_qAS^k_{n,m}(x,y;c,d) = \int_{-c}^{d}\cdots\int_{-c}^{d} {}_qas^k_{n,m}(x,y;c,d;t_1,\ldots,t_n)d_qt_1\cdots d_qt_n. \quad (1.15) $$

Askey's last conjectured Selberg q-integral [5], which has been proven by Evans [8], reduces to (1.13) when $n=1$. This is the $m=0$ case of the following theorem, which is our main result.

THEOREM 1.

$$ {}_qAS^k_{n,m}(x,y;c,d) = q^{\binom{n}{3}k^2-\binom{n}{2}\binom{k}{2}}(cd)^{\binom{n}{2}k} \times\prod_{i=1}^{n}\frac{cd(-d/c;q)_\infty(-c/d;q)_\infty}{(c+d)(-q^{x+(n-i)k+\chi(i\le m)}d/c;q)_\infty(-q^{y+(n-i)k}c/d;q)_\infty} \times\prod_{i=1}^{n}\frac{\Gamma_q(x+(n-i)k+\chi(i\le m))\Gamma_q(y+(n-i)k)\Gamma_q(1+ik)}{\Gamma_q(x+y+(2n-i-1)k+\chi(i\le m))\Gamma_q(1+k))} \quad (1.16) $$

Observe that the integrand has no poles with $(t_1,\ldots,t_n)\in[-c,d]^n_q$, *that the integrand is bounded on* $[-c,d]^n_q$, *and that the case* $cd=0$ *of* (1.16) *is Askey's first conjecture* (1.7).

Krattenthaler [22] used the case $m=0$, $k=1$ of Theorem 1 to obtain exact enumeration formulas for perfect matchings of holey Aztec rectangles.

Observe by (1.2) and (1.14) that we have the symmetry

$$ {}_qas^k_n(x,y;c,d;t_1,\ldots,t_n) = {}_qas^k_n(y,x;d,c;-t_1,\ldots,-t_n) \quad (1.17) $$

and that

$$ \int_{-c}^{d}\cdots\int_{-c}^{d} f(t_1,\ldots,t_n)d_qt_1\cdots d_qt_n = \int_{-c}^{d}\cdots\int_{-c}^{d} f(-t_1,\ldots,-t_n)d_qt_1\cdots d_qt_n. \quad (1.18) $$

Using (1.15), (1.17), and (1.18), we have the symmetry

$$ {}_qAS^k_n(x,y;c,d) = {}_qAS^k_n(y,x;d,c). \quad (1.19) $$

We follow the lines of our proofs [19, 20] of Aomoto-type extensions of the Morris constant term q-identity (1.10) and Gustafson's Askey–Wilson Selberg q-integral. We require integral forms of the q-transportation theory for the root system A_{n-1} and its alternative. These are related to the properties of the simple reflections and the minuscule weight of A_{n-1}. We use the elementary symmetry

$$ {}_q a_{n-1}^k(qt_n, t_1, \ldots, t_{n-1}) = {}_q a_{n-1}^k(t_1, \ldots, t_n). \tag{1.20}$$

See Carter [6], Grove and Benson [9], and Kadell [18–20] for details on the properties of root systems. The q-transportation theory for root systems has a long history in [16–20] and, in various forms, in many of the papers on constant term q-identities associated with root systems.

In Section 2, we use the substitution $t = qs$ to give a simple relation and, using a boundary condition, we prove the Andrews–Askey q-beta integral (1.13) when x is a positive integer.

In Section 3, we give an integral form of the q-transportation theory for the root system A_{n-1} which we express in terms of ${}_q as_n^k(x, y; c, d; t_1, \ldots, t_n)$.

In Section 4, we lift the simple relation of Section 2 to the multivariable setting, obtaining an integral form of the alternative q-transportation theory for the root system A_{n-1} which we express in terms of ${}_q as_n^k(x, y; c, d; t_1, \ldots, t_n)$.

In Section 5, we establish the dependence of ${}_q AS_{n,m}^k\ (x, y; c, d)$ on the parameters m and x.

In Section 6, we recall the global form [16, Sect. 4] of the q-transportation theory for the root system A_{n-1} and give an application.

In Section 7, we follow Askey [4] and lift the boundary condition of Section 2 to the multivariable setting, obtaining a recurrence relation involving the parameters x, c, and n.

In Section 8, we use induction on n, x, and m to evaluate ${}_q AS_{n,m}^k$ $(x, y; c, d)$ when x is a positive integer and, using Ismail's argument [15], we extend to complex x and complete the proof of Theorem 1.

2. A PROOF OF THE ANDREWS–ASKEY q-BETA INTEGRAL (1.13)

In this section, we use the substitution $t = qs$ to give a simple relation and, using a boundary condition, we prove the Andrews–Askey q-beta integral (1.13) when x is a positive integer.

Observe that the q-differential

$$ d_q(f(t)) = f(t) - f(qt) \tag{2.1}$$

satisfies the scale invariance property

$$ \frac{d_q(ct)}{ct} = \frac{d_q t}{t} = 1 - q. \tag{2.2}$$

The q-derivative is given by the ratio of q-differentials

$$\frac{d_q}{d_q t}\Big(f(t)\Big) = \frac{d_q(f(t))}{d_q t} = \frac{1}{t(1-q)}\Big(f(t) - f(qt)\Big). \tag{2.3}$$

We define

$$\int_{\alpha}^{q^N \alpha} f(t) d_q t = -\alpha(1-q) \sum_{i=0}^{N-1} q^i f(q^i \alpha), \tag{2.4}$$

to be the finite sum which results when we cancel the possibly divergent infinite series in the definition (1.11). We have the principal value of the q-integral

$$\begin{aligned} \mathrm{PV}\bigg(\int_{-c}^{d} f(t) d_q t \bigg) &= \lim_{N\to\infty} \bigg(\int_{-c}^{-q^N c} f(t) d_q t - \int_{d}^{q^N d} f(t) d_q t \bigg) \\ &= \lim_{N\to\infty} c(1-q) \sum_{i=0}^{N-1} q^i f(-q^i c) \\ &\qquad + d(1-q) \sum_{i=0}^{N-1} q^i f(q^i d), \end{aligned} \tag{2.5}$$

which extends the definition (1.11).

The following lemma gives an integral form of the alternative q-transportation theory for the root system A_{n-1}.

