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Omega theorems related to the general Euler totient function ☆

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ABSTRACT

We prove an omega estimate related to the general Euler totient function associated to a polynomial Euler product satisfying some natural analytic properties. For convenience, we work with a set of L-functions similar to the Selberg class, but in principle our results can be proved in a still more general setup. In a recent paper the authors treated a special case of Dirichlet L-functions with real characters. Greater generality of the present paper invites new technical difficulties. Effectiveness of the main theorem is illustrated by corollaries concerning Euler totient functions associated to the shifted Riemann zeta function, shifted Dirichlet L-functions and shifted L-functions of modular forms. Results are either of the same quality as the best known estimates or are entirely new.

1. Introduction

Following [3] we define the general Euler totient function associated to a polynomial Euler product

$$F(s)=\prod_p F_p(s)=\prod_p\prod_{j=1}^{d}\left(1-\frac{\alpha_j(p)}{p^s}\right)^{-1},$$

where p runs over primes and $|\alpha_j(p)|\leqslant 1$ for all p and $1\leqslant j\leqslant d$ as follows

$$\varphi(n,F)=n\prod_{p|n}F_p(1)^{-1}. \tag{1.1}$$

We assume here that d is chosen as small as possible, i.e. that there exists at least one prime number p_0 such that

$$\prod_{j=1}^{d}\alpha_j(p_0)\neq 0.$$

Then d is called the *Euler degree* of F. Of course every polynomial Euler product is also a Dirichlet series

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$$F(s)=\sum_{n=1}^{\infty}\frac{a_F(n)}{n^s}$$

which is absolutely convergent for $\sigma=\Re(s)>1$.

Let

$$\gamma(p)=p\left(1-\frac{1}{F_p(1)}\right) \tag{1.2}$$

and

$$C(F)=\frac{1}{2}\prod_p\left(1-\frac{\gamma(p)}{p^2}\right). \tag{1.3}$$

It was proved in [3, Theorem 1.1] that for a polynomial Euler product F of degree d and $x\geqslant 1$ we have

$$\sum_{n\leqslant x}\varphi(n,F)=C(F)x^2+O\left(x\left(\log(2x)\right)^d\right).$$

Let us denote the corresponding error term by

$$E(x,F)=\sum_{n\leqslant x}\varphi(n,F)-C(F)x^2. \tag{1.4}$$

As in [3] let $\mathcal{S}_0$ denote the set of all polynomial Euler products $F(s)$ belonging to the Selberg class $\mathcal{S}$ such that $F(s)\neq 0$ for

$$\sigma>1-\frac{c_0(F)}{\log(|t|+10)}\quad (s=\sigma+it,\ -\infty<t<\infty),$$

where $c_0(F)$ denotes a positive constant depending on F. We refer to [2,4,8,9] for basic definitions and results on the Selberg class. Let us remark that most probably $\mathcal{S}=\mathcal{S}_0$ but it has not been proven yet. It is convenient to introduce a more general class of L-functions as follows. We say that $F\in\tilde{\mathcal{S}}_0$ if there exists $F^*\in\mathcal{S}_0$ such that F and F^* differ by a finite number of local factors, i.e. there exists a finite set of primes T such that

$$\frac{F(s)}{F^*(s)}=\prod_{p\in T}\prod_{j=1}^{\partial_p}\left(1-\frac{\beta_j(p)}{p^s}\right)^{\epsilon_j(p)},$$

where $|\beta_j(p)|\leqslant 1$ and $\epsilon_j(p)\in\{-1,1\}$ for all $p\in T$ and $1\leqslant j\leqslant\partial_p$.

It is known that for $F\in\mathcal{S}_0$ we have

$$E(x,F)=\Omega(x) \tag{1.5}$$

as $x\to\infty$ (see [3, Corollary 1.4]). The principal goal of the present paper is to improve on this estimate. To formulate our main result we need the following auxiliary notation. For every prime number p and $\epsilon=\pm 1$ we put

$$\xi_p(F,\epsilon)=\arg\left(-\epsilon\gamma(p)\right)\quad\left(-\pi<\xi_p(F,\epsilon)\leqslant\pi\right) \tag{1.6}$$

and for every positive integer k let

$$\Psi_k(x,F,\epsilon)=\sum_{\substack{p\leqslant x\\ |\xi_p(F,\epsilon)|\leqslant\pi/2\\ p\equiv\epsilon\ (\mathrm{mod}\ k)}}\frac{|a_F(p)|}{p}\cos\xi_p(F,\epsilon). \tag{1.7}$$

Theorem 1.1. *Let $F(s)$ be a polynomial Euler product such that $F(s+i\lambda)\in\tilde{\mathcal{S}}_0$ for certain real λ. Suppose that $C(F)\neq 0$ (see (1.3)). Then, there exists a positive constant C which may depend on F such that for all integers $k>2$ and arbitrary $\epsilon=\pm 1$ we have*

$$E(x,F)=\Omega\left(x\exp\left(\Psi_k\left(C\varphi(k)\sqrt{\log x},F,\epsilon\right)\right)\right). \tag{1.8}$$

Obviously $\Psi_k(x,F,\epsilon)\geqslant 0$ for all x and hence we reprove (1.5) for a slightly larger class of L-functions but under additional but mild assumption $C(F)\neq 0$. However, in many concrete cases we have $\Psi_k(x,F,\epsilon)\to\infty$ as $x\to\infty$ for suitably chosen k and ϵ. In such cases we get an improvement of (1.5), see Theorems 2.1–2.3 below.

2. Some applications

2.1. Euler functions related to the Riemann zeta function

Euler functions related to the Riemann zeta function are defined as follows

$$\varphi(n,\lambda)=n\prod_{p|n}\left(1-\frac{1}{p^{1+i\lambda}}\right),$$

where λ is a fixed real number. The corresponding error term equals

$$E(x,\lambda)=\sum_{n\leqslant x}\varphi(n,\lambda)-\frac{1}{2\zeta(2+i\lambda)}x^2.$$

The case $\lambda=0$ is classical. The best known omega estimate due to H.L. Montgomery [7] gives oscillations of size $\Omega(x\sqrt{\log\log x})$, see also [5]. This is exactly what also follows from our Theorem 1.1. As far as we know the case $\lambda\neq 0$ has not been treated in the literature so far.

