APPLYING RUSCHEWEYH DERIVATIVE ON TWO SUB-CLASSES OF BI-UNIVALENT FUNCTIONS

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Abstract. The Ruscheweyh derivative has been applied in this paper to investigate two subclasses of the function class Σ of bi-univalent functions defined in the open unit disc. We find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses.

Keywords : Analytic and univalent functions; Bi-univalent functions; λ - convex functions; Ruscheweyh derivative; Coefficient bounds.

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1. Introduction and definitions

Let Ω denote the class of all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. Let $M(\lambda)$ denote the class of λ-convex functions in U defined as follows see [5]:
$M(\lambda) = \{f \in \Omega : Re[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda(1 + \frac{zf''(z)}{f'(z)})] > 0, \lambda \geq 0\}$.

Further, by S we shall denote the class of all functions in Ω which are univalent in U (for details, see [3],[4],[10]).

It is well known that every function $f \in S$ has an inverse f^{-1}, defined by
$f^{-1}(f(z)) = z \ \ (z \in U)$,
and
$f(f^{-1}(w)) = w \ \ (|w| < r_o(f) : r_o(f) \geq \frac{1}{4})$,
where
$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots$.

A function $f(z) \in \Omega$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U see [10].

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Let Σ denote the class of bi-univalent functions in U given by (1.1). Brannan and Taha [2] (see also [6]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\delta^*(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order $\alpha(0 < \alpha \leq 1)$, respectively (see [7]). Thus, following Brannan and Taha [2] (see also [6]), a function $f(z) \in \Omega$ is in the class $\delta^*_\Sigma(\alpha)$ of strongly bi-starlike functions of order $\alpha(o < \alpha \leq 1)$ if each of the following conditions is satisfied:

$f \in \Sigma,\ |arg\{\frac{zf'(z))}{f(z)}| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, z \in U),$

and

$|arg\{\frac{wg'(w))}{g(w)}| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, w \in U)$

where g is the extension of f^{-1} to U. The classes $\delta^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$, of bi-starlike functions of order α and bi-convex functions of order α, corresponding (respectively) to the function classes $\delta^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $\delta^*_\Sigma(\alpha)$ and $K^*_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

The object of the present paper is to introduce two subclasses of the function class Σ applying the Ruscheweyh derivative, where Ruscheweyh [9] observed that

$$D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}, \tag{1.2}$$

for $n \in \mathbb{N}_o = \{0, 1, 2, ...\}$. This symbol $D^n f(z), n \in \mathbb{N}_o$ is called by Al- Amiri [1], the n^{th} order Ruscheweyh derivative of $f(z)$.

We note that $D^o f(z) = f(z), D^1 f(z) = zf'(z)$ and

$$D^n f(z) = z + \sum_{k=2}^{\infty} \sigma(n, k) a_k z^k, \tag{1.3}$$

where

$$\sigma(n, k) = \binom{n + k - 1}{n}, \tag{1.4}$$

and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses of the function class Σ employing the techniques used by Xiao-FeiLi et al.[11].

For deriving our main results, the following lemma needed to be mentioned [8].

Lemma 1.1. *If $h \in P$ then $|c_k| \leq 2$ for each k, where P is the family of all functions h analytic in U for which $Re(h(z)) > 0, h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$ for $z \in U$.*

2. Coefficient bounds for the function class $F_\Sigma(\alpha, \lambda)$

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $F_\Sigma(\alpha, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma,\ |arg\{(1-\lambda)\frac{z(D^n f(z))'}{D^n f(z)} + \lambda[1 + \frac{z(D^n f(z))''}{(D^n f(z))'}]\}| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1, \lambda \ge 0, z \in U), \tag{2.1}$$

and

$$|arg\{(1-\lambda)\frac{w(D^n g(w))'}{D^n g(w)} + \lambda[1 + \frac{w(D^n g(w))''}{(D^n g(w))'}]\}| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1, \lambda \ge 0, w \in U), \tag{2.2}$$

where the function g is the extension of f^{-1} given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + ... \tag{2.3}$$

We note that for $n = 0$ the class $F_\Sigma(\alpha, \lambda)$ reduces to the class $B_\Sigma(\alpha, \lambda)$ introduced and studied by Xiao-FeiLi et al.[11].

