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ON NONMEASURABLE FUNCTIONS OF TWO VARIABLES AND ITERATED INTEGRALS

ALEXANDER KHARAZISHVILI

Abstract. Following the paper of Pkhakadze [7], we consider some properties of real-valued functions of two variables, which are not assumed to be measurable with respect to the two-dimensional Lebesgue measure on the plane $\mathbf{R}^2$, but for which the corresponding iterated integrals exist and are equal to each other. Close connections of these properties with certain set-theoretical axioms are emphasized.

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According to the classical Fubini theorem, if a given function $f : [0,1]^2 \to \mathbf{R}$ of two real variables is bounded and Lebesgue measurable, then there exist the corresponding iterated integrals and the equality

$$\int_0^1 \left(\int_0^1 f(x,y)dx \right) dy = \int_0^1 \left(\int_0^1 f(x,y)dy \right) dx$$

holds true. In fact, both sides of this equality are identical with the double Lebesgue integral $\int_0^1 \int_0^1 f(x,y)dxdy$.

On the other hand, it is also well known that for these iterated integrals to exist and coincide the Lebesgue measurability of f is not necessary.

Indeed, Sierpiński [9] was able to construct, by using the method of transfinite recursion, an injective function $\phi : [0,1] \to [0,1]$ whose graph $Gr(\phi)$ is thick in $[0,1]^2$ with respect to the two-dimensional Lebesgue measure λ_2. This means that $Gr(\phi)$ meets every Borel subset of $[0,1]^2$ with strictly positive λ_2-measure (notice that similar constructions are presented, e.g., in [4] and [6]). Now, denoting by g the characteristic function of $Gr(\phi)$, we see that g is not λ_2-measurable, but its iterated integrals do exist and both of them are equal to zero (hence they are equal to each other).

At the same time, assuming the Continuum Hypothesis, Sierpiński [8] constructed a subset S of $[0,1]^2$ satisfying the following relations:

(i) for every $x \in [0,1]$, the set $S \cap (\{x\} \times [0,1])$ is at most countable;

(ii) for every $y \in [0,1]$, the set $([0,1]^2 \setminus S) \cap ([0,1] \times \{y\})$ is at most countable;

Moreover, Sierpiński established in [8] that the existence of a set S with the properties (i) and (ii) is equivalent to the Continuum Hypothesis. Thus the Continuum Hypothesis is equivalent to the possibility of decomposing the unit

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square $[0,1]^2$ into two sets S and $S' = [0,1]^2 \setminus S$ such that S meets each vertical line of the plane in countably many points and S' meets every horizontal line of the plane in countably many points. This remarkable decomposition of the square implies numerous nontrivial consequences (see [5] and [10]). For example, an analogous decomposition of the product set $\omega_1 \times \omega_1$, where ω_1 stands for the least uncountable cardinal, yields the classical result of Ulam [11] stating the non-real-valued measurability of ω_1.

Remark 1. In a certain sense, one can say that both sets S and S' of Sierpiński's decomposition are absolutely nonmeasurable with respect to the class of all those nonzero σ-finite continuous measures on $[0,1]^2$ for which the assertion of the classical Fubini theorem is valid. In this context, let us especially stress that there are many measures on $\mathbf{R}^2$ strictly extending λ_2, for which the Fubini theorem remains true.

Now, denoting by h the characteristic function of S, we can readily verify that h admits both iterated integrals, but one of them is 0 and the other one is 1. This fact indicates that, in general, the existence of the iterated integrals does not imply their coincidence.

In view of these two important circumstances, it makes sense to consider the class F of all those functions $f : [0,1]^2 \to \mathbf{R}$, for which both iterated integrals exist and are equal to each other. Clearly, F is a vector space over the field of all real numbers. Of course, the study of F includes the investigation of those subsets Z of $[0,1]^2$, whose characteristic function χ_Z belongs to F, i.e., for χ_Z we have the equality

$$\int_0^1 \Big(\int_0^1 \chi_Z(x,y)dx \Big) dy = \int_0^1 \Big(\int_0^1 \chi_Z(x,y)dy \Big) dx. \qquad (*)$$

As far as we know, the first deep study of sets $Z \subset [0,1]^2$, possessing the above-mentioned property $(*)$, was carried out in the work of Pkhakadze [7]. Namely, he extensively investigated the descriptive structure of sets Z in terms of their intersections with the vertical and horizontal lines of the plane $\mathbf{R}^2$.