Theorem 2.1. *For $\lambda\neq 0$ we have*

$$E(x,\lambda)=\Omega\bigl(x(\log\log x)^{\frac{1}{2\pi}}\bigr).$$

The exponent $1/(2\pi)$ here comes as a surprise, instead one would expect something depending on λ and tending to $1/2$ as $\lambda\to 0$.

2.2. Euler functions related to Dirichlet L-functions

For a Dirichlet character χ (mod q) and a real number λ the error term we are interested in looks as follows

$$E(x,\chi,\lambda)=\sum_{n\leqslant x}\varphi(n,\chi,\lambda)-\frac{1}{2L(s+i\lambda,\chi)}x^2,$$

where

$$\varphi(n,\chi,\lambda)=n\prod_{p|n}\left(1-\frac{\chi(p)}{p^{1+i\lambda}}\right).$$

The case of $\lambda=0$ and real χ was treated in a recent paper [6]. It was proved there (Corollary 1.4 of [6]) $E(x,\chi,0)=\Omega_{\pm}(x(\log\log x)^{1/4})$ (χ-real). Now we are able to treat the general case.

Theorem 2.2. *Let χ be a Dirichlet character and let h denote its order. Then, for $x\to\infty$ we have*

$$E(x,\chi,0)=\Omega\bigl(x(\log\log x)^{\eta(\chi)}\bigr),$$

where

$$\eta(\chi)=\begin{cases}\frac{1}{2} & \text{if } h=1,\\ \frac{1}{4h\sin(\pi/(2h))} & \text{if } 2\nmid h,\ h>1,\\ \frac{1}{2h\sin(\pi/h)} & \text{if } 2\| h,\\ \frac{1}{2h}\cot\frac{\pi}{h} & \text{if } 4|h.\end{cases} \tag{2.1}$$

For $\lambda\neq 0$ we have

$$E(x,\chi,\lambda)=\Omega\bigl(x(\log\log x)^{\frac{1}{2\pi}}\bigr). \tag{2.2}$$

Observe that $\eta(\chi)\to 1/(2\pi)$ when $h\to\infty$. This is in accordance with the intuition that a Dirichlet character of large order mimics continuous character $n\mapsto n^{i\lambda}$, $\lambda\neq 0$.

2.3. Euler functions related to modular forms

Let ϕ denote a new form of weight K and level q with the normalized Fourier expansion at infinity as follows

$$\phi(z) = \sum_{n=1}^{\infty} \lambda_\phi(n) n^{\frac{K-1}{2}} e(nz) \quad (z \in \mathbb{H}).$$

The corresponding normalized L-function defined for $\sigma = \Re(s) > 1$ by the Dirichlet series

$$L(s, \phi) = \sum_{n=1}^{\infty} \frac{\lambda_\phi(n)}{n^s},$$

belongs to the class S_0 and has a polynomial Euler product of degree two. Local factors have the following form

$$L_p(s, \phi) = \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where

$$\big|\alpha(p)\big|, \big|\beta(p)\big| \leqslant 1. \tag{2.3}$$

In particular, it is possible that one or both of these coefficients vanish. But this can happen for a finite number of primes only. For primes coprime to the level we have $|\alpha(p)| = |\beta(p)| = 1$. Clearly $\lambda_\phi(p) = \alpha(p) + \beta(p)$ and hence in particular

$$\big|\lambda_\phi(p)\big| \leqslant 2 \tag{2.4}$$

for all primes p.

We define the Euler function corresponding to $L_\lambda(s, \phi) = L(s + i\lambda, \phi)$, where $\lambda \in \mathbb{R}$, and the corresponding error term according to general rules (see (1.1) and (1.4))

$$\varphi(n, \phi, \lambda) = n \prod_{p|n} \frac{1}{L_p(1 + i\lambda, \phi)}$$

and

$$E(x, \phi, \lambda) = \sum_{n \leqslant x} \varphi(n, \phi, \lambda) - C(\phi, \lambda) x^2,$$

where

$$C(\phi, \lambda) = \frac{1}{2} \prod_p \left(1 - \frac{\lambda_\phi(p)}{p^{2+i\lambda}} + \frac{\alpha(p)\beta(p)}{p^{3+2i\lambda}}\right). \tag{2.5}$$

The case $\lambda = 0$ was essentially settled in [5, Proposition 1.6], and the result is

$$E(x, \phi, 0) = \Omega_\pm\big(x(\log\log x)^{1/8}\big),$$

provided $\lambda_\phi(n) \in \mathbb{R}$ for all n. We complete this theorem with the following one.

Theorem 2.3. *Let ϕ be a new form with real Fourier coefficients. Then, for every real $\lambda \neq 0$ we have*

$$E(x, \phi, \lambda) = \Omega\big(x(\log\log x)^{1/(8\pi)}\big).$$

3. A general Ω-result

By $\mathcal{A}$ we denote the set of all arithmetic functions $\alpha(n)$ satisfying the following conditions:

(i) $\alpha(n)$ is multiplicative.
(ii) There exists a positive real number $\theta < 1$ such that

$$\alpha(n) \ll n^\theta. \tag{3.1}$$

(iii) We have

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^2} \neq 0. \tag{3.2}$$

(iv) For every $N \geqslant 1$ we have

$$\sum_{n=1}^{N} |\alpha(n)| \ll N g(N) \tag{3.3}$$

for certain non-decreasing function $g(N)$ such that $1 \leqslant g(N) \ll N^{\varepsilon}$ for every $\varepsilon > 0$.

(v) The series

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n}$$

converges.

