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REPRESENTATIONS OF A LOOP LIE ALGEBRA ASSOCIATED WITH QUANTUM PLANE*

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Abstract In this article, some modules over a loop Lie algebra associated to quantum plane are constructed. The isomorphism classes among these modules are also determined.

Key words Loop algebra; quantum plane; module; isomorphism

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1 Introduction

Quantum plane is defined as the noncommutative associative algebra $\mathcal{L}_q := \mathbb{C}_q[t_1^{\pm 1}, t_2^{\pm 1}]$ of noncommutative Laurent polynomials over the complex field $\mathbb{C}$ with the relation $t_2t_1 = qt_1t_2$, where q is a fixed nonzero complex number. As well-known, quantum plane is a fundamental ingredient in quantum groups and is the main research object in many literatures [1–10, 15, 17]. Under the usual commutator, $\mathcal{L}_q$ is a Lie algebra, usually referred to as the q-Virasoro-like algebra [1, 2], as it is the q-analog of the Virasoro-like algebra. Because the Virasoro-like algebra is closely related to the Lie algebras of Block type and the generalized Virasoro algebras [7, 11-13, 16], it would be interesting to study the representations of the q-analogs of these Lie algebras which are closely related to $\mathcal{L}_q$. Two families of modules of quantized Weyl algebra $\mathcal{A}_{2,q}$ associated to quantum plane are constructed in [17]. Another three new families of modules of $\mathcal{A}_{2,q}$ and two families of modules of $\mathcal{A}_2$ are presented in [4]. Several families of modules of a noncommuntative associative algebra related to three order quantum torus are presented in [14]. Motivated by this, it is natural to study the representations of loop algebra $\mathcal{L}(\mathcal{G}_{\tilde{q}})$, where $\tilde{q}$ is another fixed nonzero complex number (see (1.2)). We will construct two families of modules of $\mathcal{L}(\mathcal{G}_{\tilde{q}})$ over noncommutative representation spaces and two families of modules of $\mathcal{L}(\mathcal{G}_{\tilde{q}})$ over commutative representation spaces. The modules presented in this article do not belong to the category of modules in [4, 14]. Some techniques used in construction are generalizations of

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those in [1, 4, 14, 15, 17], and they also provide new tools for the research in cryptology. Here, $\mathcal{G}_{\tilde{q}}$ and $\mathcal{L}(\mathcal{G}_{\tilde{q}})$ are defined as follows.

Definition 1.1 The associative algebra $\mathcal{G}_{\tilde{q}}$ is the algebra with the underlying space spanned by $\{x^\alpha,\ \partial_k |\ \alpha=(\alpha_1,\alpha_2)\in\mathbb{Z}^2\backslash\{0\},\ k=1,2\}$, satisfying following relations:

$$x^\alpha x^\beta=\tilde{q}^{\alpha_2\beta_1}x^{\alpha+\beta},\ \partial_i\partial_j=\partial_j\partial_i,\ \partial_i x^\alpha-x^\alpha\partial_i=\alpha_i x^\alpha, \forall\alpha,\beta\in\mathbb{Z}^2\backslash\{0\},\ i,j\in\{1,2\}, \tag{1.1}$$

Under the usual commutator, $\mathcal{G}_{\tilde{q}}$ is a Lie algebra, still denoted as $\mathcal{G}_{\tilde{q}}$. Here and below, we treat x^0 as 0.

$\mathcal{L}(\mathcal{G}_{\tilde{q}}):=\mathcal{L}\otimes\mathcal{G}_{\tilde{q}}$ is the loop algebra of $\mathcal{G}_{\tilde{q}}$, Lie bracket is defined as

$$[g(t_1,t_2)u, h(t_1,t_2)v]=g(t_1,t_2)h(t_1,t_2)[u,v], \tag{1.2}$$

where $g(t_1,t_2),h(t_1,t_2)\in\mathcal{L},\mathcal{L}:=\mathcal{L}_1,u,v\in\mathcal{G}_{\tilde{q}}$.