Let us denote by $\mathcal{L}$ the class of all sets $Z \subset [0,1]^2$ having the property $(*)$. It was pointed out in [7] that:

(a) $\mathcal{L}$ is a monotone class of sets, i.e., $\mathcal{L}$ is closed under unions of increasing sequences of its members and is closed under intersections of decreasing sequences of its members;

(b) $\mathcal{L}$ is closed under unions of disjoint countable families of its members;

(c) if $\{Z_1, Z_2\} \subset \mathcal{L}$ and $Z_1 \subset Z_2$, then $(Z_2 \setminus Z_1) \in \mathcal{L}$;

(d) if all intersections of $Z \subset [0,1]^2$ with the vertical and horizontal lines are open (respectively, closed) in those lines, then $Z \in \mathcal{L}$;

Moreover, it was established in [7] that, under the Continuum Hypothesis, there exists a set $Z' \subset [0,1]^2$, whose all vertical sections are open, all horizontal sections are of type G_δ, and both iterated integrals for $\chi_{Z'}$ exist but are not equal to each other (hence Z' does not belong to $\mathcal{L}$). It follows from (a) and (d) that the set Z', being of type G_δ with respect to all vertical and horizontal lines in $\mathbf{R}^2$, is not representable as the intersection of a sequence of sets which are open with respect to all vertical and horizontal lines. In this context, the equality between the iterated integrals, corresponding to the characteristic function of a subset Z of $[0,1]^2$, may be treated as a certain invariant of descriptive properties of Z.

The work [7] contains also a number of other interesting results which still deserve to be examined and discussed from various points of view and, first of all, from the point of view of set-theoretical approaches to qualitative problems of analysis (cf., for instance, [1], [4], [5], and [6]). In particular, the following question arises: is it true that $A \cup B \in \mathcal{L}$ whenever $A \in \mathcal{L}$ and $B \in \mathcal{L}$? Surprisingly, this natural question was not considered in [7]. Nevertheless, the approach suggested and elaborated in [7] can be successfully applied in order to give an answer to the question. We are going to show here that, under the same Continuum Hypothesis (or, more generally, under Martin's Axiom), this question admits a negative solution.

Our starting point is again Sierpiński's above-mentioned decomposition of the square $[0,1]^2$. Let λ $(= \lambda_1)$ denote the Lebesgue measure on the real line $\mathbf{R}$. We will need two simple auxiliary propositions.

Lemma 1. *Assume the Continuum Hypothesis. Then, for any real number* $r \in [0, 1/2[$, *there exist two subsets* C_1 *and* C_2 *of* $[0,1]^2$ *such that:*

(1) *the set* $C_1(x) = \{y \in [0,1] : (x,y) \in C_1\}$ *is at most countable for every* $x \in [0,1]$;
(2) *the set* $C_2(x) = \{y \in [0,1] : (x,y) \in C_2\}$ *is at most countable for every* $x \in [0,1]$;
(3) *the set* $C_1(y) = \{x \in [0,1] : (x,y) \in C_1\}$ *is closed in* $\mathbf{R}$ *for any* $y \in [0,1]$ *and* $\lambda(C_1(y)) = r$;
(4) *the set* $C_2(y) = \{x \in [0,1] : (x,y) \in C_2\}$ *is closed in* $\mathbf{R}$ *for any* $y \in [0,1]$ *and* $\lambda(C_2(y)) = r$;
(5) $C_1(y) \cap C_2(y) = \varnothing$ *for each* $y \in [0,1]$.

Proof. Consider Sierpiński's set $S \subset [0,1]^2$. According to the definition of S, for every $y \in [0,1]$, the set $S \cap ([0,1] \times \{y\})$ is co-countable, so

$$\lambda(S \cap ([0,1] \times \{y\})) = 1.$$

Consequently, there exist two closed sets $C_1(y) \subset pr_1(S \cap ([0,1] \times \{y\}))$ and $C_2(y) \subset pr_1(S \cap ([0,1] \times \{y\}))$ satisfying the relations

$$C_1(y) \cap C_2(y) = \varnothing,\ \lambda(C_1(y)) = \lambda(C_2(y)) = r.$$

Now, let us define

$$C_1 = \cup\{C_1(y) \times \{y\} : y \in [0,1]\}, \quad C_2 = \cup\{C_2(y) \times \{y\} : y \in [0,1]\}.$$

Then it is obvious that the relations (3), (4), and (5) are satisfied. Besides, from the definition of the sets C_1 and C_2, we have

$$C_1 \subset S, \quad C_2 \subset S,$$

from which it immediately follows that the relations (1) and (2) are also fulfilled. This completes the proof. □

Lemma 2. *Assume the Continuum Hypothesis. Then, for any real number* $r \in [0,1[$*, there exists a subset* D *of* $[0,1]^2$ *such that:*

(1) *the set* $D(y) = \{x \in [0,1] : (x,y) \in D\}$ *is at most countable for every* $y \in [0,1]$*;*
(2) *the set* $D(x) = \{y \in [0,1] : (x,y) \in D\}$ *is closed in* $\mathbf{R}$ *for every* $x \in [0,1]$ *and* $\lambda(D(x)) = r$*.*

The proof of this proposition is very similar to the proof of Lemma 1, so it is omitted here.