LEMMA 2. *If $g(t)$ has no poles with $t \in [-c, d]_q$, then we have*

$$\begin{aligned} &\mathrm{PV}\bigg(\int_{-c}^{d} (1 + t/c)(1 - t/d) \frac{(-qt/c;q)_\infty (qt/d;q)_\infty}{(-q^x t/c;q)_\infty (q^y t/d;q)_\infty} g(t) \frac{d_q t}{t} \bigg) \\ &\quad = \mathrm{PV}\bigg(\int_{-c}^{d} (1 + q^x t/c)(1 - q^y t/d) \frac{(-qt/c;q)_\infty (qt/d;q)_\infty}{(-q^x t/c;q)_\infty (q^y t/d;q)_\infty} \\ &\qquad\qquad \times g(qt) \frac{d_q t}{t} \bigg). \end{aligned} \tag{2.6}$$

Proof. Observe that

$$f(t) = (1 + t/c)(1 - t/d) \frac{(-qt/c;q)_\infty (qt/d;q)_\infty}{(-q^x t/c;q)_\infty (q^y t/d;q)_\infty} g(t) \tag{2.7}$$

has no poles with $t \in [-c, d]_q$ and

$$f(qt) = (1 + q^x t/c)(1 - q^y t/d) \frac{(-qt/c;q)_\infty (qt/d;q)_\infty}{(-q^x t/c;q)_\infty (q^y t/d;q)_\infty} g(qt). \tag{2.8}$$

Hence we may write the result (2.6) as

$$\mathrm{PV}\left(\int_{-c}^{d} f(t)\frac{d_qt}{t}\right) = \mathrm{PV}\left(\int_{-c}^{d} f(qt)\frac{d_qt}{t}\right). \tag{2.9}$$

Observe by (2.7) that

$$0 = f(-c) = f(d). \tag{2.10}$$

Replacing $f(t)$ by $f(t)/t$ in (2.5), we obtain

$$\mathrm{PV}\left(\int_{-c}^{d} f(t)\frac{d_qt}{t}\right) = (1-q)\lim_{N\to\infty}\sum_{i=0}^{N-1} -f(-q^ic) + f(q^id). \tag{2.11}$$

Using (2.10) and observing that we may shift both of the sums on the right side of (2.11), we have

$$\mathrm{PV}\left(\int_{-c}^{d} f(t)\frac{d_qt}{t}\right) = \mathrm{PV}\left(\int_{-qc}^{qd} f(t)\frac{d_qt}{t}\right). \tag{2.12}$$

Substituting (2.12) into (2.9), we see that the result (2.6) becomes

$$\mathrm{PV}\left(\int_{-qc}^{qd} f(t)\frac{d_qt}{t}\right) = \mathrm{PV}\left(\int_{-c}^{d} f(qt)\frac{d_qt}{t}\right), \tag{2.13}$$

which follows using the substitution $t = qs$, the scale invariance (2.2) of d_qt/t, and replacing s by t. ∎

Observe that

$$\begin{aligned}
&\Big((1+t/c)(1-t/d) - (1+q^xt/c)(1-q^yt/d)\Big)\frac{q^xd}{t}\\
&\quad = \Big(1 + t(1/c-1/d) - t^2/cd - 1 - t(q^x/c - q^y/d) + q^{x+y}t^2/cd\Big)\frac{q^xd}{t}\\
&\quad = q^xd/c - q^x - q^xt/c - q^{2x}d/c + q^{x+y} + q^{2x+y}t/c\\
&\quad = 1 - q^x + q^xd/c - q^{2x}d/c - 1 + q^{x+y} - q^xt/c + q^{2x+y}t/c\\
&\quad = (1-q^x)(1+q^xd/c) - (1-q^{x+y})(1+q^xt/c).
\end{aligned} \tag{2.14}$$

Setting $g(t) = q^xd$ in Lemma 2 (2.6), moving both q-integrals to the same side of the equation, using (2.14), and observing that the principal value extends the definition (1.11), we obtain

$$\begin{aligned}
0 &= \mathrm{PV}\Bigg(\int_{-c}^{d}\Big((1-q^x)(1+q^xd/c) - (1-q^{x+y})(1+q^xt/c)\Big)\\
&\qquad\times\frac{(-qt/c;q)_\infty\quad(qt/d;q)_\infty}{(-q^xt/c;q)_\infty(q^yt/d;q)_\infty}\frac{d_qt}{t}\Bigg)\\
&= \int_{-c}^{d}\Big((1-q^x)(1+q^xd/c) - (1-q^{x+y})(1+q^xt/c)\Big)\\
&\qquad\times\frac{(-qt/c;q)_\infty\quad(qt/d;q)_\infty}{(-q^xt/c;q)_\infty(q^yt/d;q)_\infty}d_qt,
\end{aligned} \tag{2.15}$$

which we may rearrange as

$$(1-q^x)(1+q^x d/c)_q I(x,y;c,d) = (1-q^{x+y})_q I(x+1,y;c,d). \tag{2.16}$$

Using the functional equation

$$\Gamma_q(x+1) = \frac{(1-q^x)}{(1-q)}\Gamma_q(x) \tag{2.17}$$

for the q-gamma function, we see that the product on the far right side of (1.13) satisfies (2.16).

We require the following q-analogue of [16, Lemma 3].

LEMMA 3. *If $g_x(t)$ is bounded on $[-c,d]_q$ and $g_x(-c)$ is continuous at $x=0$ from the right, then we have*

$$\lim_{x\to 0^+}\frac{(1-q^x)}{(1-q)}\int_{-c}^{d}\frac{(-qt/c;q)_\infty(qt/d;q)_\infty}{(-q^x t/c;q)_\infty(q^y t/d;q)_\infty}g_x(t)d_q t$$
$$= \frac{cd}{(c+d)}\frac{(-c/d;q)_\infty}{(-q^y c/d;q)_\infty}g_0(-c). \tag{2.18}$$

Proof. Using the definition (1.11) and cancelling the factor $1-q$, we obtain

$$\lim_{x\to 0^+}\frac{(1-q^x)}{(1-q)}\int_{-c}^{d}\frac{(-qt/c;q)_\infty(qt/d;q)_\infty}{(-q^x t/c;q)_\infty(q^y t/d;q)_\infty}g_x(t)d_q t$$
$$= \lim_{x\to 0^+}(1-q^x)\Bigg(c\sum_{i=0}^{\infty}q^i\frac{(q^{i+1};q)_\infty(-q^{i+1}c/d;q)_\infty}{(q^{i+x};q)_\infty(-q^{i+x}c/d;q)_\infty}g_x(-q^i c)$$
$$+ d\sum_{i=0}^{\infty}q^i\frac{(-q^{i+1}d/c;q)_\infty(q^{i+1};q)_\infty}{(-q^{i+x}d/c;q)_\infty(q^{i+x};q)_\infty}g_x(q^i d)\Bigg)$$
$$= c\frac{(-qc/d;q)_\infty}{(-q^y c/d;q)_\infty}g_0(-c), \tag{2.19}$$

which readily simplifies to the result (2.18). ∎

Taking $g_x(t)=1$ in Lemma 3 (2.18), we see that ${}_qI(x,y;c,d)$ satisfies the boundary condition

$$\lim_{x\to 0^+}\frac{(1-q^x)}{(1-q)}{}_qI(x,y;c,d) = \frac{cd}{(c+d)}\frac{(-c/d;q)_\infty}{(-q^y c/d;q)_\infty}. \tag{2.20}$$

Using (1.4) and (2.17), we have

$$\lim_{x\to 0^+}\frac{(1-q^x)}{(1-q)}\Gamma_q(x) = \lim_{x\to 0^+}\Gamma_q(x+1) = \Gamma_q(1) = 1. \tag{2.21}$$

Hence that the product on the far right side of (1.13) also satisfies the boundary condition (2.20).