Let $\{x\}$ denote the fractional part of a real number x, and let $\mathbf{s}(x)$ be the saw-tooth function

$$\mathbf{s}(x) = \begin{cases} \frac{1}{2} - \{x\} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{otherwise}. \end{cases} \tag{3.4}$$

For $\alpha \in \mathcal{A}$ we write

$$f(x, \alpha) = \sum_{d=1}^{\infty} \frac{\alpha(d)}{d} \mathbf{s}\left(\frac{x}{d}\right), \tag{3.5}$$

where $\mathbf{s}(x)$ is defined in (3.4) and

$$R(x, \alpha) = \sup_{y \geqslant x} \left| \sum_{n > y} \frac{\alpha(n)}{n} \right| \quad (x \geqslant 1), \tag{3.6}$$

$$R^*(x, \alpha) = \sqrt{R(\sqrt{x}, \alpha)} + \frac{1}{x}. \tag{3.7}$$

It is evident that $R^*(x, \alpha)$ is positive, monotonic and according to (v) $R^*(x, \alpha) \to 0$ as $x \to \infty$. Moreover, we put

$$\rho(x, \alpha) = R^*\left(\frac{x}{g(x)}, \alpha\right) g(x) \tag{3.8}$$

and for a positive integer k and $\epsilon = \pm 1$ we write

$$\Phi_k(x, \alpha, \epsilon) = \sum_{\substack{p \leqslant x \\ p \equiv \epsilon \ (\mathrm{mod}\ k) \\ |\xi_p| \leqslant \pi/2}} \frac{|\alpha(p)|}{p} \cos \xi_p,$$

where

$$\xi_p = \xi_p(\alpha, \epsilon) = \arg\bigl(\epsilon \alpha(p)\bigr) \quad (-\pi < \xi_p \leqslant \pi). \tag{3.9}$$

Theorem 3.1. *Let $\alpha \in \mathcal{A}$ and let $b_0(x) = b_0(x, \alpha)$ and $b_1(x) = b_1(x, \alpha)$ be positive and monotonic functions satisfying the following inequalities*

$$R^*\left(b_0(x) \frac{x}{g(x)}, \alpha\right) \leqslant b_0(x), \tag{3.10}$$

$$\frac{\rho(x, \alpha)}{b_0(x)} \leqslant b_1(x) \tag{3.11}$$

for sufficiently large x, where $\rho(x, \alpha)$ is as in (3.8). Suppose that $b_0(x) = o(1)$ and $b_1(x) = o(1)$ as $x \to \infty$. Then, for every integer $k > 2$ and arbitrary $\epsilon = \pm 1$ we have

$$f(x, \alpha) = \Omega\left(\exp\left(\Phi_k\left(\frac{\varphi(k)}{3} \log \frac{1}{b_1(x^{2/3}, \alpha)}, \alpha, \epsilon\right)\right)\right) \tag{3.12}$$

as $x \to \infty$.

4. Lemmas

Lemma 4.1. *Let $\alpha \in \mathcal{A}$. For $y \geqslant \max(xR^*(x,\alpha), \sqrt{x})$ we have*

$$f(x,\alpha) = \sum_{d \leqslant y} \frac{\alpha(d)}{d} s\left(\frac{x}{d}\right) + O\big(R^*(x,\alpha)\big).$$

Proof. This is Lemma 3.1 in [5] where it has been proven for $\alpha(n)$ belonging to an another class $\mathcal{A}$ of arithmetic functions, but the proof applies to our situation without any changes. □

Lemma 4.2. *Let b and $r > 0$ be relatively prime integers, and let β be a real number. Then, for any positive N,*

$$\sum_{n=1}^{N} s\left(n\frac{b}{r} + \beta\right) = \frac{N}{r} s(r\beta) + O(r).$$

Proof. This is Lemma 3 in [7]. Note that there is a misprint in (13) of [7], and "$\ll$" there should read as "$=$". □

Lemma 4.3. *Let $\alpha \in \mathcal{A}$ and let b_0 be as in Theorem 3.1. Then there exists a positive integer $q_0 = q_0(\alpha)$ such that for square-free q satisfying*

$$q_0 \leqslant q \leqslant \frac{1}{2} \min\left(\sqrt{N}, \frac{b_0(N)}{\rho(N,\alpha)}\right) \tag{4.1}$$

and arbitrary $0 < a < q$ we have

$$\sum_{n=1}^{N} f(nq+a) = c(q,a,\alpha)N + O\big(N^{\frac{3\theta+1}{4}}\tau(q) + b_0(N)N + N^{3/4}g(N)\big),$$

where $\tau(q)$ denotes the familiar divisor function and

$$c(q,a,\alpha) = \left(\sum_{e|q} \frac{\mathbf{s}(a/e)}{e} \sum_{f_1|e^\infty} \frac{\alpha(ef_1)}{f_1^2}\right) \sum_{(f_2,q)=1} \frac{\alpha(f_2)}{f_2^2}. \tag{4.2}$$

The notation $f_1|e^\infty$ means that all the prime divisors of f_1 divide e.

Proof. We can assume that N is sufficiently large since otherwise there is nothing to prove. We apply Lemma 4.1 with $y = b_0(N)N/g(N) + N^{3/4}$ and $x = nq + a$. Since $q \leqslant \sqrt{N}/2$ we have

$$\sqrt{nq+a} \leqslant \sqrt{(N+1)q} \leqslant \sqrt{\frac{1}{2}\sqrt{N}(N+1)} \leqslant N^{3/4} \leqslant y.$$

Moreover, if $nq + a \leqslant b_0(N)N/g(N)$ then

$$(nq+a)R^*(nq+a,\alpha) \leqslant b_0(N)\frac{N}{g(N)} \leqslant y$$

since $R^*(nq+a,\alpha) \leqslant 1$ if $q \geqslant q_0$. If $b_0(N)N/g(N) < nq + a \leqslant N/g(N)$ we have using (3.10)

$$(nq+a)R^*(nq+a,\alpha) \leqslant \frac{N}{g(N)} R^*\left(b_0(N)\frac{N}{g(N)}, \alpha\right) \leqslant b_0(N)\frac{N}{g(N)} \leqslant y.$$

Finally, if $nq + a > N/g(N)$, $n \leqslant N$, using (4.1) we have

$$\begin{aligned}(nq+a)R^*(nq+a,\alpha) &\leqslant (N+1)qR^*\left(\frac{N}{g(N)}, \alpha\right) \\ &\leqslant (N+1)\frac{b_0(N)}{2}\frac{1}{R^*(N/g(N),\alpha)g(N)} R^*\left(\frac{N}{g(N)}, \alpha\right) \\ &\leqslant b_0(N)\frac{N}{g(N)} \leqslant y.\end{aligned}$$

