K-theory and derived equivalences

(Joint work with D. Dugger; S. Schwede)

Main Theorem:
If R and S are two derived equivalent rings, then $K_*(R) \cong K_*(S)$.

Neeman proved the above result for *regular rings*.

Definitions:

Ch_R : $\mathbb{Z}$-graded complexes of R-modules

$$\mathcal{H}o(\mathrm{Ch}_R) \quad = \quad \mathrm{Ch}_R[\text{q-iso}^{-1}] \quad \cong_\Delta \quad \mathcal{D}_R$$

R and S are *derived equivalent* if $\mathcal{D}_R$ and $\mathcal{D}_S$ are equivalent as triangulated categories.

$$\mathcal{D}_R \cong_\Delta \mathcal{D}_S$$

Examples:

- Morita equivalences such as R and $M_n(R)$.

- Representation theory of finite groups provides other non-Morita equivalent examples.

• For k a field the following two subalgebras of $M_3(k)$ are derived equivalent:

$$R = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \text{ and } S = \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix} \text{ for } * \in k.$$

Projectives in Mod-R: $e_{ii}R = P_i$

$$P_3 = (0\ 0\ *) \longrightarrow P_2 = (0\ \ *\ *) \longrightarrow P_1 = (*\ *\ *)$$

Indecomposables in Mod-R:

Mod-R and Mod-S are not equivalent:

Indecomposables in Mod-S:

R and S are *hereditary*, $\mathrm{Ext}^2 = 0$.
Thus $X \simeq_{\text{q-iso}} H_*(X)$ for all X in Ch_R and Ch_S.

$\mathcal{D}_R \cong_\Delta \mathcal{D}_S$:

A *stronger* notion of equivalence:

Definition: Two model categories are *Quillen equivalent* $\mathcal{M} \simeq_Q \mathcal{N}$ if there is an adjoint pair of functors $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ such that

(1) L preserves cofibrations and R fibrations.

(2) The derived functors $\overline{L} : \mathcal{H}o(\mathcal{M}) \rightleftarrows \mathcal{H}o(\mathcal{N}) : \overline{R}$ induce an equivalence of categories.

Examples:

• There exist two DGAs A and B such that $\mathcal{D}_A \cong_\Delta \mathcal{D}_B$, but d.g.$A$-mod $\not\simeq_Q$ d.g.B-mod.

• $\mathcal{H}o(K(n)\text{-mod}) \cong_\Delta \mathcal{H}o(\text{d.g.}\mathbb{F}_p[v_n, v_n^{-1}]\text{-mod})$ but there is no underlying Quillen equivalence.

The *KEY* to our main result is the following:

Proposition 1: Ch_R and Ch_S are Quillen equivalent if and only if R and S are derived equivalent.

"Proposition 2:" "A Quillen equivalence $\mathcal{M} \simeq_Q \mathcal{N}$ between stable model categories induces a K-theory equivalence, $K(\mathcal{M}) \simeq K(\mathcal{N})$."

Classical Morita Theory

Given an equivalence $F : \text{Mod-}S \rightleftarrows \text{Mod-}R : G$, consider $F(S) = P$. It follows that:

(i) $\text{Hom}_R(P, P) \cong \text{Hom}_S(S, S) \cong S$
(ii) If $\text{Hom}_R(P, X) \cong 0$ then $X \cong 0$.
(Since $\text{Hom}_R(F(S), X) \cong \text{Hom}_S(S, G(X))$.)
P is a *strong generator*.

Morita Theory: The following are equivalent:
(1) Mod- R and Mod- S are equivalent categories.
(2) There exists a f. g. projective R-module P satisfying (i) and (ii).

Proof: (1) implies (2) follows from the above.