Combining our results (2.16) and (2.20), we may establish the Andrews–Askey q-beta integral (1.13) by induction on x.

3. THE q-TRANSPORTATION THEORY FOR THE ROOT SYSTEM A_{n-1}

In this section, we give an integral form of the q-transportation theory for the root system A_{n-1} which we express in terms of ${}_q as_n^k(x, y; c, d; t_1, \ldots, t_n)$.

Using the identities

$$(z; q)_a = (-z)^a q^{\binom{a}{2}} (q^{1-a}/z; q)_a, \tag{3.1}$$

$$(z; q)_{a+b} = (z; q)_a (q^a z; q)_b \tag{3.2}$$

for reversing and splitting the q-Pockhammer symbol, respectively, and the fact that $k(1-k) + \binom{k}{2} = -\binom{k}{2}$, we have

$$\begin{aligned} s^{2k}(q^{1-k}t/s; q)_{2k} &= s^{2k}(q^{1-k}t/s; q)_k (qt/s; q)_k \\ &= s^{2k}(-q^{1-k}t/s)^k q^{\binom{k}{2}} (s/t; q)_k (qt/s; q)_k \\ &= (-st)^k q^{-\binom{k}{2}} (s/t; q)_k (qt/s; q)_k. \end{aligned} \tag{3.3}$$

The following lemma gives an integral form of the q-transportation theory for the root system A_{n-1}.

LEMMA 4. *If F is a distribution function on the domain $\mathscr{D}$ and $Y(s, t)$ is symmetric,*

$$Y(s, t) = Y(t, s), \tag{3.4}$$

in s and t, then we have

$$\begin{aligned} &\int\int_{\mathscr{D}^2} t(s-t)(s-Qt)Y(s, t)dF(s)dF(t) \\ &\qquad = Q\int\int_{\mathscr{D}^2} s(s-t)(s-Qt)Y(s, t)dF(s)dF(t). \end{aligned} \tag{3.5}$$

Proof. Since $(Qs - t)(s - t)(s - Qt)$ is antisymmetric under the substitution $s \longleftrightarrow t$, we have

$$0 = \int\int_{\mathscr{D}^2} (Qs - t)(s - t)(s - Qt)Y(s, t)dF(s)dF(t). \tag{3.6}$$

The result (3.5) follows by expanding the factor $Qs - t$ in (3.6) and rearranging the result. ∎

The following lemma expresses Lemma 4 (3.5) in terms of ${}_q as_n^k(x, y; c, d; t_1, \ldots, t_n)$.

LEMMA 5. *If $\omega(t_1, \dots, t_n)$ is symmetric,*

$$\omega(t_1, \dots, t_n) = \omega(t_1, \dots, t_{v-2}, t_v, t_{v-1}, t_{v+1}, \dots, t_n), \tag{3.7}$$

in t_{v-1} and t_v, then we have

$$\begin{aligned} &\mathrm{PV}\left(\int_{-c}^{d} \cdots \int_{-c}^{d} t_v \omega(t_1, \dots, t_n)\, {}_q as_n^k(x, y; c, d; t_1, \dots, t_n) d_q t_1 \cdots d_q t_n\right) \\ &\quad = q^k\, \mathrm{PV}\left(\int_{-c}^{d} \cdots \int_{-c}^{d} t_{v-1} \omega(t_1, \dots, t_n)_q as_n^k \right.\\ &\qquad\qquad \left. \times (x, y; c, d; t_1, \dots, t_n) d_q t_1 \cdots d_q t_n\right), \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\mathrm{PV}\left(\int_{-c}^{d} \cdots \int_{-c}^{d} \frac{1}{t_{v-1}} \omega(t_1, \dots, t_n)\, {}_q as_n^k(x, y; c, d; t_1, \dots, t_n) d_q t_1 \cdots d_q t_n\right) \\ &\quad = q^k\, \mathrm{PV}\left(\int_{-c}^{d} \cdots \int_{-c}^{d} \frac{1}{t_v} \omega(t_1, \dots, t_n)\, {}_q as_n^k \right.\\ &\qquad\qquad \left. \times (x, y; c, d; t_1, \dots, t_n)\, d_q t_1 \cdots\, d_q t_n\right). \end{aligned} \tag{3.9}$$

Proof. Define ${}_q\Delta_n^{2k}(v; t_1, \dots, t_n)$ by

$${}_q\Delta_n^{2k}(t_1, \dots, t_n) = (t_{v-1} - t_v)(t_{v-1} - q^k t_v)\, {}_q\Delta_n^{2k}(v; t_1, \dots, t_n). \tag{3.10}$$

Using (3.3) and the fact that $-st(1 - s/t)(1 - q^k t/s) = (s - t)(s - q^k t)$, we have

$$\begin{aligned} s^{2k}(q^{1-k} t/s; q)_{2k} = (s - t)(s - q^k t)(-st)^{k-1} \\ q^{-\binom{k}{2}}(qs/t; q)_{k-1}(qt/s; q)_{k-1}. \end{aligned} \tag{3.11}$$

Using (3.11) with $s = t_{v-1}$ and $t = t_v$, we see by (1.2) and (3.10) that

$$\begin{aligned} {}_q\Delta_n^{2k}(v; t_1, \dots, t_n) = (-t_{v-1} t_v)^{k-1} q^{-\binom{k}{2}}(qt_{v-1}/t_v; q)_{k-1}(qt_v/t_{v-1}; q)_{k-1} \\ \times \prod_{i=1}^{v-2} t_i^{2k}(q^{1-k} t_{v-1}/t_i; q)_{2k} t_i^{2k}(q^{1-k} t_v/t_i; q)_{2k} \\ \times \prod_{j=v+1}^{n} t_{v-1}^{2k}(q^{1-k} t_j/t_{v-1}; q)_{2k}\, t_v^{2k}(q^{1-k} t_j/t_v; q)_{2k} \\ \times \prod_{\substack{1 \le i < j \le n \\ i, j \ne v-1, v}} t_i^{2k}(q^{1-k} t_j/t_i; q)_{2k}. \end{aligned} \tag{3.12}$$

Using (1.14), (3.7), (3.10), and (3.12), we obtain

$$\omega(t_1, \ldots, t_n)\, {}_q as_n^k(x, y; c, d; t_1, \ldots, t_n) = (t_{v-1} - t_v)(t_{v-1} - q^k t_v)\, Y(t_{v-1}, t_v), \tag{3.13}$$

where

$$Y(t_{v-1}, t_v) = \omega(t_1, \ldots, t_n) \prod_{i=1}^{n} \frac{(-qt_i/c; q)_\infty}{(-q^x t_i/c; q)_\infty} \times \frac{(qt_i/d; q)_\infty}{(q^y t_i/d; q)_\infty}\, {}_q\Delta_n^{2k}(v; t_1, \ldots, t_n) \tag{3.14}$$

is symmetric in t_{v-1} and t_v. The result Lemma 5 (3.8) now follows by applying Lemma 4 (3.5) with $Y(t_{v-1}, t_v)$ given by (3.14) and $s = t_{v-1}$, $t = t_v$, $Q = q^k$.

