

Research Article

Subclasses of Harmonic Mappings Defined by Convolution

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Two new subclasses of harmonic univalent functions defined by using convolution and integral convolution are introduced. These subclasses generate several known and new subclasses of harmonic univalent functions as special cases and provide a unified treatment in the study of these classes. Coefficient bounds, extreme points, distortion bounds, convolution conditions, and convex combination are also determined.

1. Introduction

A continuous function f is said to be a complex-valued harmonic function in a simply connected domain D in complex plane \mathbb{C} if both real part of f and imaginary part of f are real harmonic in D . Such functions can be expressed as

$$f = h + \bar{g}, \quad (1)$$

where h and g are analytic in D . We call h the analytic part and g the coanalytic part of f . A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ for all z in D , see [1].

Every harmonic function $f = h + \bar{g}$ is uniquely determined by the coefficients of power series expansions in the unit disk $U = \{z : |z| < 1\}$ given by

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in U, |b_1| < 1, \quad (2)$$

where $a_n \in \mathbb{C}$ for $n = 2, 3, 4, \dots$ and $b_n \in \mathbb{C}$ for $n = 1, 2, 3, \dots$. For further information about these mappings, one may refer to [1–5].

In 1984, Clunie and Sheil-Small [1] studied the family S_H of all univalent sense-preserving harmonic functions f of the form (1) in U , such that h and g are represented by (2). Note that S_H reduces to the well-known family S , the class

of all normalized analytic univalent functions h given in (2), whenever the coanalytic part g of f is zero. Let K and K_H denote the respective subclasses of S and S_H where the images of $f(U)$ are convex. Denote by S_H^0 the subclass of S_H for which $g'(0) = 0$.

The convolution of two functions of the form

$$\Phi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n, \quad \Psi(z) = z + \sum_{n=2}^{\infty} \nu_n z^n \quad (3)$$

is given by

$$(\Phi * \Psi)(z) = \Phi(z) * \Psi(z) = z + \sum_{n=2}^{\infty} \mu_n \nu_n z^n, \quad (4)$$

and the integral convolution is defined by

$$(\Phi \diamond \Psi)(z) = \Phi(z) \diamond \Psi(z) = z + \sum_{n=2}^{\infty} \frac{\mu_n \nu_n}{n} z^n. \quad (5)$$

Towards the end of the last century, Jahangiri [3], Silverman [4], and Silverman and Silvia [5] were amongst those who focused on the harmonic starlike functions. Later Öztürk et al. [6] defined the class $S_H^*(\lambda, \alpha)$ consisting of functions $f = h + \bar{g}$ such that h and g are of the forms

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \quad (6)$$

which satisfy the condition

$$\operatorname{Re} \left\{ \frac{zh'(z) - \overline{zg'(z)}}{\lambda(zh'(z) - \overline{zg'(z)}) + (1-\lambda)(h(z) + \overline{g(z)})} \right\} > \alpha, \tag{7}$$

for some $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$ and for all $z \in U$.

Several authors [3–16] have investigated various subclasses of harmonic functions. In this work, we introduce a new subclass of harmonic functions defined by convolution.

Let σ be a real constant with $|\sigma| = 1$, then we denote $S_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$, the subclass of S_H^0 of functions of the form $f = h + \overline{g} \in S_H^0$ that satisfy the condition

$$\operatorname{Re} \left\{ \frac{H(f, \Phi, \Psi)}{\lambda H(f, \Phi, \Psi) + (1-\lambda)G(f, \Phi, \Psi)} \right\} > \alpha, \tag{8}$$

where $H(f, \Phi, \Psi) = h(z) * \Phi(z) - \sigma \overline{g(z) * \Psi(z)}$, $G(f, \Phi, \Psi) = h(z) \diamond \Phi(z) + \sigma \overline{g(z) \diamond \Psi(z)}$, $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$, and Φ, Ψ are as given in (3).