For convenience, set $\varepsilon_1=(1,0)$, $\varepsilon_2=(0,1)$.

2 $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-Modules Over Noncommutative Representation Spaces

2.1 Modules of type $\mathcal{L}_q(b_1,b_2)$

For $b_1:=b_1(t_1,t_2), b_2:=b_2(t_1,t_2)\in\mathcal{L}_q$, let $\mathcal{L}_q(b_1,b_2)$ be the vector space $\mathbb{C}_q[t_1^{\pm1},t_2^{\pm1}]$. Define an action of $\mathcal{L}(\mathcal{G}_{\tilde{q}})$ on $\mathcal{L}_q(b_1,b_2)$ by

$$g(t_1,t_2)x^\alpha\cdot f(t_1,t_2)=g(t_1,t_2)t_1^{\alpha_1}t_2^{\alpha_2}f(t_1,t_2), \tag{2.1}$$

$$g(t_1,t_2)\partial_i\cdot f=g(t_1,t_2)(b_i(t_1,t_2)f+t_i\frac{\partial}{\partial t_i}f),\ \ i=1,2, \tag{2.2}$$

where $g(t_1,t_2)\in\mathcal{L},\ f=f(t_1,t_2)\in\mathcal{L}_q,\alpha\in\mathbb{Z}^2\backslash\{0\}$.

Theorem 2.1 (1) $\mathcal{L}_q(b_1,b_2)$ is a $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module with the action defined in (2.1), (2.2) if and only if $\tilde{q}=q$, and if $q\neq1$, then, $b_1(t_1,t_2),b_2(t_1,t_2)\in\mathbb{C}$.

(2) As a $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module, $\mathcal{L}_q(b_1,b_2)$ is irreducible.

(3) As $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-modules, $\mathcal{L}_q(b_1,b_2)\cong\mathcal{L}_q(b_1',b_2')$ if and only if there exist $k_1,k_2\in\mathbb{Z}$ such that $b_1'-b_1=k_1,b_2'-b_2=k_2$.

(4) The automorphism group $\mathrm{Aut}\mathcal{L}_q(b_1,b_2)\cong\mathbb{C}^*$ (where $\mathbb{C}^*=\mathbb{C}\backslash\{0\}$).

Proof (1) The "if" part can be proved by straightforward check. Now, we prove the "only if" part. Suppose that $\mathcal{L}_q(b_1,b_2)$ is a $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module. Using $qt_1t_2=t_2t_1=x^{\varepsilon_2}x^{\varepsilon_1}\cdot1=\tilde{q}x^{\varepsilon_1}x^{\varepsilon_2}\cdot1=\tilde{q}t_1t_2$, we have

$$\tilde{q}=q.$$

From $b_1(t_1,t_2)t_2=\partial_1\cdot t_2=\partial_1x^{\varepsilon_2}\cdot1=x^{\varepsilon_2}\partial_1\cdot1=x^{\varepsilon_2}\cdot b_1=t_2b_1(t_1,t_2)$ and $q\neq1$, we have

$$b_1(t_1,t_2)\in\mathbb{C}[t_2^{\pm1}]. \tag{2.3}$$

Similarly, using $b_2(t_1,t_2)t_1=\partial_2\cdot t_1=\partial_2x^{\varepsilon_1}\cdot1=x^{\varepsilon_1}\partial_2\cdot1=x^{\varepsilon_1}\cdot b_2=t_1b_2(t_1,t_2)$ and $q\neq1$, we have

$$b_2(t_1,t_2)\in\mathbb{C}[t_1^{\pm1}]. \tag{2.4}$$

From $\partial_1\partial_2\cdot1=\partial_2\partial_1\cdot1$, we have

$$b_1(t_1,t_2)b_2(t_1,t_2)+t_1\frac{\partial}{\partial t_1}b_2(t_1,t_2)=b_2(t_1,t_2)b_1(t_1,t_2)+t_2\frac{\partial}{\partial t_2}b_1(t_1,t_2). \tag{2.5}$$