The result Lemma 5 (3.9) now follows by incorporating $1/t_{v-1}t_v$ into $\omega(t_1, \ldots, t_n)$. ∎

4. THE ALTERNATIVE Q-TRANSPORTATION THEORY FOR THE ROOT SYSTEM A_{n-1}

In this section, we lift the simple relation Lemma 2 (2.6) to the multivariable setting, obtaining an integral form of the alternative q-transportation theory for the root system A_{n-1} which we express in terms of ${}_q as_n^k(x, y; c, d; t_1, \ldots, t_n)$.

Setting $s = t_i$ and $t = t_j$ for $1 \le i < j \le n$ in (3.3), we see by (1.2) that

$$\begin{aligned} {}_q\Delta_n^{2k}(t_1, \ldots, t_n) &= \prod_{1 \le i < j \le n} (-t_i t_j)^k\, q^{-\binom{k}{2}} (t_i/t_j; q)_k (qt_j/t_i; q)_k \\ &= (-1)^{\binom{n}{2}k}\, q^{-\binom{n}{2}\binom{k}{2}} \prod_{i=1}^{n} t_i^{(n-1)k}\, {}_q a_{n-1}^k(t_1, \ldots, t_n). \end{aligned} \tag{4.1}$$

Observe by (1.20) and (4.1) that

$$\begin{aligned} {}_q\Delta_n^{2k}(qt_n, t_1, \ldots, t_{n-1}) &= (-1)^{\binom{n}{2}k}\, q^{(n-1)k - \binom{n}{2}\binom{k}{2}} \\ &\quad \times \prod_{i=1}^{n} t_i^{(n-1)k}\, {}_q a_{n-1}^k(qt_n, t_1, \ldots, t_{n-1}) \\ &= q^{(n-1)k}\, {}_q\Delta_n^{2k}(t_1, \ldots, t_n). \end{aligned} \tag{4.2}$$

The following lemma gives an integral form of the alternative q-transportation theory for the root system A_{n-1}.

LEMMA 6. *If* $g(t_1, \ldots, t_n)$ *has no poles on* $[-c, d]_q^n$, *then we have*

$$\mathrm{PV}\Big(\int_{-c}^{d} \cdots \int_{-c}^{d} (1 + t_1/c)(1 - t_1/d)\, {}_q as_n^k(x, y; c, d; t_1, \ldots, t_n) \times g(t_1, \ldots, t_n) \frac{d_q t_1}{t_1}\, d_q t_2 \cdots d_q t_n\Big)$$

$$= q^{(n-1)k}\, \mathrm{PV}\Big(\int_{-c}^{d} \cdots \int_{-c}^{d} (1 + q^x t_n/c)(1 - q^y t_n/d)_q as_n^k (x, y; c, d; t_1, \ldots, t_n) g(qt_n, t_1, \ldots, t_{n-1})\, d_q t_1 \cdots d_q t_{n-1} \frac{d_q t_n}{t_n}\Big). \quad (4.3)$$

Proof. Observe that

$$f(t_1, \ldots, t_n) = (1 + t_1/c)(1 - t_1/d) \prod_{i=1}^{n} \frac{(-qt_i/c; q)_\infty}{(-q^x t_i/c; q)_\infty} \frac{(qt_i/d; q)_\infty}{(q^y t_i/d; q)_\infty} \times {}_q\Delta_n^{2k}(t_1, \ldots, t_n)\, g(t_1, \ldots, t_n) \quad (4.4)$$

has no poles with $(t_1, \ldots, t_n) \in [-c, d]_q^n$ and

$$f(qt_n, t_1, \ldots, t_{n-1}) = (1 + q^x t_n/c)(1 - q^y t_n/d) \prod_{i=1}^{n} \frac{(-qt_i/c; q)_\infty}{(-q^x t_i/c; q)_\infty} \frac{(qt_i/d; q)_\infty}{(q^y t_i/d; q)_\infty} \times {}_q\Delta_n^{2k}(qt_n, t_1, \ldots, t_{n-1}) g(qt_n, t_1, \ldots, t_{n-1}). \quad (4.5)$$

Using (1.14), (4.2), (4.4), and (4.5), we see that we may write the result (4.3) as

$$\mathrm{PV}\Big(\int_{-c}^{d} \cdots \int_{-c}^{d} f(t_1, \ldots, t_n) \frac{d_q t_1}{t_1}\, d_q t_2 \cdots d_q t_n\Big) = \mathrm{PV}\Big(\int_{-c}^{d} \cdots \int_{-c}^{d} f(qt_n, t_1, \ldots, t_{n-1})\, d_q t_1 \cdots d_q t_{n-1} \frac{d_q t_n}{t_n}\Big). \quad (4.6)$$

Observe by (4.3) that

$$0 = f(-c, t_2, \ldots, t_n) = f(d, t_2, \ldots, t_n), \quad (4.7)$$

where $(t_2, \ldots, t_n) \in [-c, d]_q^{n-1}$. Using (4.7) and observing that we may shift both of the sums on the right side of (2.11), we have

$$\mathrm{PV}\Big(\int_{-c}^{d} \cdots \int_{-c}^{d} f(t_1, \ldots, t_n) \frac{d_q t_1}{t_1}\, d_q t_2 \cdots d_q t_n\Big) = \mathrm{PV}\Big(\int_{-qc}^{qd} \int_{-c}^{d} \cdots \int_{-c}^{d} f(t_1, \ldots, t_n) \frac{d_q t_1}{t_1}\, d_q t_2 \cdots d_q t_n\Big). \quad (4.8)$$

Substituting (4.8) into (4.6), we see that the result (4.3) becomes

$$\mathrm{PV}\left(\int_{-qc}^{qd}\int_{-c}^{d}\cdots\int_{-c}^{d} f(t_1,\ldots,t_n)\,\frac{d_qt_1}{t_1}\,d_qt_2\cdots d_qt_n\right) = \mathrm{PV}\left(\int_{-c}^{d}\cdots\int_{-c}^{d} f(qt_n,t_1,\ldots,t_{n-1})\,d_qt_1\cdots d_qt_{n-1}\,\frac{d_qt_n}{t_n}\right), \tag{4.9}$$

which follows using the substitution $(t_1,\ldots,t_n)=(qs_n,s_1,\ldots,s_{n-1})$, the scale invariance (2.2) of d_qt_1/t_1, and replacing s_i by t_i, $1\le i\le n$. ∎

5. THE DEPENDENCE OF ${}_qAS^k_{n,m}(x,y;c,d)$ ON THE PARAMETERS m AND x

In this section, we establish the dependence of ${}_qAS^k_{n,m}(x,y;c,d)$ on the parameters m and x. Throughout this section, we let $1\le m\le n$.