We see that in all cases

$$\max\big(\sqrt{nq+a},\,(nq+a)R^*(nq+a,\alpha)\big)\leqslant y$$

and we can apply Lemma 4.1. It yields

$$\sum_{n=1}^{N} f(nq+a)=\sum_{n=1}^{N}\sum_{d\leqslant y}\frac{\alpha(d)}{d}\mathbf{s}\left(\frac{nq+a}{d}\right)+O\left(\sum_{n=1}^{N}R^*(nq+a,\alpha)\right)$$
$$=\sum_{d\leqslant y}\frac{\alpha(d)}{d}\sum_{n=1}^{N}\mathbf{s}\left(n\frac{q/(d,q)}{d/(d,q)}+\frac{a}{d}\right)+O\big(b_0(N)N\big) \qquad (4.3)$$

since

$$\sum_{n=1}^{N}R^*(nq+a,\alpha)=\Bigg(\sum_{nq+a\leqslant b_0(N)N/g(N)}+\sum_{b_0(N)N/g(N)<nq+a\leqslant Nq+a}\Bigg)R^*(nq+a,\alpha)$$
$$\ll b_0(N)\frac{N}{g(N)}+NR^*\left(b_0(N)\frac{N}{g(N)},\alpha\right)\ll b_0(N)N.$$

Now we apply Lemma 4.2 with $b=q/(d,q)$, $r=d/(d,q)$ and $\beta=a/d$ to the inner sum in (4.3), and then use (3.3) getting

$$\sum_{n=1}^{N} f(nq+a)=\sum_{d\leqslant y}\frac{\alpha(d)}{d}\left(N\frac{(q,d)}{d}\mathbf{s}\left(\frac{a}{(q,d)}\right)+O(d)\right)+O\big(b_0(N)N\big)$$
$$=N\sum_{d\leqslant y}\frac{\alpha(d)}{d^2}(q,d)\mathbf{s}\left(\frac{a}{(q,d)}\right)+O\left(\sum_{n\leqslant y}|\alpha(d)|\right)+O\big(b_0(N)N\big)$$
$$=N\sum_{d\leqslant y}\frac{\alpha(d)}{d^2}(q,d)\mathbf{s}\left(\frac{a}{(q,d)}\right)+O\big(yg(y)\big)+O\big(b_0(N)N\big)$$
$$=N\sum_{d\leqslant y}\frac{\alpha(d)}{d^2}(q,d)\mathbf{s}\left(\frac{a}{(q,d)}\right)+O\big(N^{3/4}g(N)+b_0(N)N\big).$$

We replace the last sum by the series over all $d\geqslant 1$. Using (3.1) it is easy to see that this induces an error of size

$$\ll N\sum_{e|q}e\sum_{\substack{d>y\\(q,d)=e}}\frac{|\alpha(d)|}{d^2}\ll N\sum_{e|q}\frac{1}{e}\sum_{\substack{d>y/e\\(q,d)=1}}\frac{|\alpha(de)|}{d^2}$$
$$\ll N\sum_{e|q}e^{\theta-1}\sum_{d>y/e}\frac{1}{d^{2-\theta}}\ll Ny^{\theta-1}d(q)\ll N^{\frac{3\theta+1}{4}}d(q).$$

Since an easy computation shows that

$$\sum_{d=1}^{\infty}\frac{\alpha(d)}{d^2}(q,d)\mathbf{s}\left(\frac{a}{(q,d)}\right)=c(q,a,\alpha),$$

the proof is complete. □

Lemma 4.4. *Let $\alpha\in\mathcal{A}$. Then there exists a positive constant $b_1=b_1(\alpha)$ such that for every integer q we have*

$$\left|\sum_{(n,q)=1}\frac{\alpha(n)}{n^2}\right|\geqslant b_1.$$

Proof. This is Lemma 3.4 in [5]. As in the case of Lemma 4.1, it was proved in [5] for $\alpha(n)$ belonging to another class $\mathcal{A}$ of arithmetic functions, but the proof applies to our situation without any changes. □

5. Proof of Theorem 3.1

Let us fix θ' such that

$$0<\theta'<\frac{1-\theta}{2} \tag{5.1}$$

where θ is the exponent in (3.1). For every sufficiently large $x\geqslant 1$ let

$$q=q(x,\alpha,\epsilon)=\prod_{\substack{c(x)\leqslant p\leqslant x\\ |\xi_p|\leqslant\pi/2\\ |\alpha(p)|>p^{-\theta'}\\ p\equiv\epsilon\ (\mathrm{mod}\,k)}} p, \tag{5.2}$$

where ξ_p is defined in (3.9) and $c(x)$ is sufficiently large but bounded as $x\to\infty$ and chosen in such a way that $q\equiv 1 \pmod k$. Let

$$N:=\min\left\{m\geqslant 1\colon\ \frac{1}{2}\min\left(\frac{b_0(m)}{\rho(m,\alpha)},\sqrt{m}\right)\geqslant q\right\}, \tag{5.3}$$

where b_0 has the same meaning as in Lemma 4.3. Let us observe that N is well defined since $\rho(m,\alpha)=o(b_0(m))$ as $m\to\infty$. According to Lemma 4.3 for every $0<a<q$ we have

$$\sum_{n=1}^{N} f(nq+a,\alpha)=c(q,a,\alpha)N+o(N),$$

where $c(q,a,\alpha)$ is given by (4.2). Hence

$$\max_{1\leqslant x\leqslant(N+1)q}\bigl|f(x,\alpha)\bigr|\geqslant\bigl|c(q,a,\alpha)\bigr|+o(1). \tag{5.4}$$

To end the proof we show that with a proper choice of a, $|c(q,a,\alpha)|$ is large.

For $p|q$ we have using (3.1)

$$\begin{aligned}\sum_{l=0}^{\infty}\frac{\alpha(p^{l+1})}{p^{2l}}&=\alpha(p)+O\left(\sum_{l=1}^{\infty}\frac{|\alpha(p^{l+1})|}{p^{2l}}\right)\\&=\alpha(p)+O\bigl(p^{-2(1-\theta)}\bigr)\\&=\alpha(p)+\vartheta_p p^{-(1-\theta)}\end{aligned}$$

for certain $|\vartheta_p|\leqslant 1$ if p is large enough which is the case provided $c(x)$ in (5.2) is chosen sufficiently large. Since $|\alpha(p)|>p^{-\theta'}$ for $p|q$ and using (5.1) we obtain

$$\sum_{l=0}^{\infty}\frac{\alpha(p^{l+1})}{p^{2l}}=\alpha(p)\bigl(1+\vartheta'_p p^{-\theta'}\bigr)$$

for certain $|\vartheta'_p|\leqslant 1$. Hence, for $e|q$ we have

$$\begin{aligned}\sum_{f_1|e^\infty}\frac{\alpha(ef_1)}{f_1^2}&=\prod_{p|e}\sum_{l=0}^{\infty}\frac{\alpha(p^{l+1})}{p^{2l}}\\&=\alpha(e)\prod_{p|e}\bigl(1+\vartheta_p p^{-\theta'}\bigr).\end{aligned}$$