By comparing the coefficients in (2.5), we obtain

$$b_1 = b_1(t_1, t_2),\ b_2 = b_2(t_1, t_2) \in \mathbb{C}. \tag{2.6}$$

(2) Let M be any nonzero submodule of $\mathcal{L}_q(b_1, b_2)$. Take any $0 \neq f \in M$. Recall (2.1), $x^{m\varepsilon_1} \cdot x^{n\varepsilon_2} \cdot f = t_1^m t_2^n f$, $\forall m,\, n \in \mathbb{Z}$. Thus, replacing f by $t_1^m t_2^n \cdot f$ if necessary, we may suppose $0 \neq f \in \mathbb{C}[t_1, t_2]$. Note that $t_1 \frac{\partial}{\partial t_1} f = \partial_1 \cdot f - b_1 f \in M$ and $x^{-\varepsilon_1} \cdot (t_1 \frac{\partial}{\partial t_1} f) = \frac{\partial}{\partial t_1} f$. Then, replacing f by $\frac{\partial}{\partial t_1} f$ if necessary, we can suppose $f \in \mathbb{C}[t_2]$. Similarly, we may suppose $f \in \mathbb{C}^*$. Thus, $1 \in M$.

Note that $t_1^{\alpha_1} t_2^{\alpha_2} = x^{(\alpha_1, \alpha_2)} \cdot 1$ for any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$. Thus, $M = \mathcal{L}_q(b_1, b_2)$, and $\mathcal{L}_q(b_1, b_2)$ is irreducible.

(3) Suppose that $\phi : \mathcal{L}_q(b_1, b_2) \to \mathcal{L}_q(b_1', b_2')$ is a $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module isomorphism. Let $f \in \mathcal{L}_q(b_1, b_2)$, such that $\phi(f) = 1$. Then, using

$$\phi(b_1 f + t_1 \frac{\partial}{\partial t_1} f) = \phi(\partial_1 \cdot f) = \partial_1 \cdot \phi(f) = \partial_1 \cdot 1 = b_1' \phi(f) = \phi(b_1' f),$$

we obtain $b_1 f + t_1 \frac{\partial}{\partial t_1} f = b_1' f$, from which we have

$$f = c_1 t_1^{b_1' - b_1}, \quad c_1 \in \mathbb{C}[t_2^{\pm 1}].$$

Similarly, we can obtain $f = c_2 t_2^{b_2' - b_2}$, $c_2 \in \mathbb{C}[t_1^{\pm 1}]$. Thus,

$$f = c t_1^{b_1' - b_1} t_2^{b_2' - b_2}, \quad c \in \mathbb{C}^*.$$

Therefore, there exist integers k_1, k_2, such that $b_1' - b_1 = k_1, b_2' - b_2 = k_2$.

Conversely, assume that $b_1' - b_1 = k_1, b_2' - b_2 = k_2$ for some integers k_1, k_2. We define a linear map $\phi :\ \mathcal{L}_q(b_1, b_2) \to \mathcal{L}_q(b_1', b_2')$ by $\phi(f) = f t_1^{-k_1} t_2^{-k_2}$. It is checked that ϕ is a $\mathcal{L}_q(b_1, b_2)$-module isomorphism.

(4) Let $\varphi \in \mathrm{Aut}\mathcal{L}_q$, $\varphi(1) = h$, thus, $\varphi^{-1}(h) = 1$. Note that in this case $b_1 = b_1', b_2 = b_2'$. As the proof of (3), we can obtain $h \in \mathbb{C}^*$. From the irreducibility of $\mathcal{L}_q(b_1, b_2)$, we have $\varphi = h \cdot id$, where id is the identity map. Therefore, $\mathrm{Aut}\mathcal{L}_q(b_1, b_2) \cong \mathbb{C}^*$.