We set

$$\mathcal{J} = \mathrm{PV}\left(\int_{-c}^{d}\cdots\int_{-c}^{d}(1+t_1/c)(1-t_1/d)\prod_{i=n-m+2}^{n}(1+q^xt_i/c) \times {}_qas^k_n(x,y;c,d;t_1,\ldots,t_n)\,\frac{d_qt_1}{t_1}\,d_qt_2\cdots d_qt_n\right). \tag{5.1}$$

Taking $g(t_1,\ldots,t_n)=\prod_{i=n-m+2}^{n}(1+q^xt_i/c)$ in Lemma 6 (4.3), we obtain

$$\mathcal{J} = q^{(n-1)k}\ \mathrm{PV}\left(\int_{-c}^{d}\cdots\int_{-c}^{d}(1-q^yt_n/d)\prod_{i=n-m+1}^{n}(1+q^xt_i/c) \times {}_qas^k_n(x,y;c,d;t_1,\ldots,t_n)d_qt_1\cdots d_qt_{n-1}\,\frac{d_qt_n}{t_n}\right). \tag{5.2}$$

Applying Lemma 5 (3.9) $m-1$ times to the term $1/t_n$ in

$$(1-q^yt_n/d)\,\frac{1}{t_n} = 1/t_n - q^y/d \tag{5.3}$$

and the q-integral on the right side of (5.2) with v running from $n-1$ to $n-m+1$ and $\omega(t_1,\ldots,t_n)=\prod_{i=n-m+1}^{n}(1+q^xt_i/c)$ and rearranging $\omega(t_1,\ldots,t_n)$, we obtain

$$\mathcal{J} = \mathrm{PV}\left(\int_{-c}^{d}\cdots\int_{-c}^{d}(q^{(n-m)k}/t_{n-m+1} - q^{y+(n-1)k}/d)(1+q^xt_{n-m+1}/c) \times \prod_{i=n-m+2}^{n}(1+q^xt_i/c)\,{}_qas^k_n(x,y;c,d;t_1,\ldots,t_n)\,d_qt_1\cdots d_qt_n\right). \tag{5.4}$$

Applying Lemma 5 (3.8) and (3.9) $n-m$ times to the terms $-t_1/cd$ and $1/t_1$, respectively, in

$$(1+t_1/c)(1-t_1/d)\frac{1}{t_1} = 1/t_1 + 1/c - 1/d - t_1/cd \tag{5.5}$$

and the q-integral $\mathscr{I}$ on the right side of (5.1) with v running from two to $n-m+1$ and $\omega(t_1,\ldots,t_n) = \prod_{i=n-m+2}^{n}(1+q^x t_i/c)$, and factoring the integrand, we obtain

$$\mathscr{I} = \mathrm{PV}\bigg(\int_{-c}^{d}\cdots\int_{-c}^{d}\Big(q^{(n-m)k}/t_{n-m+1} + 1/c - 1/d - q^{-(n-m)k}t_{n-m+1}/cd\Big) \times \prod_{i=n-m+2}^{n}(1+q^x t_i/c)\,{}_q as_n^k(x,y;c,d;t_1,\ldots,t_n)\,d_qt_1\cdots d_qt_n\bigg). \tag{5.6}$$

Substituting

$$x, y, c, d, t \longrightarrow x+(n-m)k,\, y+(n-1)k,\, q^{(n-m)k}c,\, q^{(n-m)k}d,\, t_{n-m+1} \tag{5.7}$$

into (2.14) and rearranging the left side, we obtain

$$\begin{aligned}\Big(&(q^{(n-m)k}/t_{n-m+1} + 1/c)(1 - q^{-(n-m)k}t_{n-m+1}/d)\\ &-(1+q^x t_{n-m+1}/c)(q^{(n-m)k}/t_{n-m+1} - q^{y+(n-1)k}/d)\Big)q^{x+(n-m)k}d\\ &= (1-q^{x+(n-m)k})(1+q^{x+(n-m)k}d/c)\\ &\quad -(1-q^{x+y+(2n-m-1)k})(1+q^x t_{n-m+1}/c).\end{aligned} \tag{5.8}$$

Equating (5.4) and (5.6), multiplying by $q^{x+(n-m)k}d$, factoring the integrand of (5.6), moving both q-integrals to the same side of the equation, using (5.8), and observing that the principal value extends the definition (1.11), we obtain

$$\begin{aligned}0 &= (1-q^{x+(n-m)k})(1+q^{x+(n-m)k}d/c)\int_{-c}^{d}\cdots\int_{-c}^{d}\\ &\quad\times\prod_{i=n-m+2}^{n}(1+q^x t_i/c)\,{}_q as_n^k(x,y;c,d;t_1,\ldots,t_n)d_qt_1\cdots d_qt_n\\ &\quad -(1-q^{x+y+(2n-m-1)k})\int_{-c}^{d}\cdots\int_{-c}^{d}(1+q^x t_{n-m+1}/c)\\ &\quad\times\prod_{i=n-m+2}^{n}(1+q^x t_i/c)\times{}_Q as_n^k(x,y;c,d;t_1,\ldots,t_n)d_qt_1\cdots d_qt_n.\end{aligned} \tag{5.9}$$

Using (1.14) and (1.15), we may rearrange (5.9) as

$$ {}_qAS^k_{n,m}(x,y;c,d) = (1+q^{x+(n-m)k}d/c) \times \frac{(1-q^{x+(n-m)k})}{(1-q^{x+(2n-m-1)k})}\, {}_qAS^k_{n,m-1}(x,y;c,d), \tag{5.10} $$

which gives the dependence of ${}_qAS^k_{n,m}(x,y;c,d)$ on the parameter m.

Using (5.10), we see by induction on m that

$$ {}_qAS^k_{n,m}(x,y;c,d) = \prod_{i=1}^{m}(1+q^{x+(n-i)k}d) \times \frac{(1-q^{x+(n-i)k})}{(1-q^{x+(2n-i-1)k})}\, {}_qAS^k_{n}(x,y;c,d). \tag{5.11} $$

Setting $m=n$ in (5.11) and using the fact that

$$ {}_qAS^k_{n,n}(x,y;c,d) = {}_qAS^k_{n}(x+1,y;c,d), \tag{5.12} $$

we obtain

$$ {}_qAS^k_{n}(x+1,y;c,d) = \prod_{i=1}^{n}(1+q^{x+(n-i)k}d) \times \frac{(1-q^{x+(n-i)k})}{(1-q^{x+(2n-i-1)k})}\, {}_qAS^k_{n}(x,y;c,d), \tag{5.13} $$

which gives the dependence of ${}_qAS^k_{n}(x,y;c,d)$ on the parameter x.

