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CONFOUNDING STRUCTURE OF TWO-LEVEL NONREGULAR FACTORIAL DESIGNS*

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Abstract In design theory, the alias structure of regular fractional factorial designs is elegantly described with group theory. However, this approach cannot be applied to nonregular designs directly. For an arbitrary nonregular design, a natural question is how to describe the confounding relations between its effects, is there any inner structure similar to regular designs? The aim of this article is to answer this basic question. Using coefficients of indicator function, confounding structure of nonregular fractional factorial designs is obtained as linear constrains on the values of effects. A method to estimate the sparse significant effects in an arbitrary nonregular design is given through an example.

Key words Nonregular design; alias set; partial aliasing

2000 MR Subject Classification 62K15

1 Introduction

One purpose of factorial experiments is to estimate the main and interaction effects of the factors relating to a factorial system. For a regular design, all effects can be classified into different alias sets by defining contrast subgroup; and an effect can be estimated if and only if other effects in the same alias set are negligible. This structure of regular designs was elegantly described with group theory as stated in Mukerjee and Wu [1]; unfortunately, the theoretic tools used in regular designs cannot be directly applied to nonregular designs.

In recent years, researches on nonregular designs became more popular for the awareness of its practical and theoretical importance and some advantages over regular ones. Nonregular designs can be used to estimate 2nd order effects by use of its partial aliasing as discovered by Hamada and Wu [2]. Several extensions of the minimum aberration criterion for regular designs to nonregular ones were proposed by Deng and Tang [3]. Tang and Deng [4], Xu and Wu [5], Xu [6], and references therein. Through the use of indicator function, Fontana et al [7] and Ye [8] investigated nonregular design from a different viewpoint. A natural question about nonregular design is how effects are confounded with each other; is there any inner structure similar to that of a regular design? To explore this problem is the purpose of this article. The

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confounding structure as a group of linear constrains on the effects will be obtained; through an example, a method to estimate the sparse significant effects of a factorial system by use of the structure will be illustrated.

This article is organized as follows. In Section 2, concepts and notations of factorial designs and some theories on indicator function are reviewed. In Section 3, some results on the alias structure of regular designs are recalled from a new point of view. The main result on confounding structure of nonregular designs is given in Section 4. A conclusion remark is presented in Section 5.

2 Some Concepts and Notations on Factorial Designs

A full 2^p factorial design consists of all the 2^p treatment combinations of p factors each taking two levels -1 and $+1$. It is represented by a $2^p \times p$ matrix $\mathbf{D}$ with entries -1 and $+1$, where the columns are respectively assigned to the factors denoted by $1, \cdots, p$ and every row represents a treatment combination. The rows are respectively denoted as $\mathbf{x}_i = (x_{i,1}, \cdots, x_{i,p})$, $i = 1, \cdots, 2^p$. By running a treatment combination $\mathbf{x}_i$, an experimenter can obtain a response, which can be considered as an observation of the treatment effect $\tau(\mathbf{x}_i)$, where $\tau(\mathbf{x}_i)$ is an unknown parameter in the context of the factorial experiment.

Let e be the interaction effect of r factors $j_1, \cdots, j_r$, denoted by $e_{j_1 \cdots j_r}$, for $r = 0, 1, \cdots, p$. With Yates algorithm, the value of e can be defined as

$$V(e) = \frac{1}{2^p} \Bigg(\sum_{\prod_{s=1}^{r} x_{i,j_s} = +1} \tau(\mathbf{x}_i) - \sum_{\prod_{s=1}^{r} x_{i,j_s} = -1} \tau(\mathbf{x}_i) \Bigg), \tag{1}$$

the number r is called the order of the effect. When $r = 1$, e turns out to be a main effect. When $r = 0$, e is called the grand mean, denoted by I, we have

$$V(I) = \frac{1}{2^p} \sum_{i=1}^{2^p} \tau(\mathbf{x}_i). \tag{2}$$

Interaction effects of all orders are collectively called factorial effects. All the factorial effects form a group with product operation, the product of letters $1, \cdots, p$ with squares dropped. This group is denoted by $\mathcal{E}$, apparently the cardinality of $\mathcal{E}$ is 2^p.

We rank all the effects in an order as follows. Effect $e_{i_1 \cdots i_k}$ is said to be "smaller" than $e_{j_1 \cdots j_l}$ if either $k < l$ or when $k = l$ and $i_1 \cdots i_k$ is lexicographically ahead of $j_1 \cdots j_l$. Then, from all the smallest to the largest effects are denoted as $e_0, e_1, \cdots, e_{2^p - 1}$ in a sequence.