2.2 Modules of type $\mathcal{L}_{q^2}(b)$

For $b := b(t_1, t_2) \in \mathcal{L}_q$, we define a $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module $\mathcal{L}_{q^2}(b)$ on the space $\mathbb{C}_q[t_1^{\pm 1}, t_2^{\pm 1}]$ with the actions:

$$g(t_1, t_2) x^{\alpha} \cdot f(t_1, t_2) = g(t_1, t_2) t_1^{\alpha_1} t_2^{\alpha_2} f(q^{\alpha_2} t_1, t_2), \tag{2.7}$$

$$g(t_1, t_2) \partial_1 \cdot f = g(t_1, t_2) t_1 \frac{\partial}{\partial t_1} f, \tag{2.8}$$

$$g(t_1, t_2) \partial_2 \cdot f = g(t_1, t_2)(b f + t_2 \frac{\partial}{\partial t_2} f), \tag{2.9}$$

where $g(t_1, t_2) \in \mathcal{L}$, $f = f(t_1, t_2) \in \mathcal{L}_{q^2}(b), \alpha \in \mathbb{Z}^2 \backslash \{0\}$.

Theorem 2.2 (1) $\mathcal{L}_{q^2}(b)$ is a $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module with action defined by (2.7), (2.8), (2.9) if and only if $\tilde{q} = q^2$, and if $q = 1$, then, $b = b(t_2) \in \mathbb{C}[t_2^{\pm 1}]$; if $q \neq 1$, then, $b \in \mathbb{C}$.

(2) As a $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module, $\mathcal{L}_{q^2}(b)$ is irreducible.

(3) As $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-modules, $\mathcal{L}_{q^2}(b_1) \cong \mathcal{L}_{q^2}(b_2) (q \neq 0)$ if and only if there exists an integer k such that $b_2 = b_1 + k$.

(4) $\mathrm{Aut}\mathcal{L}_{q^2}(b) \cong \mathbb{C}^*$.

Proof (1) The "if" part can be verified by straightforward check. Now, we prove the "only if" part. Assume that $\mathcal{L}_{q^2}(b)$ is a $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module. Using

$$q^2 t_1 t_2 = q t_2 t_1 = x^{\varepsilon_2} x^{\varepsilon_1} \cdot 1 = \tilde{q} x^{\varepsilon_1} x^{\varepsilon_2} \cdot 1 = \tilde{q} t_1 t_2,$$

we obtain

$$\tilde{q} = q^2.$$

From $\partial_1 \partial_2 \cdot 1 = \partial_2 \partial_1 \cdot 1$, (2.8), and (2.9), we have

$$t_1 \frac{\partial}{\partial t_1} b(t_1, t_2) = 0.$$

Thus,

$$b \in \mathbb{C}[t_2^{\pm 1}].$$

If $q \neq 1$, then, from $bt_1 = \partial_2 x^{\varepsilon_1} \cdot 1 = x^{\varepsilon_1} \partial_2 \cdot 1 = t_1 b$, we obtain $b = b(t_1, t_2) \in \mathbb{C}[t_1^{\pm 1}]$. Thus, $b \in \mathbb{C}$.

The rest of the proof is similar to that of Theorem 2.1.

Remark 2.3 If $q = 1$, we denote $\mathcal{L}_{q^2}(b)$ as $\mathcal{L}(b)$.

Theorem 2.4 Suppose that $\mathcal{L}_q(b_1, b_2)$ and $\mathcal{L}_{q^2}(b)$ are $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-modules. (1) if $q \neq 1$, then, $\mathcal{L}_q(b_1, b_2)$ is not isomorphic to $\mathcal{L}_{q^2}(b)$. (2) If $q = 1$, then, $\mathcal{L}_q(b_1, b_2) \cong \mathcal{L}_{q^2}(b)$ if and only if b_1, $b - b_2 \in \mathbb{Z}$.