We also denote $SP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$, the subclass of S_H^0 of functions of the form $f = h + \overline{g} \in S_H^0$ that satisfy the condition

$$\operatorname{Re} \left\{ (1 + e^{i\gamma}) \frac{H(f, \Phi, \Psi)}{\lambda H(f, \Phi, \Psi) + (1-\lambda)G(f, \Phi, \Psi)} - e^{i\gamma} \right\} > \alpha, \tag{9}$$

where γ is real.

We note that the families $S_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ and $SP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ are of special interest, because they contain various classes of well-known harmonic univalent functions as well as many new ones. For different choice of Φ, Ψ, λ , and α we obtain the following various classes introduced by other authors:

- (1) $S_H^0((z/(1-z)^2), (z/(1-z)^2), 1, \lambda, \alpha) = S_H^0(\lambda, \alpha)$ (see Öztürk et. al [6]).
- (2) $S_H^0((z/(1-z)^2), (z/(1-z)^2), 1, 0, \alpha) = S_H^0(\alpha)$ (see Jahangiri [3]).
- (3) $S_H^0((z/(1-z)^2), (z/(1-z)^2), 1, 0, 0) = S_H^0$ (see Silverman and Silvia [5]).
- (4) $S_H^0((z/(1-z)^2), (z/(1-z)^2), 1, 0, 0) = S_H^{*0}$, with $b_1 = 0$ (see Avcı and Zlotkiewicz [17] and Silverman [4]).
- (5) $S_H^0(((z+z^2)/(1-z)^3), ((z+z^2)/(1-z)^3), 1, 0, \alpha) = K_H^0(\alpha)$ (see Jahangiri [3]).
- (6) $S_H^0(((z+z^2)/(1-z)^3), ((z+z^2)/(1-z)^3), 1, 0, 0) = K_H^{*0}$ (see Silverman [4]).
- (7) $S_H^0(\Phi, \Psi, 1, 0, \alpha) = S_H^0(\Phi, \Psi, \alpha)$ (see Dixit et al. [11]).
- (8) $S_H^0(\Phi, \Phi, \sigma, 0, \alpha) = S_H^0(\Phi, \Phi, \alpha)$ (see Ali et al. [7]).

(9) $S_H^0(\Phi, \Psi, 1, \lambda, \alpha) = S_H^0(\Phi, \Psi, \lambda, \alpha)$ (see Joshi et al. [14]).

(10) $S_H^0(\Phi, \Psi, (1)^l, 0, \alpha) = H(l, \alpha)$, where $\mu_n = \nu_n = n^l$ (see Jahangiri et al. [13]).

(11) $S_H^0((z/(1-z)^{\gamma+1}), (z/(1-z)^{\gamma+1}), 1, 0, \alpha) = R_H(\gamma, \alpha)$, where $\gamma > -1$ (see Murugusundaramoorthy [15]).

(12) $S_H^0(\Phi, \Psi, (-1)^l, 0, \alpha) = M_H(l, \gamma, \alpha)$, where $\mu_n = \nu_n = n^l \cdot (((\gamma+1)_{n-1})/(n-1)!)$ and $(\gamma+1)_{n-1} = (\gamma+1)(\gamma+2) \cdots (\gamma+n-1)$ (see Al-Shaqsi and Darus [10]).

(13) $S_H^0(\Phi, \Psi, 1, \lambda, \alpha) = R_H^{p,q}(\Phi, \Psi, \lambda, \alpha)$, where $\mu_n = \nu_n = (((\gamma_1)_{n-1} \cdots (\gamma_p)_{n-1})/((\beta_1)_{n-1} \cdots (\beta_p)_{n-1}))(1/(n-1)!)$ (see Murugusundaramoorthy et al. [16]).