Theorem 2.2. *Let $f(z)$ given by (1.1) be in the class $F_\Sigma(\alpha, \lambda), 0 < \alpha \le 1$ and $\lambda \ge 0$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha(1 - 4n\lambda - \lambda^2) - \alpha n^2(\lambda^2 + 4\lambda + 1) + (n+1)^2(\lambda+1)^2}}, \tag{2.4}$$

and

$$|a_3| \le \frac{2\alpha}{(n+1)(n+2)(1+2\lambda)} + \frac{4\alpha^2}{(n+1)^2(\lambda+1)^2}. \tag{2.5}$$

Proof:

We can write the argument inequalities in (2.1) and (2.2) equivalently as follows:

$$(1-\lambda)\frac{z(D^n f(z))'}{D^n f(z)} + \lambda[1 + \frac{z(D^n f(z))''}{(D^n f(z))'}] = [p(z)]^\alpha, \tag{2.6}$$

and

$$(1-\lambda)\frac{w(D^n g(w))'}{D^n g(w)} + \lambda[1 + \frac{w(D^n g(w))''}{(D^n g(w))'}] = [q(w)]^\alpha, \tag{2.7}$$

where $p(z)$ and $q(w)$ satisfy the following inequalities
$Re(p(z)) > 0 \ (z \in U)$ and $Re(q(w)) > 0 \ (w \in U)$.
Furthermore, the functions $p(z)$ and $q(w)$ have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ..., \tag{2.8}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + ... \tag{2.9}$$

And

$g(w)$ is given as in (2.3).

Now equating the coefficients in equations (2.6) and (2.7), we get

$$(n+1)(1+\lambda)a_2 = p_1\alpha, \tag{2.10}$$

$$(n+1)(n+2)(1+2\lambda)a_3 = p_2\alpha + \frac{\alpha(\alpha-1)}{2}p_1^2 + \frac{1+3\lambda}{(1+\lambda)^2}p_1^2\alpha^2. \tag{2.11}$$

and

$$-(n+1)(1+\lambda)a_2 = q_1\alpha, \tag{2.12}$$

$$(n+1)(n+2)(1+2\lambda)(2a_2^2 - a_3) = q_2\alpha + \frac{\alpha(\alpha-1)}{2}q_1^2 + \frac{1+3\lambda}{(1+\lambda)^2}q_1^2\alpha^2. \tag{2.13}$$

From equations (2.10) and (2.12), we get

$$p_1 = -q_1, \tag{2.14}$$

also we get

$$2(n+1)^2(\lambda+1)^2a_2^2 = \alpha^2(p_1^2+q_1^2). \tag{2.15}$$

From (2.11),(2.13) and (2.15) we obtain

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{\alpha(1-4n\lambda-\lambda^2) - \alpha n^2(\lambda^2+4\lambda+1) + (n+1)^2(\lambda+1)^2}. \tag{2.16}$$

Applying lemma (1.1) for the coefficients p_2 and q_2, we get

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha(1-4n\lambda-\lambda^2) - \alpha n^2(\lambda^2+4\lambda+1) + (n+1)^2(\lambda+1)^2}}. \tag{2.17}$$

Next, in order to find the bound on $|a_3|$, by subtracting (2.13) from (2.11), we get

$$\begin{aligned} &2(n+1)(n+2)(1+2\lambda)a_3 - 2(n+1)(n+2)(1+2\lambda)a_2^2 = \\ &\alpha(p_2-q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2) + \frac{1+3\lambda}{(1+\lambda)^2}\alpha^2(p_1^2-q_1^2). \end{aligned} \tag{2.18}$$

Upon substituting the value of a_2^2 from (2.15) and observing that $p_1^2 = q_1^2$ it follows that

$$a_3 = \frac{\alpha(p_2-q_2)}{2(n+1)(n+2)(1+2\lambda)} + \frac{\alpha^2(p_1^2+q_1^2)}{2(n+1)^2(\lambda+1)^2}, \tag{2.19}$$

Applying lemma (1.1) once again for the coefficients p_1, p_2, q_1 *and* q_2, we readily get

$$|a_3| \leq \frac{2\alpha}{(n+1)(n+2)(1+2\lambda)} + \frac{4\alpha^2}{(n+1)^2(\lambda+1)^2}, \tag{2.20}$$

This completes the proof of Theorem 2.2. □

Putting $n = 0$ in Theorem 2.2 we have

Corollary 2.3. *Let $f(z)$ given by (1.1) be in the class* $\mathrm{M}_\Sigma(\alpha,\lambda), 0 < \alpha \leq 1, \lambda \geq 0, z \in U$. *Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha(1-\lambda^2)+(1+\lambda)^2}}, \tag{2.21}$$

and

$$|a_3| \leq \frac{\alpha}{1+2\lambda} + \frac{4\alpha^2}{(1+\lambda)^2}. \tag{2.22}$$

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3. Coefficient bounds for the function class $F_\Sigma(\beta,\lambda)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $F_\Sigma(\beta,\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma,\ Re\{(1-\lambda)\frac{z(D^n f(z))'}{D^n f(z)} + \lambda[1+\frac{z(D^n f(z))''}{(D^n f(z))'}]\} > \beta \quad (0 \leq \beta < 1, \lambda \geq 0, z \in U), \tag{3.1}$$

and

$$Re\{(1-\lambda)\frac{w(D^n g(w))'}{D^n g(w)} + \lambda[1+\frac{w(D^n g(w))''}{(D^n g(w))'}]\} > \beta \quad (0 \leq \beta < 1, \lambda \geq 0, w \in U), \tag{3.2}$$

where the function $g(w)$ is given as in (2.3).