The values of all factorial effects are the coefficients of a linear model

$$\tau(\mathbf{x}_i) = \sum_{e_{j_1 \cdots j_r} \in \mathcal{E}} V(e_{j_1 \cdots j_r}) x_{i,j_1} \cdots x_{i,j_r}, \ i = 1, \cdots, 2^p. \tag{3}$$

A full factorial design is rarely used in practice when the number of combinations is large for economical reason. Instead, a fractional factorial design, consisting of a subset of all treatment combinations of the full design, is commonly used. If a fractional design $\mathbf{F}$ consists of n treatment combinations $\mathbf{x}_{i_j}, j = 1, \cdots, n$, of the full design $\mathbf{D}$, then, $\mathbf{F}$ is represented by the $n \times p$ matrix with these $\mathbf{x}_{i_j}$'s as its rows.

Now, the jth column of $\mathbf{F}$ is defined as the expression of main effect e_j. The expression of effect $e_{j_1\cdots j_r}$, the interactions of factors $j_1,\cdots,j_r$, in design $\mathbf{F}$ is defined as the dot production of the j_1th, j_2th, $\cdots$, j_rth columns of $\mathbf{F}$, denoted by $E_{\mathbf{F}}(e_{j_1\cdots j_r})$. Specially, the expression of e_0 or I is an n-dimensional column vector with all entries being $+1$.

All n-dimensional vectors with -1 and $+1$ as its entries constitute a multiplicative group with dot production as its operation, denoted as $\{-1,+1\}^n$. Thus, an homeomorphism from the effect group $\mathcal{E}$ into $\{-1,+1\}^n$:

$$E_{\mathbf{F}}:\mathcal{E}\mapsto\{-1,+1\}^n$$

is defined by the above definition of expressions. Note that, the expressions of an effect in different fractions of $\mathbf{D}$ might be different, and two different effects may have the same expression in a fraction.

Definition 1 Two effects are said to be aliased with each other in a fraction if their expressions are the same or the same after multiplying -1 to one of them.

Definition 2 A fraction is said to be normal if no effect has expression $(-1,\cdots,-1)^T$ in it.

Any fraction can be turned to be a normal one by a permutation of symbol levels for some factors, that is, any fraction is isomorphic to a normal one. So that fractions mentioned in this article are assumed to be normal.

$E_{\mathbf{F}}$ is also used to represent the matrix

$$E_{\mathbf{F}}=(E_{\mathbf{F}}(e_0),E_{\mathbf{F}}(e_1),\cdots,E_{\mathbf{F}}(e_{2^p-1})).$$

V, $\tau(\mathbf{F})$, and $\tau(\mathbf{D})$ are used to denote

$$\begin{aligned}V&=(V(e_0),V(e_1),\cdots,V(e_{2^p-1}))^T,\\ \tau(\mathbf{F})&=(\tau(\mathbf{x}_{i_1}),\tau(\mathbf{x}_{i_2}),\cdots,\tau(\mathbf{x}_{i_n}))^T,\\ \tau(\mathbf{D})&=(\tau(\mathbf{x}_1),\tau(\mathbf{x}_2),\cdots,\tau(\mathbf{x}_{2^p}))^T,\end{aligned}$$

respectively. Then, for any effect $e\in\mathcal{E}$, formulas (1) and (2) can be commonly represented as

$$V(e)=\frac{1}{2^p}\tau(\mathbf{D})^TE_{\mathbf{D}}(e). \tag{4}$$

By (3), for a fractional design $\mathbf{F}$, we have

$$\tau(\mathbf{F})=E_{\mathbf{F}}V, \tag{5}$$

and especially, if $\mathbf{F}=\mathbf{D}$, then, it is just $\tau(\mathbf{D})=E_{\mathbf{D}}V$, which is a matrix form of model (3).

Example 1 If a 4 run 4 factors two level experiment is done and 4 results are obtained, as in Table 1, then expression of effect e_1 is $E_{\mathbf{F}_1}(e_1)=(+1,-1,-1,-1)^T$, the expression of interactional effect of factor 2 and factor 3 is $E_{\mathbf{F}_1}(e_{23})=(+1,+1,-1,-1)^T$, the dot production of $(+1,-1,-1,+1)^T$ and $(+1,-1,+1,-1)^T$. As no effect has the expression $(-1,-1,-1,-1)^T$, the fraction is normal. Values of effects can not be calculated by (4) for only results of 4 runs of total 16 runs are obtained, they can only be somehow estimated.

For details of effects and relevant linear model, one may refer to Wu and Hamada [9].

Table 1 A 4×4 nonregular fractional factorial design $\mathbf{F}_1$

treatment combinations $\mathbf{x}_{i_j}$	1	2	3	4
$\mathbf{x}_{i_1}$	+1	+1	+1	+1
$\mathbf{x}_{i_2}$	−1	−1	−1	−1
$\mathbf{x}_{i_3}$	−1	−1	+1	+1
$\mathbf{x}_{i_4}$	−1	+1	−1	+1

Next, let us analyze a fractional design $\mathbf{F}$ further. From an approach proposed by Fontana et al [7], a fractional design $\mathbf{F}$ can be characterized by its indicator function

$$F(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} = \mathbf{x}_{i_j}, \quad j = 1, \cdots, n, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

which is defined over the set of all treatments of $\mathbf{D}$. Using polynomial algebraic methods, Fontana et al [7] and Pistone et al [10] reached the following results.