6. THE GLOBAL FORM OF THE q-TRANSPORTATION THEORY FOR THE ROOT SYSTEM A_{n-1}

In this section, we recall the global form [16, Sect. 4] of the q-transportation theory for the root system A_{n-1} and give an application.

Throughout this section, we follow the notation and recall results of [16, Sect. 4].

We set

$$ {}_{(Q_{i,j})}\Delta_n(t_1,\ldots,t_n) = \prod_{1\le i<j\le n}(t_i - Q_{i,j}t_j) \tag{6.1} $$

and we define the antisymmetrization operator

$$ \Xi_n\Big(f(t_1,\ldots,t_n)\Big) = \sum_{\pi\in S_n}\operatorname{sign}(\pi)\, f(t_{\pi(1)},\ldots,t_{\pi(n)}). \tag{6.2} $$

The case $m = 0$ of [16, Lemma 4 (4.4)] is

$$\Xi_n\Big({}_{(Q_{i,j})}\Delta_n(t_1, \ldots, t_n)\Big) = \Bigg(\sum_{\pi\in S_n} \prod_{\substack{1\le i<j\le n\\ \pi(i)\pi(j)}} Q_{i,j}\Bigg)\Delta_n(t_1, \ldots, t_n). \tag{6.3}$$

Using (3.1) and the fact that $(1-k)(2k-1) + \binom{2k-1}{2} = 0$, we see that

$$\begin{aligned} s^{2k-1}(q^{1-k}t/s; q)_{2k-1} &= s^{2k-1}(-q^{1-k}t/s)^{2k-1}q^{\binom{2k-1}{2}}(q^{1-k}s/t; q)_{2k-1} \\ &= -t^{2k-1}(q^{1-k}s/t; q)_{2k-1} \end{aligned} \tag{6.4}$$

is antisymmetric in s and t. Using (6.4) with $s = t_i$ and $t = t_j$ for $1 \le i < j \le n$, we see that

$${}_q\Delta_n^{2k-1}(t_1, \ldots, t_n) = \prod_{1\le i<j\le n} t_i^{2k-1}(q^{1-k}t_j/t_i; q)_{2k-1} \tag{6.5}$$

is antisymmetric in $t_1, \ldots, t_n$.

If F is a distribution function on the domain $\mathscr{D}$, then we have

$$\begin{aligned} &\int\cdots\int_{\mathscr{D}^n} {}_{(Q_{i,j})}\Delta_n(t_1, \ldots, t_n)\,{}_q\Delta_n^{2k-1}(t_1, \ldots, t_n)dF(t_1)\cdots dF(t_n) \\ &\quad= \frac{1}{n!}\Bigg(\sum_{\pi\in S_n} \prod_{\substack{1\le i<j\le n\\ \pi(i)>\pi(j)}} Q_{i,j}\Bigg)\int\cdots\int_{\mathscr{D}^n} \Delta_n(t_1, \ldots, t_n)\,{}_q\Delta_n^{2k-1} \\ &\qquad\times (t_1, \ldots, t_n)\,dF(t_1)\cdots dF(t_n). \end{aligned} \tag{6.6}$$

Observe that (6.6) gives the effect of $(Q_{i,j})$ on the value of the integral on the left side of (6.6). This is the case $m = 0$ of the global q-transportation theory for the root system A_{n-1} developed in [16, Sect. 4].

Observe that

$$s^{2k}(q^{1-k}t/s; q)_{2k} = s^{2k-1}(q^{1-k}t/s; q)_{2k-1}\,(s - q^k t). \tag{6.7}$$

Using (6.7) with $s = t_i$ and $t = t_j$ for $1 \le i < j \le n$, we see that

$${}_q\Delta_n^{2k}(t_1, \ldots, t_n) = \prod_{1\le i<j\le n} (t_i - q^k t_j)\,{}_q\Delta_n^{2k-1}(t_1, \ldots, t_n). \tag{6.8}$$

We have MacMahon [24]

$$\sum_{\pi\in S_n} \prod_{\substack{1\le i<j\le n\\ \pi(i)>\pi(j)}} Q = \prod_{i=1}^{n} \frac{(1-Q^i)}{(1-Q)}. \tag{6.9}$$

We set

$$\begin{aligned} {}_qh_n^k(x, y; c, d; t_1, \ldots, t_n) &= \prod_{i=1}^{n} \frac{(-qt_i/c; q)_\infty(qt_i/d; q)_\infty}{(-q^x t_i/c; q)_\infty(q^y t_i/d; q)_\infty}\Delta_n(t_1, \ldots, t_n) \\ &\quad\times\ {}_q\Delta_n^{2k-1}(t_1, \ldots, t_n)\,d_qt_1\cdots d_qt_n, \end{aligned} \tag{6.10}$$

which is symmetric,

$$ {}_qh_n^k(x, y; c, d; t_1, \ldots, t_n) = {}_qh_n^k(x, y; c, d; t_{\pi(1)}, \ldots, t_{\pi(n)}), \qquad \pi \in S_n, \tag{6.11}$$

in $t_1, \ldots, t_n$, and we use capital letters to denote the q-integral

$$ {}_qH_n^k(x, y; c, d) = \int_{-c}^{d} \cdots \int_{-c}^{d} {}_qh_n^k(x, y; c, d; t_1, \ldots, t_n) d_qt_1 \cdots d_qt_n. \tag{6.12}$$

Setting $Q_{i,j} = Q$, $1 \le i < j \le n$, in (6.3) and using (6.9), we obtain

$$ \Xi_n\Big(\prod_{1\le i<j\le n} (t_i - Qt_j)\Big) = \prod_{i=1}^{n} \frac{(1-Q^i)}{(1-Q)} \Delta_n(t_1, \ldots, t_n), \tag{6.13}$$

which is [16, (4.13)]. See Macdonald [23] and Carter [6, Theorems 10.2.1, 10.2.3].

Setting $Q_{i,j} = q^k$, $1 \le i < j \le n$, in (6.6), setting $Q = q^k$ in (6.9), and using (6.8) and (6.10), we obtain

$$ {}_qAS_n^k(x, y; c, d) = \frac{1}{n!} \prod_{i=1}^{n} \frac{(1-q^{ik})}{(1-q^k)} {}_qH_n^k(x, y; c, d). \tag{6.14}$$

7. A RECURRENCE RELATION

In this section, we follow Askey [4] and lift the boundary condition of Section 2 to the multivariable setting, obtaining a recurrence relation involving the paramters x, c, and n.