Theorem 1. *Under the Continuum Hypothesis, there exist two subsets* A *and* B *of the square* $[0,1]^2$*, both belonging to* $\mathcal{L}$ *but such that their union* $A \cup B$ *does not belong to* $\mathcal{L}$*.*

Proof. Fix a real number $r \in]0,1/2[$. Let C_1 and C_2 be as in Lemma 1, and let D be as in Lemma 2. We put

$$A = C_1 \cup D, \quad B = C_2 \cup D$$

and we are going to show that A and B are the required sets. Indeed, for each $x \in [0,1]$ we have

$$\mathrm{card}(C_1(x)) \leq \omega, \quad \mathrm{card}(C_2(x)) \leq \omega,$$

the set $D(x)$ is closed and $\lambda(D(x)) = r$. These relations imply that

$$\int_0^1 \Big(\int_0^1 \chi_A(x,y)dy\Big)dx = \int_0^1 \Big(\int_0^1 \chi_B(x,y)dy\Big)dx = r.$$

At the same time, for each $y \in [0,1]$ we have $\mathrm{card}(D(y)) \leq \omega$, the sets $C_1(y)$ and $C_2(y)$ are closed, and the equalities

$$\lambda(C_1(y)) = \lambda(C_2(y)) = r$$

are satisfied. Obviously, these relations imply that

$$\int_0^1 \Big(\int_0^1 \chi_A(x,y)dx\Big)dy = \int_0^1 \Big(\int_0^1 \chi_B(x,y)dx\Big)dy = r$$

and, consequently, both sets A and B belong to the class $\mathcal{L}$.

Now, consider the union $A \cup B$. For each $x \in [0,1]$ we may write

$$(A \cup B)(x) = C_1(x) \cup C_2(x) \cup D(x).$$

Since $C_1(x)$ and $C_2(x)$ are at most countable, $D(x)$ is closed and $\lambda(D(x)) = r$, we readily deduce

$$\int_0^1 \Big(\int_0^1 \chi_{A\cup B}(x,y)dy\Big)dx = r.$$

Further, for each $y \in [0,1]$ we may write

$$(A \cup B)(y) = C_1(y) \cup C_2(y) \cup D(y), \quad C_1(y) \cap C_2(y) = \varnothing.$$

Since $D(y)$ is at most countable, $C_1(y)$ and $C_2(y)$ are closed and $\lambda(C_1(y)) = \lambda(C_2(y)) = r$, we see that

$$\lambda((A \cup B)(y)) = \lambda(C_1(y) \cup C_2(y)) = \lambda(C_1(y)) + \lambda(C_2(y)) = 2r,$$

whence it follows that

$$\int_0^1 \Big(\int_0^1 \chi_{A\cup B}(x,y)dx \Big) dy = 2r.$$

This immediately implies that

$$\int_0^1 \Big(\int_0^1 \chi_{A\cup B}(x,y)dy \Big) dx \neq \int_0^1 \Big(\int_0^1 \chi_{A\cup B}(x,y)dx \Big) dy,$$

i.e., $(A \cup B) \notin \mathcal{L}$. Theorem 1 has thus been proved. □

Recall that F denotes the class of all those functions $f : [0,1]^2 \to \mathbf{R}$, which satisfy the relation $(*)$. If $f \in F$ and $g \in F$, then $f + g \in F$. In this context, the next statement is of interest.

Theorem 2. *Under the Continuum Hypothesis, there exist two functions h_1 and h_2 belonging to F such that:*
(1) *$h_1 \cdot h_2$ does not belong to F;*
(2) *$min(h_1, h_2)$ does not belong to F.*

Proof. Let A and B be as in Theorem 1. Consider the sets

$$A' = [0,1]^2 \setminus A, \ B' = [0,1]^2 \setminus B.$$

It can be easily verified that $A' \in \mathcal{L}$, $B' \in \mathcal{L}$ and $A' \cap B' \notin \mathcal{L}$ (for this purpose, it suffices to apply Theorem 1 and relation (c) indicated at the beginning of the paper). Let us put:
h_1 = the characteristic function of A';
h_2 = the characteristic function of B'.
Then we have the equalities

$$h_1 \cdot h_2 = \min(h_1, h_2) = \chi_{A' \cap B'},$$

which yield at once the desired result.
By the way, we get $(h_1 + h_2) \in F$ but $(h_1 + h_2)^2 \notin F$. □

Remark 2. As already mentioned, Theorems 1 and 2 are also valid under Martin's Axiom which is significantly weaker than the Continuum Hypothesis. Assuming **MA** instead of **CH**, the proofs of Theorems 1 and 2 do not essentially change (actually, the argument remains almost the same). In addition, it should be noticed that none of these two theorems can be proved within **ZFC** theory. This follows from the fact that in [3] a model of set theory is indicated, in which, for every function $f : [0,1]^2 \to [0,1]$, the existence of iterated integrals for f

necessarily implies the equality between them. Clearly, in such a model the negation of the Continuum Hypothesis holds true. In this context, see also [2] where certain axioms of symmetry are discussed, which turn out to be closely connected with the negation of the Continuum Hypothesis and with iterated integrals.

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Author's addresses:

A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia
and
I. Vekua Inst. Appl. Math.
2, University St., Tbilisi 0186
Georgia
E-mail: kharaz2@yahoo.com