Lemma 1 Denote $F(\mathbf{D}) = (F(\mathbf{x}_1), F(\mathbf{x}_2), \cdots, F(\mathbf{x}_{2^p}))^T$. Then, $F(\mathbf{D})$ is a linear combination of the expressions of all effects in $\mathbf{D}$:

$$F(\mathbf{D}) = \frac{n}{2^p} \sum_{i=0}^{2^p-1} b_{e_i} E_{\mathbf{D}}(e_i), \tag{7}$$

where the b_{e_i}'s are uniquely determined by the fraction $\mathbf{F}$.

Lemma 2 For every e_i, coefficient b_{e_i} in (7) takes value in $[-1, 1]$ and can be calculated as

$$b_{e_i} = \frac{1}{n} E_{\mathbf{F}}(I)^T E_{\mathbf{F}}(e_i), \ i = 0, \cdots, 2^p - 1, \tag{8}$$

so b_{e_i} is the average of the entries in the expression of effect e_i in $\mathbf{F}$.

At the end of this section, let us go back to examine (3) again. If all coefficients of the equations are known, then the factorial system is clear to us. From (4), we know that those coefficients, that is, values of effects, can be estimated by running all treatments of $\mathbf{D}$ to get entries of $\tau(\mathbf{D})$. When we do not have enough resources to run all the treatments, we should consider to eliminate some insignificant terms of (3) and estimate remaining important coefficients as a model.

In this section, no new results are presented, but some concepts are restated in different words. Expression of effects is defined as convenient notation for later proofing, and normal fraction is defined for later description of confounding structures.

3 Review of Confounding Structure of Regular Fractions

A regular 2^{p-m} design is a 2^{-m} fraction of the full factorial 2^p treatment combinations. It is obtained by adding m factors or columns to a full factorial design with $p-m$ factors, a new column is obtained as a dot production of some columns of the full factorial design.

Definition 3 Let $\mathbf{F}$ be a regular fraction of a full factorial design $\mathbf{D}$. All effects aliased with effect I, whose expressions in $\mathbf{F}$ being $(+1, \cdots, +1)^T$ or $(-1, \cdots, -1)^T$, constitute a subset of $\mathcal{E}$, denoted by $\mathcal{D}_{\mathbf{F}}$. It is obvious under the homeomorphism that $\mathcal{D}_{\mathbf{F}}$ is a subgroup of $\mathcal{E}$, it is called the defining contrast subgroup of $\mathbf{F}$. We always have $I \in \mathcal{D}_{\mathbf{F}}$.

A quotient group of $\mathcal{E}$ by $\mathcal{D}_{\mathbf{F}}$ can be obtained, denoted by $\mathcal{E}/\mathcal{D}_{\mathbf{F}}$, whose elements are the cosets of $\mathcal{D}_{\mathbf{F}}$.

Definition 4 A coset of $\mathcal{D}_{\mathbf{F}}$ in $\mathcal{E}$, denoted by $\mathcal{A}_i$, is called an alias set of $\mathbf{F}$.

So that two effects in the same alias set are aliased with each other by the homeomorphism.

For more details about regular designs, one may refer to Murkejee and Wu [1] in which regular fractional factorial designs of s levels were studied in depth, and in Lemma 2.4.3 the confounding structure of regular s-level designs was well described. The following lemma gives a restatement of the structure for a two level regular design.

For normal fraction, it is easy to find out that any effect in $\mathcal{A}_i$ has exactly the same expression, which is denoted as $E_{\mathbf{F}}(\mathcal{A}_i)$.

Lemma 3 Let $\mathbf{F}$ be a normal regular 2^{p-m} design. For any alias set $\mathcal{A}_i$, we have

$$\sum_{e\in\mathcal{A}_i} V(e) = \frac{1}{2^{p-m}}\tau(\mathbf{F})^T E_{\mathbf{F}}(\mathcal{A}_i),\ i = 0, \cdots, 2^{p-m}-1. \tag{9}$$

Lemma 3 plays an important role on regular fractional factorial designs. It justifies the use of a regular fractional of full factorial designs to get knowledge of the factorial system of (3). $\tau(\mathbf{F})$ on the right side of (9) can be estimated by running treatments of $\mathbf{F}$. So, if every $\mathcal{A}_i$ contains at most one non-negligible effect, then by (9), we can estimate it and the system of (3) can be figured out. Under the hierarchical principle, many criteria more or less aim at finding optimal designs that reach the above requirement. These criteria include maximum resolution criterion, minimum aberration criterion and clear effects criterion, and also a recent general minimum lower order confounding (GMC) criterion proposed by Zhang et al [11].