Proof (1) Suppose $\phi : \mathcal{L}_q(b_1, b_2) \to \mathcal{L}_{q^2}(b)$ is a $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module isomorphism. Then, there exists $f \in \mathcal{L}_q(b_1, b_2)$, such that $\phi(f) = t_1$. Using $\phi(\partial_1 \cdot f) = \partial_1 \cdot \phi(f) = \partial_1 \cdot t_1$, we obtain

$$b_1 f + t_1 \frac{\partial}{\partial t_1} f = f.$$

Solving this equation, we have $f = c_1 t_1^{1-b_1}$, $c_1 \in \mathbb{C}[t_2^{\pm 1}]$. Thus,

$$b_1 \in \mathbb{Z}. \tag{2.10}$$

From $\phi(\partial_2 \cdot f) = \partial_2 \cdot \phi(f) = \partial_2 \cdot t_1$, we obtain

$$b_2 f + t_2 \frac{\partial}{\partial t_2} f = bf.$$

So, $f = c_2 t_2^{b-b_2}$, $c_2 \in \mathbb{C}[t_1^{\pm 1}]$. Thus,

$$\begin{aligned} & b - b_2 \in \mathbb{Z}, \\ & f = c t_1^{1-b_1} t_2^{b-b_2}, \quad c \in \mathbb{C}^*, \end{aligned} \tag{2.11}$$

and $\phi(c t_1^{-b_1} t_2^{b-b_2}) = \phi(x^{-\varepsilon_1} \cdot f) = x^{-\varepsilon_1} \cdot t_1 = 1$. Therefore,

$$\begin{aligned} \phi(c) &= \phi(x^{(b-b_2)\varepsilon_2} \cdot x^{b_1 \varepsilon_1} \cdot c t_1^{-b_1} t_2^{b_2 - b}) = x^{(b-b_2)\varepsilon_2} \cdot x^{b_1 \varepsilon_1} \cdot 1 \\ &= t_2^{b-b_2} (q^{(b-b_2)} t_1)^{b_1} = q^{2(b-b_2)b_1} t_1^{b_1} t_2^{b-b_2}, \end{aligned}$$

and for all $\alpha \in \mathbb{Z}^2\backslash\{0\}$,

$$\begin{aligned}\phi(ct_1^{\alpha_1}t_2^{\alpha_2}) = \phi(x^\alpha \cdot c) &= x^\alpha \cdot q^{2(b-b_2)b_1}t_1^{b_1}t_2^{b-b_2} \\ &= q^{2(b-b_2)b_1}q^{2\alpha_2 b_1}t_1^{\alpha_1+b_1}t_2^{\alpha_2+b-b_2}.\end{aligned}$$

Now, for any $\alpha, \beta \in \mathbb{Z}^2\backslash\{0\}$, from

$$\begin{aligned}\phi(ct_1^{\beta_1}t_2^{\beta_2}t_1^{\alpha_1}t_2^{\alpha_2}) &= \phi(x^\beta \cdot ct_1^{\alpha_1}t_2^{\alpha_2}) \\ &= x^\beta \cdot q^{2(b-b_2)b_1}q^{2\alpha_2 b_1}t_1^{\alpha_1+b_1}t_2^{\alpha_2+b-b_2} \\ &= q^{2(b-b_2)b_1}q^{2\alpha_2 b_1}t_1^{\beta_1}t_2^{\beta_2}(q^{\beta_2}t_1)^{\alpha_1+b_1}t_2^{\alpha_2+b-b_2} \\ &= q^{2(b-b_2)b_1}q^{2\alpha_2 b_1}q^{2\beta_2(\alpha_1+b_1)}t_1^{\alpha_1+\beta_1+b_1}t_2^{\alpha_2+\beta_2+b-b_2},\end{aligned}$$