Using the symmetry (6.11) of the integrand ${}_qh_n^k(x, y; c, d; t_1, \ldots, t_n)$ and the fact that the integrand vanishes when $t_i = t_j$ for $1 \le i < j \le n$, we have

$$ {}_qH_n^k(x, y; c, d) = n \int_{-c}^{d} \frac{(-qt_n/c; q)_\infty (qt_n/d; q)_\infty}{(-q^x t_n/c; q)_\infty (q^x t_n/d; q)_\infty} g_x(t_n) d_qt_n, \tag{7.1}$$

where

$$ g_x(t_n) = \int_{t_n}^{d} \cdots \int_{t_n}^{d} \prod_{i=1}^{n-1} \frac{(-qt_i/c; q)_\infty (qt_i/d; q)_\infty}{(-q^x t_i/c; q)_\infty (q^y t_i/d; q)_\infty} \times \Delta_n(t_1, \ldots, t_n) {}_q\Delta_n^{2k-1}(t_1, \ldots, t_n) d_qt_1 \cdots d_qt_{n-1}. \tag{7.2}$$

Since $t + c = c(1 + t/c)$, we have

$$ \Delta_n(t_1, \ldots, t_{n-1}, -c) = c^{n-1} \prod_{i=1}^{n-1} (1 + t_i/c) \Delta_{n-1}(t_1, \ldots, t_{n-1}) \tag{7.3}$$

and, using the antisymmetry (6.4),

$$
\begin{aligned}
{}_q\Delta_n^{2k-1}(t_1,\ldots,t_{n-1},-c) &= \prod_{i=1}^{n-1} t_i^{2k-1}(-q^{1-k}c/t_i;q)_{2k-1}\,{}_q\Delta_{n-1}^{2k-1}(t_1,\ldots,t_{n-1}) \\
&= c^{(2k-1)(n-1)}\prod_{i=1}^{n-1}(-q^{1-k}t_i/c;q)_{2k-1} \\
&\quad\times\ {}_q\Delta_{n-1}^{2k-1}(t_1,\ldots,t_{n-1}). \qquad (7.4)
\end{aligned}
$$

Observe that

$$
\prod_{i=1}^{n-1}(1+t_i/c)\frac{(-qt_i/c;q)_\infty}{(-q^x t_i/c;q)_\infty}\bigg|_{x=0} = 1. \qquad (7.5)
$$

Substituting (7.3) and (7.4) into (7.2), setting $x = 0$ and $t_n = -c$, and using (7.5) and the fact that $(\alpha;q)_{2k-1} = (\alpha;q)_\infty/(q^{2k-1}\alpha;q)_\infty$, we obtain

$$
\begin{aligned}
g_0(-c) &= c^{2(n-1)k}\int_{-c}^{d}\cdots\int_{-c}^{d}\prod_{i=1}^{n-1}\frac{(-q^{1-k}t_i/c;q)_\infty(qt_i/d;q)_\infty}{(-q^k t_i/c;q)_\infty(q^y t_i/d;q)_\infty} \\
&\quad\times\Delta_{n-1}(t_1,\ldots,t_{n-1})_q\Delta_{n-1}^{2k-1}(t_1,\ldots,t_{n-1})d_qt_1\cdots d_qt_{n-1}. \qquad (7.6)
\end{aligned}
$$

Since the integrand on the right side of (7.6) vanishes for $t_i = -q^j c$ where $1 \le i \le n-1$ and $0 \le j \le k-1$, we may write (7.6) as

$$
\begin{aligned}
g_0(-c) &= c^{2k(n-1)}\int_{-q^k c}^{d}\cdots\int_{-q^k c}^{d}\prod_{i=1}^{n-1}\frac{(-q^{1-k}t_i/c;q)_\infty(qt_i/d;q)_\infty}{(-q^k t_i/c;q)_\infty(q^y t_i/d;q)_\infty} \\
&\quad\times\Delta_{n-1}(t_1,\ldots,t_{n-1})_q\Delta_{n-1}^{2k-1}(t_1,\ldots,t_{n-1})d_qt_1\cdots d_qt_{n-1} \\
&= c^{2k(n-1)}\,{}_qH_{n-1}^k(2k,y;q^kc,d). \qquad (7.7)
\end{aligned}
$$

Since g_x satisfies the hypotheses of Lemma 3, we see using (7.1) and (7.7) that Lemma 3 (2.18) gives

$$
\begin{aligned}
&\lim_{x\to 0^+}\frac{(1-q^x)}{(1-q)}\,{}_qH_n^k(x,y;c,d) \\
&\quad = n\frac{cd}{(c+d)}\frac{(-c/d;q)_\infty}{(-q^y c/d;q)_\infty}c^{2k(n-1)}\,{}_qH_{n-1}^k(2k,y;q^kc,d), \qquad (7.8)
\end{aligned}
$$

which is [16, (9.17)]. Using (6.14), we see that (7.8) becomes the recurrence relation

$$
\begin{aligned}
\lim_{x\to 0^+}\frac{(1-q^x)}{(1-q)}\,{}_qAS_n^k(x,y;c,d) &= \frac{(1-q^{nk})}{(1-q^k)}\frac{cd}{(c+d)}\frac{(-c/d;q)_\infty}{(-q^y c/d;q)_\infty} \\
&\quad\times c^{2(n-1)k}\,{}_qAS_{n-1}^k(2k,y;q^kc,d). \qquad (7.9)
\end{aligned}
$$

8. A PROOF OF THEOREM 1

In this section, we use induction on n, x, and m to evaluate ${}_qAS^k_{n,m}(x, y; c, d)$ when x is a positive integer and, using Ismail's argument [15], we extend to complex x and complete the proof of Theorem 1.

Using the fact that $cd(-d/c;q)_\infty/(c+d) = d(-qd/c;q)_\infty$, we may write the function on the right side of (1.16) as

$$ {}_qR^k_{n,m}(x, y; c, d) = {}_q\mathcal{J}^k_n(c, d)\, {}_q\mathcal{K}^k_{n,m}(x, y; c, d)\, {}_q\mathcal{L}^k_{n,m}(x, y), \qquad (8.1) $$

where

$$ {}_q\mathcal{J}^k_n(c, d) = q^{\binom{n}{3}k^2 - \binom{n}{2}\binom{k}{2}}(cd)^{\binom{n}{2}k}, \qquad (8.2) $$

$$ \begin{aligned} &{}_q\mathcal{K}^k_{n,m}(x, y; c, d) \\ &\quad = d^n \prod_{i=1}^{n} \frac{(-qd/c;q)_\infty(-c/d;q)_\infty}{(-q^{x+(n-i)k+\chi(i\le m)}d/c;q)_\infty(-q^{y+(n-i)k}c/d;q)_\infty}, \end{aligned} \qquad (8.3) $$

$$ \begin{aligned} &{}_q\mathcal{L}^k_{n,m}(x, y) \\ &\quad = \prod_{i=1}^{n} \frac{\Gamma_q(x+(n-i)k+\chi(i\le m))\Gamma_q(y+(n-i)k)\Gamma_q(1+ik)}{\Gamma_q(x+y+(2n-i-1)k+\chi(i\le m))\Gamma_q(1+k)}. \end{aligned} \qquad (8.4) $$