In contrast, from (9), we can see that two non-negligible aliased effects can not be estimated individually if no follow-up experiment is made. So that hierarchical principal or empirical knowledge have to be employed to assume the negligibility of the effects aliased to an important effect.

Table 2 A regular fractional factorial design $\mathbf{F}_2$

treatment combinations $\mathbf{x}_{i_j}$	1	2	3	4=123
$\mathbf{x}_{i_1}$	+1	−1	−1	+1
$\mathbf{x}_{i_2}$	+1	−1	+1	−1
$\mathbf{x}_{i_3}$	+1	+1	−1	−1
$\mathbf{x}_{i_4}$	+1	+1	+1	+1
$\mathbf{x}_{i_5}$	−1	−1	−1	−1
$\mathbf{x}_{i_6}$	−1	−1	+1	+1
$\mathbf{x}_{i_7}$	−1	+1	−1	+1
$\mathbf{x}_{i_8}$	−1	+1	+1	−1

Example 2 If a regular 2^{4-1} design is obtained by adding a new collum to a full three factor design, the new collum is a dot production of the existing three collumns, as in Table 2, then there are eight alias sets: $\mathcal{A}_0 = \{I, e_{1234}\}$, $\mathcal{A}_1 = \{e_1, e_{234}\}$, $\mathcal{A}_2 = \{e_2, e_{134}\}$, $\mathcal{A}_3 = \{e_3, e_{124}\}$,

$\mathcal{A}_4 = \{e_4, e_{123}\}$, $\mathcal{A}_5 = \{e_{12}, e_{34}\}$, $\mathcal{A}_6 = \{e_{13}, e_{24}\}$, and $\mathcal{A}_7 = \{e_{14}, e_{23}\}$. If interactional effects of three or four factors are negligible, then, $V(I)$, $V(e_1)$, $V(e_2)$, $V(e_3)$, and $V(e_4)$ can be calculated by (9), but interactional effects of two factors cannot. Those main effects and grand mean are clear factors (for the concept of clear effect, one may also see Yang et al [12]).

4 Confounding Structure of Nonregular Fractions

Let **F** be any fraction of the full factorial design **D**, and $b_0, b_1, \cdots, b_{2^p-1}$ are the coefficients of its indicator function in Lemma 1, which can be calculated by Lemma 2. An explicit linear constrain on the $V(e_i)$'s can be obtained as the following result.

Theorem 1 Let **F** be a fraction with n combinations from **D**, then we have

$$\sum_{i=0}^{2^p-1} b_{e_i} V(e_i) = \frac{1}{n}\sum_{j=1}^{n} \tau(\mathbf{x}_{i_j}). \tag{10}$$

Proof By left multiplying $E_{\mathbf{F}}(I)^T$ to both sides of (5), we obtain

$$E_{\mathbf{F}}(I)^T \tau(\mathbf{F}) = E_{\mathbf{F}}(I)^T E_{\mathbf{F}} V.$$

Thus,

$$\begin{aligned}\sum_{j=1}^{n} \tau(\mathbf{x}_{i_j}) = E_{\mathbf{F}}(I)^T\tau(\mathbf{F}) &= \Big(E_{\mathbf{F}}(I)^T E_{\mathbf{F}}(e_0), \cdots, E_{\mathbf{F}}(I)^T E_{\mathbf{F}}(e_{2^p-1})\Big)V \\ &= n(b_{e_0}, \cdots, b_{e_{2^p-1}})\Big(V(e_0), \cdots, V(e_{2^p-1})\Big)^T \\ &= n\sum_{i=o}^{2^p-1} b_{e_i} V(e_i).\end{aligned}$$

Definition 5 For a general fraction **F** of **D**, all effects aliased with grand mean I, whose expressions being $(+1, \cdots, +1)^T$ or $(-1, \cdots, -1)^T$, constitute a subgroup of $\mathcal{E}$, called the unit subgroup of **F** and denoted by $\mathcal{U}_{\mathbf{F}}$. Any coset of $\mathcal{U}_{\mathbf{F}}$, including itself, is called an alias set of **F**.

As a nonregular fraction **F** cannot be obtained by adding columns to some full design, so cannot be defined only by unit subgroup. This is why we don't follow the name of defining contrast subgroup to name it in nonregular situation.

Lemma 4 For nonregular fraction **F**, the regular fraction $\hat{\mathbf{F}}$ defined by $\mathcal{U}_{\mathbf{F}}$ as its defining contrast subgroup contains all treatments of **F**.

The fraction $\hat{\mathbf{F}}$ is called the minimal regular fraction containing **F** in Fontana et al [7].