It is clear that the class $S_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ generates a number of known subclasses and provides a unified treatment of these subclasses of harmonic mappings. Motivated by work of Ali et al. [7], we obtain convolution characterization for functions in the class $S_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ and $SP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$. We also obtain sufficient coefficient condition for these two classes, and the last section is devoted to determine growth estimates and extreme points for the class $TS_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ and $TSP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$.

2. Main Results

We now derive a convolution characterization for functions in the class $S_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$.

Theorem 1. *Let $f = h + \overline{g} \in S_H^0$. Then $f \in S_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ if and only if*

$$\begin{aligned} & (h * \Phi) \diamond \left[\frac{z + (((x+2\alpha-1)(1-\lambda))/(2-2\alpha))z^2}{(1-z)^2} \right] \\ & - \sigma \overline{(g * \Psi)} \\ & \diamond \left[\left(\frac{x + \alpha - \lambda(x+2\alpha-1)\overline{z}}{1-\alpha} - \frac{(x+2\alpha-1)(1-\lambda)\overline{z}^2}{2-2\alpha} \right) ((1-\overline{z})^2)^{-1} \right] \neq 0, \\ & |x| = 1, |z| \neq 0. \end{aligned} \tag{10}$$

Proof. A necessary and sufficient condition for $f = h + \overline{g}$ to be in the class $S_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$, with h and g of the form (1) is given by (8). The condition (8) holds if and only if

$$\begin{aligned} & \frac{1}{(1-\alpha)} \left\{ \frac{H(f, \Phi, \Psi)}{\lambda H(f, \Phi, \Psi) + (1-\lambda)G(f, \Phi, \Psi)} - \alpha \right\} \\ & \neq \frac{x-1}{x+1}; \quad |x| = 1, x \neq -1, 0 < |z| < 1. \end{aligned} \tag{11}$$

By simple algebraic manipulation, (11) gives

$$\begin{aligned}
 & 0 \neq (x+1) \left[h(z) * \Phi(z) - \overline{\sigma g(z) * \Psi(z)} \right] \\
 & - (x+2\alpha-1) \left[\lambda \left(h(z) * \Phi(z) - \overline{\sigma g(z) * \Psi(z)} \right) \right. \\
 & \quad \left. + (1-\lambda) (h(z) \diamond \Phi(z) + \overline{\sigma g(z) \diamond \Psi(z)}) \right] \\
 & = (h * \Phi) \diamond \left[\frac{2(1-\alpha)z + (x-1+2\alpha)(1-\lambda)z^2}{(1-z)^2} \right] \\
 & \quad - \overline{\sigma(g * \Psi)} \\
 & \quad \diamond \left[\frac{2(\bar{x}+\alpha - \lambda(\bar{x}+2\alpha-1))z - (\bar{x}+2\alpha-1)(1-\lambda)z^2}{(1-z)^2} \right]. \tag{12}
 \end{aligned}$$

The latter condition, along with (8) for $x = -1$, establishes the result for all $|x| = 1$. \square

An application of the convolution condition in Theorem 1 gives sufficient condition for harmonic functions to belong to the class $S_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$.

Theorem 2. Let $f = h + \bar{g} \in S_H^0$. Then $f \in S_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ if

$$\begin{aligned}
 & \sum_{n=2}^{\infty} \left(\frac{n-\alpha-\lambda\alpha(n-1)}{1-\alpha} \right) \frac{|\mu_n| |a_n|}{n} \\
 & + \sum_{n=2}^{\infty} \left(\frac{n+\alpha-\lambda\alpha(n+1)}{1-\alpha} \right) \frac{|\nu_n| |b_n|}{n} \leq 1. \tag{13}
 \end{aligned}$$