Thus, recalling Lemma 4.4, we have

$$\bigl|c(q,a,\alpha)\bigr|\gg\left|\sum_{e|q}\frac{\mathbf{s}(a/e)}{e}\alpha(e)\prod_{p|e}\bigl(1+\vartheta_p p^{-\theta'}\bigr)\right|. \tag{5.5}$$

Let now $a=q/k$. Then, since q is square-free, $q\equiv 1 \pmod k$ and $p\equiv\epsilon \pmod k$ for every prime $p|q$, we have

$$\mathbf{s}\left(\frac{a}{e}\right)=\epsilon^{\omega(e)}\left|\mathbf{s}\left(\frac{a}{e}\right)\right|=\epsilon^{\omega(e)}\left(\frac{1}{2}-\frac{1}{k}\right)$$

for every $e|q$. Consequently recalling (5.5) we have

$$\begin{aligned}\big|c(q,a,\alpha)\big| &\gg \left|\sum_{e|q}\frac{\epsilon^{\omega(e)}}{e}\alpha(e)\prod_{p|e}\big(1+\vartheta_p p^{-\theta'}\big)\right| \\ &= \prod_{p|q}\left|1+\frac{|\alpha(p)|e^{i\xi_p}}{p}\big(1+\vartheta_p p^{-\theta'}\big)\right| \\ &= \prod_{p|q}|c_p|,\end{aligned}$$

say. Easy computation shows that

$$|c_p|^2 = 1+2\frac{|\alpha(p)|}{p}\cos\xi_p + O\left(\frac{|\alpha(p)|}{p^{1+\theta'}}\right)$$

and thus

$$\begin{aligned}\log|c_p| &= \frac{1}{2}\log\left(1+2\frac{|\alpha(p)|}{p}\cos\xi_p + O\left(\frac{|\alpha(p)|}{p^{1+\theta'}}\right)\right) \\ &= \frac{|\alpha(p)|}{p}\cos\xi_p + O\left(\frac{|\alpha(p)|}{p^{1+\theta'}}\right) + O\left(\frac{|\alpha(p)|^2}{p^2}\right) \\ &= \frac{|\alpha(p)|}{p}\cos\xi_p + O\left(\frac{|\alpha(p)|}{p^{1+\theta'}}\right)\end{aligned}$$

because of (3.1) and (5.1). Since, using (3.3) it is easy to see that

$$\sum_{p|q}\frac{|\alpha(p)|}{p^{1+\theta'}} \leqslant \sum_{n=1}^{\infty}\frac{|\alpha(n)|}{n^{1+\theta'}} \ll 1,$$

we have

$$\begin{aligned}\big|c(a,q,\alpha)\big| &\gg \exp\left(\sum_{p|q}\frac{|\alpha(p)|}{p}\cos\xi_p\right) \\ &\gg \exp\big(\Phi_k(x,\alpha,\epsilon)\big). \end{aligned} \tag{5.6}$$

By the Prime Number Theorem in arithmetic progressions we have $q \leqslant \exp(\frac{4}{3\varphi(k)}x)$ for large x and thus, recalling (5.3), (3.11) and using

$$1/\rho(N-1,\alpha) \leqslant (N-1)/g(N-1)^2 \leqslant N-1,$$

we obtain

$$\begin{aligned}x &\geqslant \frac{3\varphi(k)}{4}\min\left(\log\frac{b_0(N-1)}{2\rho(N-1,\alpha)}, \log\frac{\sqrt{N-1}}{2}\right) \\ &\geqslant \frac{\varphi(k)}{3}\log\frac{b_0(N-1)}{\rho(N-1,\alpha)} \geqslant \frac{\varphi(k)}{3}\log\frac{1}{b_1(N-1)}\end{aligned} \tag{5.7}$$

if N is sufficiently large. On the other hand, we know that $b_1(t)$ is monotonic and

$$(N+1)q \leqslant \frac{1}{2}(N+1)\sqrt{N-1} \leqslant (N-1)^{3/2}$$

for $N \geqslant 5$. Thus taking into account (5.4), (5.6) and (5.7) we obtain (3.12), and the result follows.

6. Proof of Theorem 1.1

For a positive integer n we put

$$\alpha(n) = \mu(n)\prod_{p|n}\gamma(p), \tag{6.1}$$

where $\mu(n)$ denotes the familiar Möbius function and $\gamma(p)$ is defined in (1.2). We have

$$\left|\alpha(n)\right| \leqslant \tau_{d+1}(n) \tag{6.2}$$

where $\tau_{d+1}(n)$ denotes the well-known divisor function of order $d+1$ (see [3, Lemma 2.4]). For $\sigma > 1$ we have

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} = \frac{H(s)}{F(s)},$$

where $H(s)$ is a Dirichlet series which is absolutely convergent for $\sigma > 1/2$ (see [3, Lemma 2.3]). Thus applying the known complex integration method it is easy to see that

$$\sum_{n\leqslant x} \alpha(n) \ll x \exp(-c\sqrt{\log x}) \tag{6.3}$$

for certain positive $c = c(F)$. The proof is standard and we skip it. Using the above properties and also the fact that

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^2} = \prod_p \left(1 - \frac{\gamma(p)}{p^2}\right) = 2C(F) \neq 0,$$

it is easy to check that α belongs to the class $\mathcal{A}$ defined in Section 3, and that

$$R^*(x,\alpha) \ll \exp\left(-\frac{c}{4}\sqrt{\log x}\right) \tag{6.4}$$

and

$$g(x) = \log^d(3x).$$

Consequently,

$$\rho(x,\alpha) \ll \exp\left(-\frac{c}{5}\sqrt{\log x}\right). \tag{6.5}$$

By Lemma 2.5 from [3] we have

$$\sum_{n\leqslant x} \varphi(n, F) = \frac{x^2}{2} \sum_{n\leqslant x} \frac{\alpha(n)}{n^2} + x \sum_{n\leqslant x} \frac{\alpha(n)}{n} \mathbf{s}\left(\frac{x}{n}\right) - R(x,\alpha), \tag{6.6}$$

where

$$R(x,\alpha) = \frac{1}{2} \sum_{n\leqslant x} \alpha(n) \left\{\frac{x}{n}\right\}\left(1 - \left\{\frac{x}{n}\right\}\right). \tag{6.7}$$