Let 2^r be the cardinality of $\mathcal{U}_{\mathbf{F}}$, the number of effects contained in the unit subgroup of fraction **F**, so the fraction **F** has 2^{p-r} alias sets, with subscripts of $0, 1, \cdots, 2^{p-r}-1$ respectively ($\mathcal{A}_0 = \mathcal{U}_{\mathbf{F}}$).

Theorem 2 For any alias set $\mathcal{A}_j$ of a normal fraction **F**, we have

$$\sum_{i=0}^{2^p-1} b_{ke_i} V(e_i) = \frac{1}{n}\tau(\mathbf{F})^T E_{\mathbf{F}}(\mathcal{A}_j),\ j = 0, \cdots, 2^{p-r}-1, \tag{11}$$

where k is any effect in $\mathcal{A}_j$.

Proof Note that left multiplying both sides of (5) by $E_{\mathbf{F}}(\mathcal{A}_j)^T$ leads to

$$E_{\mathbf{F}}(\mathcal{A}_j)^T\tau(\mathbf{F}) = E_{\mathbf{F}}(\mathcal{A}_j)^T E_{\mathbf{F}}V.$$

Thus,

$$\begin{aligned}\tau(\mathbf{F})^T E_{\mathbf{F}}(\mathcal{A}_j) &= \Big(E_{\mathbf{F}}(k)^T E_{\mathbf{F}}(e_0),\cdots,E_{\mathbf{F}}(k)^T E_{\mathbf{F}}(e_{2^p-1})\Big)V\\ &= \Big(E_{\mathbf{F}}(I)^T E_{\mathbf{F}}(ke_0),\cdots,E_{\mathbf{F}}(I)^T E_{\mathbf{F}}(ke_{2^p-1})\Big)V\\ &= n(b_{ke_0},\cdots,b_{ke_{2^p-1}})\Big(V(e_0),\cdots,V(e_{2^p-1})\Big)^T\\ &= n\sum_{i=0}^{2^p-1} b_{ke_i}V(e_i).\end{aligned}$$

Noticing that the right side of (11) can be estimated by running $\mathbf{F}$, Theorem 2 gives all the linear constrains on the effects which can serve as the confounding structure of general fractions. This is the main result of this article, the proof of which seems not hard while notations and tools are prepared. It generalizes the result of Lemma 4 from regular situation to nonregular case. It will be useful in explaining the confounding relationship among effects and in estimating them.

We denote $b_{ij} = \frac{1}{n}E_{\mathbf{F}}(\mathcal{A}_i)^T E_{\mathbf{F}}(\mathcal{A}_j)$ which reflects the confounding degree between any two effects respectively in $\mathcal{A}_i$ and $\mathcal{A}_j$.

Let us denote the sum of aliased effect values as

$$V(\mathcal{A}_i) = \sum_{e\in\mathcal{A}_i} V(e),$$

which is called the value of alias set $\mathcal{A}_i$.

Theorem 2 can be rephrased as follows.

Corollary 1 For any fraction $\mathbf{F}$, let $\mathcal{A}_i$ be any alias set of it, then we have

$$V(\mathcal{A}_i) + \sum_{j=0,j\neq i}^{2^{p-r}-1} b_{ij}V(\mathcal{A}_j) = \frac{1}{n}\tau(\mathbf{F})^T E_{\mathbf{F}}(\mathcal{A}_i), i=0,\cdots,2^{p-r}-1. \tag{12}$$

Note that (9) is a special case of (12).

When $b_{ij}\neq 0$, that is, $E_{\mathbf{F}}(\mathcal{A}_i)$ and $E_{\mathbf{F}}(\mathcal{A}_j)$ are not mutually orthogonal, the two alias sets are called to be partially aliased.

Let $\mathbf{B}$ denote the matrix (b_{ij}), then we can rewrite (12) as

$$\mathbf{B}(V(\mathcal{A}_0),\cdots,V(\mathcal{A}_{2^{p-r}-1}))^T = \frac{1}{n}(E_{\mathbf{F}}(\mathcal{A}_0),\cdots,E_{\mathbf{F}}(\mathcal{A}_{2^{p-r}-1}))^T\tau(\mathbf{F}). \tag{13}$$

If $\mathbf{B}$ is nonsingular, every $V(\mathcal{A}_i)$ can be estimated with (13) and the design can be analyzed in the same way for a regular one. But by the following theorem, $\mathbf{B}$ is probably not of full rank.

Theorem 3 Among the 2^{p-r} equations of (12), there are n of them are linearly independent, and other equations can be dropped for being linear combinations of the n independent equations.