Proof. For h and g given by (2), Theorem 1 gives

$$\begin{aligned}
 & \left| (h * \Phi) \diamond \left[\left(z + \frac{(x+2\alpha-1)(1-\lambda)}{2-2\alpha} z^2 \right) \right. \right. \\
 & \quad \left. \left. \times ((1-z)^2)^{-1} \right] \right. \\
 & \quad \left. - \overline{\sigma(g * \Psi)} \diamond \left[\frac{x+\alpha-\lambda(x+2\alpha-1)}{1-\alpha} \bar{z} \right. \right. \\
 & \quad \quad \left. \left. - \frac{(x+2\alpha-1)(1-\lambda)}{2-2\alpha} \bar{z}^2 \right. \right. \\
 & \quad \quad \left. \left. \times ((1-\bar{z})^2)^{-1} \right] \right| \\
 & = \left| z + \sum_{n=2}^{\infty} \frac{\mu_n}{n} \left[n + (n-1) \frac{(x+2\alpha-1)(1-\lambda)}{2-2\alpha} \right] a_n z^n \right. \\
 & \quad \left. - \sigma \sum_{n=2}^{\infty} \frac{\nu_n}{n} \left[n \frac{x+\alpha-\lambda(x+2\alpha-1)}{1-\alpha} \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & \quad \left. - (n-1) \frac{(x+2\alpha-1)(1-\lambda)}{2-2\alpha} \right] \bar{b}_n \bar{z}^n \Big| \\
 & > |z| \left[1 - \sum_{n=2}^{\infty} \left(\frac{n-\alpha-\lambda\alpha(n-1)}{1-\alpha} \right) \frac{|\mu_n| |a_n|}{n} \right. \\
 & \quad \left. - \sum_{n=2}^{\infty} \left(\frac{n+\alpha-\lambda\alpha(n+1)}{1-\alpha} \right) \frac{|\nu_n| |b_n|}{n} \right]. \tag{14}
 \end{aligned}$$

The last expression is nonnegative by hypothesis, and hence by Theorem 1, it follows that $f \in S_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$. \square

Theorem 3. Let $f = h + \bar{g} \in S_H^0$. Then $f \in SP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ if and only if

$$\begin{aligned}
 & (h * \Phi) \diamond \left[z + \frac{[(x+1)e^{i\gamma} + x + 2\alpha - 1](1-\lambda)}{2-2\alpha} z^2 \right. \\
 & \quad \left. \times ((1-z)^2)^{-1} \right] \\
 & \quad - \overline{\sigma(g * \Psi)} \\
 & \quad \diamond \left[\left(\frac{(x+1)e^{i\gamma} + x + \alpha - \lambda((x+1)e^{i\gamma} + x + 2\alpha - 1)}{1-\alpha} \bar{z} \right. \right. \\
 & \quad \left. \left. - \frac{[(x+1)e^{i\gamma} + x + 2\alpha - 1](1-\lambda)}{2-2\alpha} \bar{z}^2 \right) ((1-\bar{z})^2)^{-1} \right] \neq 0, \\
 & \quad |x| = 1, \quad |z| \neq 0. \tag{15}
 \end{aligned}$$

Proof. A necessary and sufficient condition for $f = h + \bar{g}$ to be in the class $S_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$, with h and g of the form (1) is given by (9). The condition (9) holds if and only if

$$\begin{aligned}
 & \frac{1}{(1-\alpha)} \left\{ (1+e^{i\gamma}) \frac{H(f, \Phi, \Psi)}{\lambda H(f, \Phi, \Psi) + (1-\lambda)G(f, \Phi, \Psi)} \right. \\
 & \quad \left. - e^{i\gamma} - \alpha \right\} \neq \frac{x-1}{x+1}; \tag{16}
 \end{aligned}$$

$$|x| = 1, \quad x \neq -1, \quad 0 < |z| < 1.$$

By simple algebraic manipulation, we get the desired result. \square

Now sufficient coefficient condition for the class $SP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ is easily obtained.