and

$$\begin{aligned}\phi(ct_1^{\beta_1}t_2^{\beta_2}t_1^{\alpha_1}t_2^{\alpha_2}) &= \phi(cq^{\beta_2\alpha_1}t_1^{\alpha_1+\beta_1}t_2^{\alpha_2+\beta_2}) \\ &= q^{\beta_2\alpha_1}q^{2(b-b_2)b_1}q^{2(\alpha_2+\beta_2)b_1}t_1^{\alpha_1+\beta_1+b_1}t_2^{\alpha_2+\beta_2+b-b_2},\end{aligned}$$

we have

$$q^{\beta_2\alpha_1} = 1, \quad \forall \alpha = (\alpha_1, \alpha_2),\ \beta = (\beta_1, \beta_2) \in \mathbb{Z}^2\backslash\{0\}.$$

Thus, $q = 1$, which is absurd.

(2) The "only if" part can be obtained from (2.10) and (2.11). Now, we prove the "if" part. Suppose b_1, $b - b_2 \in \mathbb{Z}$. Define a linear map $\phi : \mathcal{L}_q(b_1, b_2) \to \mathcal{L}_{q^2}(b)$ by

$$\phi(t_1^{\alpha_1}t_2^{\alpha_2}) = t_1^{\alpha_1+b_1}t_2^{\alpha_2+b_2-b}, \quad \forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2\backslash\{0\}.$$

One can check that ϕ is an isomorphism.

3 $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-Modules Over Commutative Representation Spaces

3.1 Modules of type $\mathcal{L}^-$

Let $p \in \mathbb{C}^*$, $b := b(t_1, t_2) \in \mathbb{C}_q[t_1^{\pm 1}, t_2]$. We define a $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module $\mathcal{L}^-$ over the space $\mathbb{C}_q[t_1^{\pm 1}, t_2]$ by

$$g(t_1, t_2)x^\alpha \cdot f(t_1, t_2) = g(t_1, t_2)p^{\alpha_2}t_1^{\alpha_1}f(q^{\alpha_2}t_1, t_2 - \alpha_2), \tag{3.1}$$

$$g(t_1, t_2)\partial_1 \cdot f = g(t_1, t_2)(b + t_1\frac{\partial}{\partial t_1}f), \quad g(t_1, t_2)\partial_2 \cdot f = g(t_1, t_2)t_2 f, \tag{3.2}$$

where $g(t_1, t_2) \in \mathcal{L}$, $f = f(t_1, t_2) \in \mathcal{L}^-, \alpha \in \mathbb{Z}^2\backslash\{0\}$.

Theorem 3.1 (1) $\mathcal{L}^-$ is a $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module with the action defined by (3.1), (3.2) if and only if $q = \tilde{q} = 1$, $b = 0$.

(2) $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module $\mathcal{L}^-$ is irreducible.

(3) $\mathrm{Aut}\mathcal{L}^- \cong \mathbb{C}^*$.

Proof (1) We only prove the "only if" part. Assume that $\mathcal{L}^-$ is a $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module. Using

$$p\tilde{q}t_1 = p\tilde{q}x^{\varepsilon_1} \cdot 1 = \tilde{q}x^{\varepsilon_1}x^{\varepsilon_2} \cdot 1 = x^{\varepsilon_2}x^{\varepsilon_1} \cdot 1 = x^{\varepsilon_2} \cdot t_1 = pqt_1,$$

we obtain

$$\tilde{q} = q.$$

From

$$t_1t_2 = x_1^{\varepsilon} \cdot t_2 = x^{\varepsilon_1}\partial_2 \cdot 1 = \partial_2 x^{\varepsilon_1} \cdot 1 = \partial_2 \cdot t_1 = t_2t_1 = qt_1t_2,$$

we have

$$q = 1.$$

Then, from $\partial_1\partial_2 \cdot 1 = \partial_2\partial_1 \cdot 1$ and (3.2), we obtain $b(t_1, t_2) = t_2 b(t_1, t_2)$. Thus,

$$b = 0.$$

The rest of the proof is obvious.