Proof Consider the matrix

$$E_{\mathbf{D}} = (E_{\mathbf{D}}(e_0), E_{\mathbf{D}}(e_1), \cdots, E_{\mathbf{D}}(e_{2^p-1})) \tag{14}$$

with expressions of all effects in $\mathbf{D}$ as its columns, which is a matrix on the real number field. It is well known that all the columns of $E_{\mathbf{D}}$ are mutually orthogonal, hence they are linearly independent, that is, matrix (14) has full rank. Consider another matrix

$$E_{\mathbf{F}} = (E_{\mathbf{F}}(e_0), E_{\mathbf{F}}(e_1), \cdots, E_{\mathbf{F}}(e_{2^p-1})), \tag{15}$$

which is composed by n rows of (14), so (15) has a full row rank. We can pick out n linearly independent columns, say, $E_{\mathbf{F}}(e_{i_1}), E_{\mathbf{F}}(e_{i_2}), \cdots, E_{\mathbf{F}}(e_{i_n})$, which are mutually different (as there are 2^{p-r} such different expressions, we have $n \leq 2^{p-r}$). Then, each e_{i_j} is in different alias set, say $\mathcal{A}_{i_j}$, and we have n linearly independent vectors

$$E_{\mathbf{F}}(\mathcal{A}_{i_1}), E_{\mathbf{F}}(\mathcal{A}_{i_2}), \cdots, E_{\mathbf{F}}(\mathcal{A}_{i_n}), \tag{16}$$

which can serve as a base of n-dimensional vector space. By (12), the n equations are obtained as

$$\sum_{j=0}^{2^{p-r}-1} b_{i_k j} V(\mathcal{A}_j) = \frac{1}{n}\tau(\mathbf{F})^T E_{\mathbf{F}}(\mathcal{A}_{i_k}),\ k = 1, \cdots, n. \tag{17}$$

Any other equation not in (17), say,

$$\sum_{j=0}^{2^{p-r}-1} b_{lj} V(\mathcal{A}_j) = \frac{1}{n}\tau(\mathbf{F})^T E_{\mathbf{F}}(\mathcal{A}_l), \tag{18}$$

can be obtained as a linear combination of (17). $E_{\mathbf{F}}(\mathcal{A}_l)$ can be expressed as a linear combination of (16):

$$E_{\mathbf{F}}(\mathcal{A}_l) = \alpha_1 E_{\mathbf{F}}(\mathcal{A}_{i_1}) + \alpha_2 E_{\mathbf{F}}(\mathcal{A}_{i_2}) + \cdots + \alpha_n E_{\mathbf{F}}(\mathcal{A}_{i_n}). \tag{19}$$

The right sides of (18) and (17) have the same combination coefficients, so we have

$$\begin{aligned}
&\frac{1}{n}\tau(\mathbf{F})^T E_{\mathbf{F}}(\mathcal{A}_l) \\
=& \frac{1}{n}\tau(\mathbf{F})^T(\alpha_1 E_{\mathbf{F}}(\mathcal{A}_{i_1}) + \alpha_2 E_{\mathbf{F}}(\mathcal{A}_{i_2}) + \cdots + \alpha_n E_{\mathbf{F}}(\mathcal{A}_{i_n})) \\
=& \alpha_1(\frac{1}{n}\tau(\mathbf{F})^T E_{\mathbf{F}}(\mathcal{A}_{i_1})) + \alpha_2(\frac{1}{n}\tau(\mathbf{F})^T E_{\mathbf{F}}(\mathcal{A}_{i_2})) + \cdots + \alpha_n(\frac{1}{n}\tau(\mathbf{F})^T E_{\mathbf{F}}(\mathcal{A}_{i_n}))
\end{aligned}$$

and

$$\begin{aligned}
b_{lj} &= \frac{1}{n} E_{\mathbf{F}}(\mathcal{A}_j)^T E_{\mathbf{F}}(\mathcal{A}_l) \\
&= \frac{1}{n} E_{\mathbf{F}}(\mathcal{A}_j)^T(\alpha_1 E_{\mathbf{F}}(\mathcal{A}_{i_1}) + \alpha_2 E_{\mathbf{F}}(\mathcal{A}_{i_2}) + \cdots + \alpha_n E_{\mathbf{F}}(\mathcal{A}_{i_n})) \\
&= \alpha_1 \frac{1}{n} E_{\mathbf{F}}(\mathcal{A}_j)^T E_{\mathbf{F}}(\mathcal{A}_{i_1}) + \alpha_2 \frac{1}{n} E_{\mathbf{F}}(\mathcal{A}_j)^T E_{\mathbf{F}}(\mathcal{A}_{i_2}) + \cdots + \alpha_n \frac{1}{n} E_{\mathbf{F}}(\mathcal{A}_j)^T E_{\mathbf{F}}(\mathcal{A}_{i_n}) \\
&= \alpha_1 b_{i_1 j} + \alpha_2 b_{i_2 j} + \cdots + \alpha_n b_{i_n j}.
\end{aligned}$$