(2) implies (1), the equivalence is given by:

$$\text{Hom}_R(P, -) : \text{Mod-}R \rightleftarrows \text{Mod-}S : - \otimes_S P.$$

Check for $X = S$:

$$X \xrightarrow{\cong} Hom_R(P, X \otimes_S P).$$

and for $Y = P$:

$$\text{Hom}_R(P, Y) \otimes_S P \xrightarrow{\cong} Y$$

Homotopical Morita Theory

Given a Quillen equivalence $L : \mathrm{Ch}_S \rightleftarrows \mathrm{Ch}_R : R$, consider $L(S) = P$. It follows that:

(i) $\mathcal{D}_R(P, P)_* \cong S$, concentrated in degree 0.
(ii) If $\mathcal{D}_R(P, X) \cong 0$, then $X \cong 0$.
(iii) $\oplus_i \mathcal{D}_R(P, X_i) \cong \mathcal{D}_R(P, \oplus_i X_i)$.
(Since $\mathcal{D}_S(S, -)$ also has this property.)

(ii) and (iii) imply P is a *compact, weak generator*.

Lemma: (Bökstedt-Neeman) In $\mathrm{C}h_R$, the compact objects are the *perfect complexes*, those complexes quasi-isomorphic to bounded complexes of finitely generated projectives.

Proposition 1: The following are equivalent:
(1) $\mathrm{Ch}_R \simeq_Q \mathrm{Ch}_S$
(2) $\mathcal{D}_R \cong_\Delta \mathcal{D}_S$ (R and S are derived equivalent.)
(3) $(\mathcal{D}_R)_{\text{compact}} \cong_\Delta (\mathcal{D}_S)_{\text{compact}}$
(4) $K_b(\text{proj-}R) \cong_\Delta K_b(\text{proj-}S)$
(5) There is a bounded complex of finitely generated projectives P in Ch_R such that (i) and (ii) hold. P is a *tilting complex*.

Rickard showed (4) and (5) are equivalent.

Proposition 1: The following are equivalent:
(1) $\mathrm{Ch}_R \simeq_Q \mathrm{Ch}_S$
(2) $\mathcal{D}_R \cong_\Delta \mathcal{D}_S$ (R and S are derived equivalent.)
(5) There is a tilting complex P in Ch_R.

Proof: (1) $\longrightarrow$ (2) $\longrightarrow$ (3) = (4) $\longrightarrow$ (5) as above.

(5) $\longrightarrow$ (1):
Step 1: Let $\mathcal{E}(P) = \mathrm{Hom}_R(P, P)$.
There is a Quillen equivalence:

$$\mathrm{Hom}_R(P, -) : \mathrm{Ch}_R \rightleftarrows \text{Mod-}\,\mathcal{E}(P) : - \otimes_{\mathcal{E}(P)} P$$

Check for $X = \mathcal{E}(P)$:

$$X \xrightarrow{\cong} Hom_R(P, X \otimes_{\mathcal{E}(P)} P).$$

and for $Y = P$:

$$\mathrm{Hom}_R(P, Y) \otimes_{\mathcal{E}(P)} P \xrightarrow{\cong} Y$$

Step 2: $\mathcal{E}(P)$ is quasi-isomorphic to S (since $H_*\mathcal{E}(P) = S$). Hence,

$$\mathrm{Ch}_R \;\simeq_Q\; \text{Mod-}\,\mathcal{E}(P) \;\simeq_Q\; \mathrm{Ch}_S$$

Recall: $R = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$ and $S = \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix}$

Recall the indecomposables in Mod-R:

Claim: $T = P_1 \oplus P_2 \oplus P_2/P_3$ is a *tilting module.*

(i) $\mathrm{Hom}_R(T, T) \cong S =$

$$\begin{pmatrix} P_1 \to P_1 & P_2 \to P_1 & 0 \\ 0 & P_2 \to P_2 & 0 \\ 0 & P_2 \to P_2/P_3 & P_2/P_3 \to P_2/P_3 \end{pmatrix}$$

(ii) T is a weak generator:
There is a short exact sequence
$0 \to R \to P_1 \oplus P_2 \oplus P_2 \to 0 \oplus 0 \oplus P_2/P_3 \to 0.$
If $\mathcal{D}_R(T, X) \cong 0$, then $\mathcal{D}_R(R, X) \cong 0$ and $X \cong 0$.