We extend the range of summations in (6.6) to infinity. By (6.3) the induced error term is $\ll x\exp(-c\sqrt{\log x})$. Moreover, splitting the range of summation in (6.7) into the initial part $[1, [\sqrt{x}]]$ and $\ll \sqrt{x}$ disjoint subintervals $I \subset ([\sqrt{x}], x]$ where the function $n \mapsto \{\frac{x}{n}\}(1 - \{\frac{x}{n}\})$ is monotonic, estimating the sum over the initial segment trivially by

$$\sum_{n\leqslant\sqrt{x}} \left|\alpha(n)\right| \leqslant \sum_{n\leqslant\sqrt{x}} \tau_{d+1}(n) \ll \sqrt{x}\log^d x$$

and using partial summation combined with (6.3) in every subinterval I, we see that $R(x,\alpha) \ll x\exp(-c'\sqrt{\log x})$ for certain positive $c' < c$. Finally, we see that

$$E(x, F) = xf(x,\alpha) + O\left(xe^{-c'\sqrt{\log x}}\right),$$

where $f(x,\alpha)$ is defined in (3.5). We apply Theorem 3.1 with $b_0(x) = b_1(x) = \exp(-\frac{c}{11}\sqrt{\log x})$. Using (6.4) and (6.5) it is easy to verify (3.10) and (3.11) for x large enough. Hence, for arbitrary $\epsilon = \pm 1$ we have

$$\begin{aligned} E(x, F) &= \Omega\left(x\exp\left(\Phi_k\left(\frac{\varphi(k)}{3}\log\frac{1}{b_1(x^{2/3},\alpha)}, \alpha, \epsilon\right)\right)\right) \\ &= \Omega\left(x\exp\left(\Phi_k\left(C\varphi(k)\sqrt{\log x}, \alpha, \epsilon\right)\right)\right), \end{aligned} \tag{6.8}$$

where

$$C = \frac{c}{33}\sqrt{\frac{2}{3}}.$$

Observe that for primes p we have

$$\alpha(p) = -a_F(p) + O\left(\frac{1}{p}\right)$$

and due to (1.6), (3.9), (6.1) and (1.2) also

$$\xi_p(\alpha, \epsilon) = \xi_p(F, \epsilon).$$

Thus

$$\Phi_k(x, \alpha, \epsilon) = \Psi_k(x, F, \epsilon) + O(1).$$

Inserting this into (6.8) we obtain (1.8). The proof is complete.

7. Proof of Theorem 2.1

We apply Theorem 1.1 to $F(s) = \zeta(s + i\lambda)$ with parameters $\epsilon = -1$ and $k = 4$. The corresponding Ψ-function (see (1.7)) equals

$$\Psi(x, \lambda) = \sum_{\substack{p \leqslant x \\ p \equiv 3 \ (\mathrm{mod}\ 4)}} \frac{1}{p} \cos^+(\lambda \log p),$$

where, as usual, $\cos^+(t) = \max(0, \cos t)$ for all real t. We have

$$\Psi(x, \lambda) = \frac{1}{2} \sum_{p \leqslant x} \frac{1}{p} \cos^+(\lambda \log p) - \frac{1}{2} \sum_{p \leqslant x} \frac{\chi_4(p)}{p} \cos^+(\lambda \log p)$$
$$= \frac{1}{2} S_1 - \frac{1}{2} S_2, \tag{7.1}$$

say, where χ_4 denotes the unique non-principal Dirichlet character (mod 4). We have the following Fourier expansion

$$\cos^+ x = \sum_{\nu \in \mathbb{Z}} a(\nu) e^{i\nu x}, \tag{7.2}$$

where

$$a(\nu) = \begin{cases} \frac{1}{4} & \text{if } |\nu| = 1, \\ \frac{(-1)^{(\nu/2)+1}}{\pi} \frac{1}{\nu^2 - 1} & \text{if } 2|\nu, \\ 0 & \text{otherwise}. \end{cases} \tag{7.3}$$

In particular we have $a(0) = 1/\pi$.

Lemma 7.1. *For every positive constant c_0 and any real number $|\lambda| \geqslant c_0$ we have*

$$\sum_{p \leqslant x} \frac{1}{p^{1+i\lambda}} \ll \log\log\bigl(|\lambda| + 10\bigr),$$

where the implied constant depends only on c_0. For a non-principal Dirichlet character χ we have

$$\sum_{p \leqslant x} \frac{\chi(p)}{p^{1+i\lambda}} \ll \log\log\bigl(|\lambda| + 10\bigr)$$

uniformly for all real λ. The implied constant depends on χ.

This can be proved using basic analytic properties of Dirichlet L-functions by a standard application of the complex integration method (along the lines of the proof of Satz 5.3 in [10] for instance). We skip details.

Inserting (7.2) into (7.1) and using (7.3) together with Lemma 7.1 we obtain

$$S_1 = \sum_{\nu \in \mathbb{Z}} a(\nu) \sum_{p \leqslant x} \frac{1}{p^{1-i\nu\lambda}}$$
$$= \frac{1}{\pi} \sum_{p \leqslant x} \frac{1}{p} + O_\lambda\left(\sum_{\nu \geqslant 1} \frac{1}{\nu^2} \log\log\bigl(\nu|\lambda| + 10\bigr)\right)$$
$$= \frac{1}{\pi} \log\log x + O_\lambda(1),$$

and similarly $S_2 \ll_\lambda 1$. Thus

$$\Psi(x,\lambda) = \frac{1}{2\pi}\log\log x + O_\lambda(1)$$

and the result follows from (1.8) after some trivial calculations.

8. Proof of Theorem 2.2

Let us adopt the following convention. For two Dirichlet characters $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ we write $\chi_1 = \chi_2$ when the corresponding induced characters mod $[q_1, q_2]$ are equal. Hence, if $\chi_1 = \chi_2$ the values $\chi_1(p)$ and $\chi_2(p)$ still can differ for a finite number of primes p.