In the end, we prove that (17) is an independent equation system. It is sufficient to verify that the sub-matrix

$$\begin{aligned}(b_{i_j i_k})_{n\times n} &= \frac{1}{n}(E_{\mathbf{F}}(\mathcal{A}_{i_j})^T E_{\mathbf{F}}(\mathcal{A}_{i_k}))_{n\times n}\\ &= \frac{1}{n}(E_{\mathbf{F}}(\mathcal{A}_{i_1}), E_{\mathbf{F}}(\mathcal{A}_{i_2}), \cdots, E_{\mathbf{F}}(\mathcal{A}_{i_n}))^T (E_{\mathbf{F}}(\mathcal{A}_{i_1}), E_{\mathbf{F}}(\mathcal{A}_{i_2}), \cdots, E_{\mathbf{F}}(\mathcal{A}_{i_n}))\end{aligned}$$

has full row rank. It is an obvious fact because of the linear independence of (16).

Theorem 2 or Corollary 1 is also an extension of the partial aliasing theory of Plackett-Burman designs proposed by Hamada and Wu [2].

For arbitrary nonregular fractional design, including nonorthogonal designs, the problem of estimating a few significant effects can be explored by the confounding theory. Following example illustrates this point.

Example 3 Consider the nonregular fraction $\mathbf{F}_1$ in Example 1, as shown in Table 1. It is nonregular and even nonorthogonal, so has no value for experimenters, but for models with only few sparse significant effects things may be different.

As only I and e_{1234} have the expression $(+1,+1,+1,+1)^T$, it is normal. There are total 8 alias sets with different expressions, listed in Table 3.

Table 3 The alias sets and their expressions of $\mathbf{F}_1$

alias set	with expression	alias set	with expression
$\mathcal{A}_0=\{I,e_{1234}\}$	$(+1,+1,+1,+1)^T$	$\mathcal{A}_4=\{e_4,e_{123}\}$	$(+1,-1,+1,+1)^T$
$\mathcal{A}_1=\{e_1,e_{234}\}$	$(+1,-1,-1,-1)^T$	$\mathcal{A}_5=\{e_{12},e_{34}\}$	$(+1,+1,+1,-1)^T$
$\mathcal{A}_2=\{e_2,e_{134}\}$	$(+1,-1,-1,+1)^T$	$\mathcal{A}_6=\{e_{13},e_{24}\}$	$(+1,+1,-1,+1)^T$
$\mathcal{A}_3=\{e_3,e_{124}\}$	$(+1,-1,+1,-1)^T$	$\mathcal{A}_7=\{e_{14},e_{23}\}$	$(+1,+1,-1,-1)^T$

By (12), there are 8 equations, but on the basis of Theorem 3 only 4 of them are independent. Let τ_i be the treatment-effect of ith row, then, we have

$$\begin{aligned}V(\mathcal{A}_0)-\frac{1}{2}V(\mathcal{A}_1)+\frac{1}{2}V(\mathcal{A}_4)+\frac{1}{2}V(\mathcal{A}_5)+\frac{1}{2}V(\mathcal{A}_6)&=\frac{1}{4}(\tau_1+\tau_2+\tau_3+\tau_4),\\ V(\mathcal{A}_1)-\frac{1}{2}V(\mathcal{A}_0)+\frac{1}{2}V(\mathcal{A}_2)+\frac{1}{2}V(\mathcal{A}_3)+\frac{1}{2}V(\mathcal{A}_7)&=\frac{1}{4}(\tau_1-\tau_2-\tau_3-\tau_4),\\ V(\mathcal{A}_2)+\frac{1}{2}V(\mathcal{A}_1)+\frac{1}{2}V(\mathcal{A}_4)-\frac{1}{2}V(\mathcal{A}_5)+\frac{1}{2}V(\mathcal{A}_6)&=\frac{1}{4}(\tau_1-\tau_2-\tau_3+\tau_4),\\ V(\mathcal{A}_3)+\frac{1}{2}V(\mathcal{A}_1)+\frac{1}{2}V(\mathcal{A}_4)+\frac{1}{2}V(\mathcal{A}_5)-\frac{1}{2}V(\mathcal{A}_6)&=\frac{1}{4}(\tau_1-\tau_2+\tau_3-\tau_4).\end{aligned}$$

By effect sparsity principle, only a few $V(\mathcal{A}_i)$'s are significant. Suppose that only $V(\mathcal{A}_0)$, $V(\mathcal{A}_1)$, $V(\mathcal{A}_2)$, and $V(\mathcal{A}_7)$ are significant, then,

$$\begin{aligned}&V(\mathcal{A}_0)-\frac{1}{2}V(\mathcal{A}_1)=\frac{1}{4}(\tau_1+\tau_2+\tau_3+\tau_4),\\ &V(\mathcal{A}_1)-\frac{1}{2}V(\mathcal{A}_0)+\frac{1}{2}V(\mathcal{A}_2)+\frac{1}{2}V(\mathcal{A}_7)=\frac{1}{4}(\tau_1-\tau_2-\tau_3-\tau_4),\\ &V(\mathcal{A}_2)+\frac{1}{2}V(\mathcal{A}_1)=\frac{1}{4}(\tau_1-\tau_2-\tau_3+\tau_4),\\ &\frac{1}{2}V(\mathcal{A}_1)=\frac{1}{4}(\tau_1-\tau_2+\tau_3-\tau_4).\end{aligned} \tag{20}$$