We note that for $n = 0$ the class $F_\Sigma(\beta,\lambda)$ reduces to the class $B_\Sigma(\beta,\lambda)$ introduced and studied by Xiao-FeiLi et al.[11].

Theorem 3.2. *Let $f(z)$ given by (1.1) be in the class $F_\Sigma(\beta,\lambda), 0 \leq \beta < 1$ and $\lambda \geq 0$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{(n+1)((1-n)\lambda+1)}}, \tag{3.3}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{(n+1)(n+2)(1+2\lambda)} + \frac{4(1-\beta)^2}{(1+n)^2(1+\lambda)^2}. \tag{3.4}$$

Proof:

The argument inequalities in (3.1) and (3.2) equivalently can be written as follows:

$$(1-\lambda)\frac{z(D^n f(z))'}{D^n f(z)} + \lambda[1+\frac{z(D^n f(z))''}{(D^n f(z))'}] = \beta + (1-\beta)p(z), \tag{3.5}$$

and

$$(1-\lambda)\frac{w(D^n g(w))'}{D^n g(w)} + \lambda[1+\frac{w(D^n g(w))''}{(D^n g(w))'}] = \beta + (1-\beta)q(w), \tag{3.6}$$

where $g(w), p(z)$, and $q(w)$ have the forms (2.3),(2.8) and (2.9) respectively. Equating coefficients in equations (3.5) and (3.6) yields

$$(n+1)(1+\lambda)a_2 = p_1(1-\beta), \tag{3.7}$$

$$(n+1)(n+2)(1+2\lambda)a_3 = p_2(1-\beta) + \frac{1+3\lambda}{(1+\lambda)^2}p_1^2(1-\beta)^2. \tag{3.8}$$

and

$$-(n+1)(1+\lambda)a_2 = q_1(1-\beta), \tag{3.9}$$

$$(n+1)(n+2)(1+2\lambda)(2a_2^2 - a_3) = q_2(1-\beta) + \frac{1+3\lambda}{(1+\lambda)^2}q_1^2(1-\beta)^2. \tag{3.10}$$

From equations (3.7) and (3.9), we get

$$p_1 = -q_1, \tag{3.11}$$

also we get

$$2(n+1)^2(\lambda+1)^2 a_2^2 = (1-\beta)^2(p_1^2 + q_1^2). \tag{3.12}$$

Now adding (3.8) to (3.10) gives

$$2(n+1)(n+2)(1+2\lambda)a_2^2 = (1-\beta)(p_2+q_2) + \frac{1+3\lambda}{(1+\lambda)^2}(p_1^2+q_1^2)(1-\beta)^2, \tag{3.13}$$

substituting value of $p_1^2 + q_1^2$ from (3.12) in (3.13) we get

$$a_2^2 = \frac{(1-\beta)(p_2+q_2)}{2(n+1)(1+\lambda(1-n))}. \tag{3.14}$$

Applying lemma (1.1) for the coefficients p_2 and q_2 we have

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{(n+1)((1-n)\lambda+1)}}, \tag{3.15}$$

Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8) we get

$$2(n+1)(n+2)(1+2\lambda)a_3 = (1-\beta)(p_2-q_2) + 2(n+1)(n+2)(1+2\lambda)a_2^2, \tag{3.16}$$

putting value of a_2^2 from (3.12) in (3.16) we get

$$a_3 = \frac{(1-\beta)(p_2-q_2)}{2(n+1)(n+2)(1+2\lambda)} + \frac{(1-\beta)^2(p_1^2+q_1^2)}{2(n+1)^2(\lambda+1)^2}, \tag{3.17}$$

Applying lemma (1.1) once again for the coefficients p_1, p_2, q_1 *and* q_2, we readily get

$$|a_3| \leq \frac{2(1-\beta)}{(n+1)(n+2)(1+2\lambda)} + \frac{4(1-\beta)^2}{(1+n)^2(1+\lambda)^2}, \tag{3.18}$$

This completes the proof of Theorem 3.2. □

Putting $n = 0$ in Theorem 3.2 we have

Corollary 3.3. *Let $f(z)$ given by (1.1) be in the class* $\mathrm{M}_\Sigma(\beta,\lambda), 0 \leq \beta < 1, \lambda \geq 0, z \in U$. *Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{1+\lambda}}, \tag{3.19}$$

and

$$|a_3| \leq \frac{1-\beta}{1+2\lambda} + \frac{4(1-\beta)^2)}{(1+\lambda)^2}. \tag{3.20}$$

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