Lemma 8.1. *For $k \in \{3,4\}$ let χ_k denote the unique non-principal Dirichlet character* $\pmod{k}$*. Then, for every Dirichlet character χ we have*

$$\chi^\nu \neq \chi_3 \quad \textit{for all integers } \nu \geqslant 1 \tag{8.1}$$

or

$$\chi^\nu \neq \chi_4 \quad \textit{for all integers } \nu \geqslant 1. \tag{8.2}$$

Proof. Suppose that $\chi^\nu = \chi_3$ and $\chi^\mu = \chi_4$ for certain positive integers ν and μ. Then, for $d = (\nu,\mu)$, $d = m\nu + n\mu$ we have

$$\chi^d = \chi_3^m \chi_4^n = \begin{cases} \chi_3 & \text{if } 2|n, \\ \chi_4 & \text{if } 2|m, \\ \chi_3\chi_4 & \text{if } 2\nmid nm. \end{cases} \tag{8.3}$$

Observe that the case $2|n$ and $2|m$ cannot occur since otherwise we would have $\chi^d = \chi_0$ and thus also $\chi_3 = (\chi^d)^{\nu/d} = \chi_0$, a contradiction. Suppose that $\chi^d = \chi_3$ Then

$$\chi_4 = \chi^\mu = \chi^{d\frac{\mu}{d}} = \chi_3^{\frac{\mu}{d}} = \begin{cases} \chi_3 & \text{if } 2\nmid \frac{\mu}{d}, \\ \chi_0 & \text{otherwise}. \end{cases}$$

Since $\chi_4 \neq \chi_0$ and $\chi_4 \neq \chi_3$ in both cases we obtain a contradiction. In a similar way we exclude other possibilities in (8.3) and the lemma follows. □

Now we can give a proof of Theorem 2.2. We apply Theorem 1.1 to $F(s) = L(s+i\lambda,\chi)$ with parameters $\epsilon = -1$ and $k \in \{3,4\}$, where k is chosen in such a way that (8.1) or (8.2) of Lemma 8.1 holds. The corresponding Ψ-function (see (1.7)) equals

$$\Psi_k(x,\chi,\lambda) = \sum_{\substack{p\leqslant x \\ p\equiv -1 \,(\mathrm{mod}\, k)}} \frac{1}{p}\cos^+\bigl(\xi_p(\chi)\bigr),$$

where

$$e^{i\xi_p(\chi)} = \frac{\chi(p)}{p^{i\lambda}}.$$

We proceed as in the proof of Theorem 2.1. Using Fourier expansion (7.2) we obtain

$$\begin{aligned}\Psi_k(x,\chi,\lambda) &= \sum_{\nu\in\mathbb{Z}} a(\nu) \sum_{\substack{p\leqslant x \\ p\equiv -1 \,(\mathrm{mod}\, k)}} \frac{\chi^\nu(p)}{p^{1+i\nu\lambda}} \\ &= \frac{1}{2}\sum_{\nu\in\mathbb{Z}} a(\nu)\sum_{p\leqslant x}\frac{\chi^\nu(p)}{p^{1+i\nu\lambda}} - \frac{1}{2}\sum_{\nu\in\mathbb{Z}} a(\nu)\sum_{p\leqslant x}\frac{\chi^\nu(p)\chi_k(p)}{p^{1+i\nu\lambda}}.\end{aligned}$$

Since $\chi^\nu\chi_k \neq \chi_0$ for all $\nu \geqslant 1$, arguing as in the proof of Theorem 2.1 we see that the second summand contributes at most $O(1)$ in both cases $\lambda \neq 0$ and $\lambda = 0$. Hence

$$\Psi_k(x,\chi,\lambda) = \frac{1}{2}\sum_{\nu\in\mathbb{Z}} a(\nu)\sum_{p\leqslant x}\frac{\chi^\nu(p)}{p^{1+i\nu\lambda}} + O(1). \tag{8.4}$$

Let $\lambda \neq 0$. Using Lemma 7.1 again we see that the total contribution of all summands with $\nu \neq 0$ is $O(1)$ and hence

$$\Psi_k(x,\chi,\lambda) = \frac{1}{2}a(0)\sum_{p\leqslant x}\frac{1}{p} + O(1) = \frac{1}{2\pi}\log\log x + O(1).$$

A direct application of Theorem 1.1 yields (2.2) and the proof in the case $\lambda \neq 0$ is complete.

Let now $\lambda = 0$. In this case also other summands in (8.4) can provide a non-trivial contribution. They are exactly these with $h|\nu$. We have

$$\Psi_k(x,\chi,0) = A(\chi)\log\log x + O(1),$$

where

$$A(\chi) = \frac{1}{2}\sum_{\substack{\nu\in\mathbb{Z}\\ h|\nu}} a(\nu) = \frac{1}{2\pi} + \sum_{\substack{\nu\geqslant 1\\ h|\nu}} a(\nu). \tag{8.5}$$

Now an application of Theorem 1.1 gives

$$E(x,\chi,0) = \Omega\big(x(\log\log x)^{A(\chi)}\big),$$

and to conclude the proof it suffices to show that $A(\chi) = \eta(\chi)$, see (2.1).

Case 1: $2\nmid h$. Suppose in addition that $h>1$. Then, using (8.5) and (7.3) we obtain

$$A(\chi) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{\nu\geqslant 1}\frac{(-1)^\nu}{1-4h^2\nu^2} = \frac{1}{2\pi}\left(1 + 4\sum_{\nu=1}^{\infty}\frac{1}{1-16h^2\nu^2} - 2\sum_{\nu=1}^{\infty}\frac{1}{1-4h^2\nu^2}\right).$$

Using the following well-known formula valid for complex $z \notin \pi\mathbb{Z}$

$$\cot(z) = \frac{1}{z} + 2z\sum_{\nu=1}^{\infty}\frac{1}{z^2-\nu^2\pi^2},$$

we see after some simple computations that

$$A(\chi) = \frac{1}{4h}\left(\cot\left(\frac{\pi}{4h}\right) - \cot\left(\frac{\pi}{2h}\right)\right) = \frac{1}{4h\sin(\pi/(2h))}.$$

This proves the second case in (2.1). When $h=1$ we have one additional term corresponding to $\nu=1$. Then

$$A(\chi) = \frac{1}{4\sin(\pi/2)} + a(1) = \frac{1}{2},$$

and the first case in (2.1) follows as well.