By now, the values of $V(\mathcal{A}_0) = V(I) + V(e_{1234})$, $V(\mathcal{A}_1) = V(e_1) + V(e_{234})$, $V(\mathcal{A}_2) = V(e_2) + V(e_{134})$, and $V(\mathcal{A}_7) = V(e_{14}) + V(e_{23})$ can be obtained easily. If the 3rd or higher order effects are negligible, then we can obtain $V(I)$, $V(e_1)$, and $V(e_2)$. If only one of $V(e_{14})$ and $V(e_{23})$ is significant, then it can be obtained too; when both of them are significant, a follow-up experiment is required to distinguish them; this requirement might be a disadvantage of nonregular designs with cardinality of its unit subgroup larger than 1. This disadvantage is the same as that of regular 2^{4-1} design $\mathbf{F}_2$ in Example 2, the relevant $\hat{\mathbf{F}}_1$ mentioned in Lemma 4, but nonregular design have an advantage of run-size flexibility over regular ones.

To identify those sparse significant $V(\mathcal{A}_i)$'s mentioned in the above example in (20) for orthogonal designs, if no empirical model exists, typical Half Normal Plot method can be employed. Suppose that the 3rd or higher order effects are negligible, let $\mathcal{A}_{i_0}$ be the unit subgroup and $\mathcal{A}_{i_j}$, $j = 1, \cdots, k$, are alias sets containing at least one effect with order lower than 3, expressions $E(\mathcal{A}_{i_j})$ are contrasts (with equal number of -1's and $+1$'s) by the orthogonality. If $E(\mathcal{A}_{i_j}) = (x_{j1}, \cdots, x_{jn})^T$ and $y_1, \cdots, y_n$ are responses of the n runs, then use $\frac{1}{n}(x_{j1}y_1 + \cdots + x_{jn}y_n)$ as a rough estimate of $V(\mathcal{A}_{i_j})$; with those estimates, one can draw an Half Normal Plot to identify the few sparse significant $V(\mathcal{A}_i)$'s.

Just like Plackett-Burman designs, almost all nonregular orthogonal designs mentioned in literature have $2^r = 1$. One exception is Sun and Wu [13], which gives a study of four 16 run nonregular orthogonal designs with both aliasing and partial aliasing existence.

Is this kind of orthogonal designs worth of using? According to Sun and Wu [13], there are exactly five non-isomorphic orthogonal designs with 15 factors, H16.I, H16.II, H16.III, H16.IV, and H16.V with only H16.I being regular. The cardinality of defining contrast subgroup of H16.I is 2^{11}, and cardinalities of unit subgroups of H16.II, H16.III, H16.IV, and H16.V are 2^{10}, 2^9, 2^8, and 2^8, respectively. Then, values of all 16 alias sets can be estimated by regular H16.I, each can be ascribed to one effect only if its aliased $2^{11} - 1$ effects are assumed negligible; that is, under the assumption of negligibility of all the 2nd or higher order effects, grand mean and 15 main effects can be estimated. Up to values of 16 of total 2^5, 2^6, 2^7, and 2^7 alias sets can be estimated by H16.II, H16.III, H16.IV, and H16.V, and each can be ascribed to one effect while its aliased $2^{10} - 1$, $2^9 - 1$, $2^8 - 1$ and $2^8 - 1$ effects being neglected; this means some of the 16 estimated effects could be of order 2 or 3. So, the four nonregular designs have more model robustness with fewer aliased effects assumed negligible, at the cost that the values of other $2^5 - 16$, $2^6 - 16$, $2^7 - 16$, and $2^7 - 16$ alias sets have to be neglected.

5 Conclusion

In this article, using the coefficients of indicator function, the confounding structure of arbitrary nonregular fractional designs is obtained as linear constrains on the values of effects. The concepts of alias sets for arbitrary fractions, along with the concept of expressions of effects, are useful for nonregular fractions with cardinality of unit subgroup larger than 1, which could be of practical and theoretical interest.

We also figure out a way for estimating sparse significant effects in an arbitrary fractional design. This method can find its usage when some runs are ruined while executing a regular experiment such as an agriculture experiment. And apart from Plackett-Burman designs, construction of other kind of useful nonregular designs such as Hall's 16 run nonregular orthogonal designs can be feasible.

From this generalization of confounding property of regular designs, the counterparts of optimal criteria for regular designs in general nonregular situation are expected.

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