3.2 Module type $\mathcal{L}^+(p_1, p_2)$

Let $p_1, p_2 \in \mathbb{C}^*$. We define a $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module $\mathcal{L}^+(p_1, p_2)$ over the space $\mathbb{C}_q[t_1, t_2]$ by

$$g(t_1, t_2)x^{\alpha} \cdot f(t_1, t_2) = g(t_1, t_2)p_1^{\alpha_1}p_2^{\alpha_2} f(t_1 - \alpha_1, t_2 - \alpha_2), \tag{3.3}$$

$$g(t_1, t_2)\partial_i \cdot f = g(t_1, t_2)t_i f, \quad i = 1, 2, \tag{3.4}$$

where $g(t_1, t_2) \in \mathcal{L}$, $f = f(t_1, t_2) \in \mathcal{L}^+(p_1, p_2), \alpha \in \mathbb{Z}^2\backslash\{0\}$.

Theorem 3.2 (1) $\mathcal{L}^+(p_1, p_2)$ is a $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module with the action defined by (3.3) and (3.4) if and only if $q = \tilde{q} = 1$.

(2) $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-module $\mathcal{L}^+(p_1, p_2)$ is irreducible.

(3) As $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-modules, $\mathcal{L}^+(p_1, p_2) \cong \mathcal{L}^+(p_1', p_2')$ if and only if $p_1 = p_1', p_2 = p_2'$.

(4) $\mathrm{Aut}\mathcal{L}^+(p_1, p_2) \cong \mathbb{C}^*$.

Theorem 3.3 As $\mathcal{L}(\mathcal{G}_{\tilde{q}})$-modules, $\mathcal{L}(b)$ (recall Remark 2.3.), $\mathcal{L}^-$, $\mathcal{L}^+(p_1, p_2)$ are not isomorphic to each other.

Proof It is checked that $\mathcal{L}^+(p_1, p_2)$ is not isomorphic to $\mathcal{L}^-$ and $\mathcal{L}(b)$. Now, we want to prove $\mathcal{L}(b) \ncong \mathcal{L}^-$. Suppose $\phi : \mathcal{L}(b) \to \mathcal{L}^-$ is an isomorphism. Assume $\phi(1) = f$ for some $f \in \mathcal{L}^-$. Using $\phi(\partial_1 \cdot 1) = \partial_1 \cdot f$, we obtain $t_1\frac{\partial}{\partial t_1} f = 0$. Thus, $f := f(t_2) \in \mathbb{C}[t_2]$. For any $n \in \mathbb{Z}$, using $\phi(x^{n\varepsilon_2} \cdot 1) = x^{n\varepsilon_2} \cdot f$, we have

$$\phi(t_2^n) = p^n f(t_2 - n), \quad \forall n \in \mathbb{Z}. \tag{3.5}$$

Note that $f := f(t_2) \in \mathbb{C}[t_2]$. Suppose that the degree of $f(t_2)$ is m. Then, for different $n_1, \cdots, n_{m+2} \in \mathbb{Z}$, there exist $m + 2$ constants $a_1, \cdots, a_{m+2}$, not all zeros, such that

$$a_1p^{n_1} f(t_2 - n_1) + a_2p^{n_2} f(t_2 - n_2) + \cdots + a_{m+2}p^{n_{m+2}} f(t_2 - n_{m+2}) = 0,$$

but

$$a_1t^{n_1} + a_2t^{n_2} + \cdots + a_{m+2}t^{n_{m+2}} \neq 0.$$

Thus, we get a contradiction. Therefore, $\mathcal{L}(b) \ncong \mathcal{L}^-$.

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