Using (2.17), we see that ${}_qR^k_{n,m}(x, y; c, d)$ satisfies (5.10) and (5.12). Observe that

$$ \begin{aligned} {}_q\mathcal{J}^k_{n-1}(q^k c, d) &= q^{\binom{n-1}{3}k^2 - \binom{n-1}{2}\binom{k}{2}}(q^k cd)^{\binom{n-1}{2}k} \\ &= q^{\binom{n}{3}k^2 - \binom{n-1}{2}\binom{k}{2}}(cd)^{\binom{n-1}{2}k}. \end{aligned} \qquad (8.5) $$

Comparing (8.2) and (8.5), we have

$$ {}_q\mathcal{J}^k_n(c, d) = q^{-(n-1)\binom{k}{2}}(cd)^{(n-1)k}\, {}_q\mathcal{J}^k_{n-1}(q^k c, d). \qquad (8.6) $$

Observe that (8.3) gives

$$ {}_q\mathcal{K}^k_n(0, y; c, d) = d^n \prod_{i=1}^{n} \frac{(-qd/c;q)_\infty(-c/d;q)_\infty}{(-q^{(n-i)k}d/c;q)_\infty(-q^{y+(n-i)k}c/d;q)_\infty}, \qquad (8.7) $$

$$ \begin{aligned} &{}_q\mathcal{K}^k_{n-1}(2k, y; q^k c, d) \\ &\quad = d^{n-1} \prod_{i=1}^{n-1} \frac{(-q^{1-k}d/c;q)_\infty(-q^k c/d;q)_\infty}{(-q^{(n-i)k}d/c;q)_\infty(-q^{y+(n-i)k}c/d;q)_\infty}. \end{aligned} \qquad (8.8) $$

Using (3.1) and (3.2), we obtain

$$\frac{(-qd/c;q)_\infty(-c/d;q)_\infty}{(-q^{1-k}d/c;q)_\infty(-q^kc/d;q)_\infty} = \frac{(-c/d;q)_k}{(-q^{1-k}d/c;q)_k} = q^{\binom{k}{2}}(c/d)^k. \quad (8.9)$$

Using (8.9) and the fact that $d(-qd/c;q)_\infty/(-d/c;q)_\infty = cd/(c+d)$ to compare (8.7) and (8.8), we obtain

$$\begin{aligned} {}_q\mathcal{K}_n^k(0,y;c,d) = q^{(n-1)\binom{k}{2}}(c/d)^{(n-1)k}\frac{cd}{(c+d)}\frac{(-c/d;q)_\infty}{(-q^y c/d;q)_\infty} \\ \times {}_q\mathcal{K}_{n-1}^k(2k,y;q^kc,d). \end{aligned} \quad (8.10)$$

Using (2.17) and (2.21), we obtain

$$\begin{aligned} &\lim_{x\to 0^+}\frac{(1-q^x)}{(1-q)}{}_q\mathcal{L}_n^k(x,y) \\ &= \prod_{i=1}^{n-1}\Gamma_q((n-i)k)\prod_{i=1}^{n}\frac{\Gamma_q(y+(n-i)k)\Gamma_q(1+ik)}{\Gamma_q(y+(2n-i-1)k)\Gamma_q(1+k)} \\ &= \frac{\Gamma_q(1+nk)}{\Gamma_q(1+k)}\prod_{i=1}^{n-1}\frac{\Gamma_q((n-i)k)\Gamma_q(y+(n-1-i)k)\Gamma_q(1+ik)}{\Gamma_q(y+(2n-i-1)k)\Gamma_q(1+k)} \\ &= \frac{(1-q^{nk})}{(1-q^k)}\prod_{i=1}^{n-1}\frac{\Gamma_q((n+1-i)k)\Gamma_q(y+(n-1-i)k)\Gamma_q(1+ik)}{\Gamma_q(y+(2n-i-1)k)\Gamma_q(1+k)}, \end{aligned} \quad (8.11)$$

$${}_q\mathcal{L}_{n-1}^k(2k,y) = \prod_{i=1}^{n-1}\frac{\Gamma_q((n+1-i)k)\Gamma_q(y+(n-1-i)k)}{\Gamma_q(y+(2n-i-1)k)}\frac{\Gamma_q(1+ik)}{\Gamma_q(1+k)}. \quad (8.12)$$

Comparing (8.11) and (8.12), we have

$$\lim_{x\to 0^+}\frac{(1-q^x)}{(1-q)}{}_q\mathcal{L}_n^k(x,y) = \frac{(1-q^{nk})}{(1-q^k)}{}_q\mathcal{L}_{n-1}^k(2k,y). \quad (8.13)$$

Multiplying (8.6), (8.10), and (8.13), we see that ${}_qR_n^k(x,y;c,d)$ satisfies the recurrence relation (7.9).

Using (5.10), (5.12), and (7.9), and proceeding by induction on n, x, and m, we may evaluate ${}_qAS_n^k(x,y;c,d)$ when x is a positive integer.

Recall the identity theorem by Hille [14, Sect. 8.1] that two functions which are analytic in the domain $\mathcal{D}$ and agree at infinitely many points of $\mathcal{D}$ which include an accumulation point in $\mathcal{D}$ agree throughout $\mathcal{D}$.

Let w and z be complex numbers. Observe that $(wz;q)_\infty$ is an entire function of z and that if $|w| \le 1$ then $1/(wz;q)_\infty$ is an analytic function of z in the unit disc around zero. Observe that

$$s^{2k}(q^{1-k}t/s;q)_{2k} = (s-q^{1-k}t)\cdots(s-q^kt). \quad (8.14)$$

Setting $s = t_i$ and $t = t_j$ for $1 \leq i < j \leq n$ in (8.14), we see by (1.2) that ${}_q\Delta_n^{2k}(t_1, \ldots, t_n)$ is a polynomial. Since finite sums and products of analytic functions are analytic, we see that ${}_qAS_{n,m}^k(x, y; c, d)$ and ${}_qR_{n,m}^k(x, y; c, d)$ are analytic functions of $z = q^x$ in the unit disc around zero. Since they have the same values at $z = q^x$ where x is a positive integer and they are both analytic at $z = 0$, we see by the identity theorem that they are equal for z in the unit disc around zero. Since $|q| < 1$, this is the half plane where the real part of x is positive. This completes the evaluation of ${}_qAS_{n,m}^k(x, y; c, d)$ and establishes Theorem 1.

Our argument was introduced by Ismail [15] who gave a simple proof of Ramanujan's ${}_1\psi_1$ summation formula. Kaneko [21] used Ismail's argument to give a multivariable ${}_1\psi_1$ summation theorem using the Macdonald polynomials.

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