(iii) T is compact:
T is quasi-isomorphic to $P_1 \oplus P_2 \oplus (P_3 \to P_2)$.

K-theory and Proposition 2

Definition:

(1) A *Waldhausen subcategory* $\mathcal{U}$ of a pointed model category $\mathcal{M}$ is a full subcategory of cofibrant objects containing $*$ such that if $A \rightarrowtail B$ and $A \to X$ are in $\mathcal{U}$ then $(A \coprod_B X)_{\text{in } \mathcal{M}}$ is also in $\mathcal{U}$.

(2) $\overline{\mathcal{U}}$ is the full subcategory of cofibrant objects in $\mathcal{M}$ weakly equivalent to an object in $\mathcal{U}$.

(3) If $\mathcal{U} \cong \overline{\mathcal{U}}$ then we say $\mathcal{U}$ is *complete*.

Main Example: $\mathcal{M} = \text{Ch}_R$ and $\mathcal{U}_R$ contains the bounded complexes of f. g. projectives.
Thus, $\overline{\mathcal{U}_R} = \mathcal{M}_{\text{compact, cofibrant}}$.

Lemma: Any Waldhausen subcategory is a "category of cofibrations and weak equivalences" in the sense of Waldhausen.

Definition: $K(\mathcal{U})$ is the Waldhausen K-theory of $\mathcal{U}$. (Using the S-dot construction.)
Note, $K(\overline{\mathcal{U}_R}) \cong K(R)$.

Proposition 2: Assume $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ is a Quillen equivalence and $\mathcal{U}$ is a complete Waldhausen subcategory of $\mathcal{M}$. Then

- $\mathcal{V} = \overline{L\mathcal{U}}$ is a complete Waldhausen subcategory of $\mathcal{N}$ and
- $L : K(\mathcal{U}) \xrightarrow{\simeq} K(\mathcal{V})$.

Corollary: If $L : \mathrm{Ch}_R \rightleftarrows \mathrm{Ch}_S : R$ is a Quillen equivalence, then $K(R) \xrightarrow{\simeq} K(S)$.

Proof: One can show that: $\overline{L(\overline{\mathcal{U}_R})} = \overline{\mathcal{U}_S}$. The forward inclusion follows since L preserves compact and cofibrant objects. For the other direction, if $X \in \overline{\mathcal{U}_S}$, then $\overline{R}(X) \in \overline{\mathcal{U}_R}$. Since $L(\overline{R}(X)) \simeq X$ it follows that $X \in \overline{L(\overline{\mathcal{U}_R})}$.

Proof of Main Theorem: Since R and S are derived equivalent if and only if they are Quillen equivalent, the main result that derived equivalence implies K-theory equivalence follows from the corollary.

Corollary: If $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ is a Quillen equivalence between stable model categories, then $K_c(\mathcal{M}) \xrightarrow{\simeq} K_c(\mathcal{N})$ (where K_c is the K-theory of the compact objects).

General result: Assume $\mathcal{A}$ and $\mathcal{B}$ are cocomplete abelian categories with sets of small projective *strong* generators.

(1) $\mathcal{A}$ and $\mathcal{B}$ are derived equivalent if and only if $\mathrm{Ch}_{\mathcal{A}}$ and $\mathrm{Ch}_{\mathcal{B}}$ are Quillen equivalent.

(2) If $\mathcal{A}$ and $\mathcal{B}$ are derived equivalent, then $K_c(\mathcal{A}) \simeq K_c(\mathcal{B})$ (where K_c is the K-theory of the compact objects.)

Neeman considered instead small abelian categories and the associated Quillen K-theory of exact categories.