Case 2: $2\|h$. Now we have

$$A(\chi) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{1-h^2\nu^2}$$

and this is the same sum as in Case 1 but with h replaced by $h/2$. Hence

$$A(\chi) = \frac{1}{2h\sin(\pi/h)},$$

and the third case in (2.1) follows.

Case 3: $4|h$. Now

$$A(\chi) = \frac{1}{2\pi}\left(1 + 2\sum_{\nu=1}^{\infty}\frac{1}{1-h^2\nu^2}\right) = \frac{1}{2h}\cot\frac{\pi}{h},$$

and the proof is complete.

9. Proof of Theorem 2.3

We shall use the following auxiliary result.

Lemma 9.1. *For a cusp form ϕ we have*

$$\sum_{p\leqslant x}\frac{|\lambda_\phi(p)|^2}{p}=\log\log x+O_\phi(1), \tag{9.1}$$

and

$$\sum_{p\leqslant x}\frac{\lambda_\phi(p)}{p^{1+i\lambda}}\ll_\phi \log\log\bigl(|\lambda|+10\bigr), \tag{9.2}$$

$$\sum_{p\leqslant x}\frac{|\lambda_\phi(p)|^2\chi_4(p)}{p^{1+i\lambda}}\ll_\phi \log\log\bigl(|\lambda|+10\bigr) \tag{9.3}$$

uniformly for all real λ. Moreover, for every fixed positive b_0 we have

$$\sum_{p\leqslant x}\frac{|\lambda_\phi(p)|^2}{p^{1+i\lambda}}\ll_\phi \log\log\bigl(|\lambda|+10\bigr), \tag{9.4}$$

uniformly for $|\lambda|\geqslant b_0$.

We skip the proof for the same reasons as in the case of Lemma 7.1. Required analytic properties of $L(s,\phi)$, $L(s,\phi\otimes\chi_4)$, $L(s,\phi\otimes\overline{\phi})$ and $L(s,\phi\otimes\overline{\phi}\otimes\chi_4)$ are either easy to establish or are explicitly stated in [1, Chapter 5].

In order to prove Theorem 2.3 we apply Theorem 1.1 with $(\epsilon,k)=(-1,4)$. Using (2.3) and (2.4) we trivially check that

$$\left|1-\frac{\lambda_\phi(p)}{p^{2+i\lambda}}+\frac{\alpha(p)\beta(p)}{p^{3+2i\lambda}}\right|\geqslant 1-\frac{2}{p^2}-\frac{1}{p^3}>0$$

for all primes p and real numbers λ. Thus, recalling (2.5) we have $C(\phi,\lambda)\neq 0$, and Theorem 1.1 can be applied.

In our case the corresponding Ψ-function equals

$$\Psi(x,\phi,\lambda)=\sum_{\substack{p\leqslant x\\ p\equiv -1\ (\mathrm{mod}\ 4)}}\frac{|\lambda_\phi(p)|}{p}\cos^+\bigl(\xi_p(\phi,\lambda)\bigr), \tag{9.5}$$

where

$$e^{i\xi_p(\phi,\lambda)}=\frac{\operatorname{sgn}\lambda_\phi(p)}{p^{i\lambda}}.$$

We use the following trivial identity $\cos^+(t)=\frac{1}{2}\cos(t)+\frac{1}{2}|\cos(t)|$ and split the sum on the right hand side of (9.5) into two summands $\Psi_1+\Psi_2$, say. We have

$$\Psi_1=\frac{1}{4}\sum_{\substack{p\leqslant x\\ p\equiv -1\ (\mathrm{mod}\ 4)}}\frac{\lambda_\phi(p)}{p^{1+i\lambda}}+\frac{1}{4}\sum_{\substack{p\leqslant x\\ p\equiv -1\ (\mathrm{mod}\ 4)}}\frac{\lambda_\phi(p)}{p^{1-i\lambda}}$$
$$=\frac{1}{4}\Re\left\{\sum_{p\leqslant x}\frac{\lambda_\phi(p)}{p^{1+i\lambda}}+\sum_{p\leqslant x}\frac{\lambda_\phi(p)\chi_4(p)}{p^{1+i\lambda}}\right\},$$

where, as before, χ_4 denotes the non-principal Dirichlet character (mod 4). According to (9.2) applied to ϕ and $\phi\otimes\chi_4$ we see that $\Psi_1=O(1)$. Now we estimate Ψ_2. Recalling (2.4) and using

$$\bigl|\cos(x)\bigr|=\sum_{2|\nu}a(\nu)e^{i\nu x},$$

where coefficients $a(\nu)$ are the same as in (7.3), we have

$$\begin{aligned}\Psi_2 &\geqslant \frac{1}{4}\sum_{\substack{p\leqslant x\\ p\equiv -1 \ (\mathrm{mod}\ 4)}} \frac{|\lambda_\phi(p)|^2}{p}\left|\cos\bigl(\xi_p(\phi,\lambda)\bigr)\right| \\ &= \frac{1}{4}\sum_{2|\nu} a(\nu) \sum_{\substack{p\leqslant x\\ p\equiv -1 \ (\mathrm{mod}\ 4)}} \frac{|\lambda_\phi(p)|^2(\operatorname{sgn}\lambda_\phi(p))^\nu}{p^{1+i\nu\lambda}} \\ &= \frac{1}{8\pi}\sum_{p\leqslant x}\frac{|\lambda_\phi(p)|^2}{p} + \frac{1}{8\pi}\sum_{p\leqslant x}\frac{|\lambda_\phi(p)|^2\chi_4(p)}{p} + \frac{1}{4}\sum_{2|\nu\neq 0} a(\nu)\left(\sum_{p\leqslant x}\frac{|\lambda_\phi(p)|^2}{p^{1+i\nu\lambda}} + \sum_{p\leqslant x}\frac{|\lambda_\phi(p)|^2\chi_4(p)}{p^{1+i\nu\lambda}}\right).\end{aligned}$$

Formula (9.1) shows that the first sum is $\log\log x + O(1)$. Using (9.2)–(9.4) we see that all the remaining sums contribute $O(1)$. Thus

$$\Psi(x,\phi,\lambda) \geqslant \frac{1}{8\pi}\log\log x + O(1),$$

and the result follows.

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