Theorem 4. Let $f = h + \bar{g} \in S_H^0$. Then $f \in SP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ if

$$\begin{aligned}
 & \sum_{n=2}^{\infty} \left(\frac{2n-1-\alpha-\lambda(1+\alpha)(n-1)}{1-\alpha} \right) \frac{|\mu_n| |a_n|}{n} \\
 & + \sum_{n=2}^{\infty} \left(\frac{2n+1+\alpha-\lambda(1+\alpha)(n+1)}{1-\alpha} \right) \frac{|\nu_n| |b_n|}{n} \leq 1. \tag{17}
 \end{aligned}$$

We further let $TS_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ and $TSP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ denote the subclasses of $S_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ and $SP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$, respectively, consisting of functions $f = h + \bar{g} \in S_H^0$ such that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \tag{18}$$

$$g(z) = \sigma \sum_{n=2}^{\infty} b_n z^n, \quad a_n \geq 0, b_n \geq 0.$$

Let

$$\Phi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n, \quad \Psi(z) = z + \sum_{n=2}^{\infty} \nu_n z^n, \tag{19}$$

with $\mu_n \geq 0, \nu_n \geq 0$.

Theorem 5. *Let $f = h + \bar{g}$ of the form (18). Then $f \in TS_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{\mu_n}{n} \left(\frac{n - \alpha - \lambda \alpha (n - 1)}{1 - \alpha} \right) a_n + \sum_{n=2}^{\infty} \frac{\nu_n}{n} \left(\frac{n + \alpha - \lambda \alpha (n + 1)}{1 - \alpha} \right) b_n \leq 1. \tag{20}$$

Proof. If $f \in TS_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$, then (8) is equivalent to

$$\begin{aligned} \operatorname{Re} \left\{ \left((1 - \alpha) z - \sum_{n=2}^{\infty} \mu_n \frac{[n - \alpha - \lambda \alpha (n - 1)]}{n} a_n z^n \right. \right. \\ \left. \left. - \sum_{n=2}^{\infty} \nu_n \frac{[n + \alpha - \lambda \alpha (n + 1)]}{n} b_n \bar{z}^n \right) \right. \\ \left. \times \left(z - \sum_{n=2}^{\infty} \mu_n \left[\lambda + \left(\frac{1 - \lambda}{n} \right) \right] a_n z^n \right. \right. \\ \left. \left. + \sum_{n=2}^{\infty} \nu_n \left[\left(\frac{1 - \lambda}{n} \right) - \lambda \right] b_n \bar{z}^n \right)^{-1} \right\} > 0, \end{aligned} \tag{21}$$

for $z \in U$. Letting $z \rightarrow 1^-$ through real values, we get

$$\begin{aligned} \left((1 - \alpha) - \sum_{n=2}^{\infty} \mu_n \left(\frac{[n - \alpha - \lambda \alpha (n - 1)]}{n} \right) a_n \right. \\ \left. - \sum_{n=2}^{\infty} \nu_n \left(\frac{[n + \alpha - \lambda \alpha (n + 1)]}{n} \right) b_n \right) \\ \times \left(1 - \sum_{n=2}^{\infty} \mu_n \left[\lambda + \left(\frac{1 - \lambda}{n} \right) \right] a_n \right. \\ \left. + \sum_{n=2}^{\infty} \nu_n \left[\left(\frac{1 - \lambda}{n} \right) - \lambda \right] b_n \right)^{-1} \geq 0, \end{aligned} \tag{22}$$

which gives required condition (20).

Conversely, for h and g given by (18),

$$\begin{aligned} \left| (h * \Phi) \diamond \left[\left(z + \frac{(x + 2\alpha - 1)(1 - \lambda)}{2 - 2\alpha} z^2 \right) \right. \right. \\ \left. \left. \times ((1 - z)^2)^{-1} \right] \right. \\ \left. - \overline{\sigma(g * \Psi)} \diamond \left[\left(\frac{x + \alpha - \lambda(x + 2\alpha - 1)}{1 - \alpha} \bar{z} \right. \right. \right. \\ \left. \left. - \frac{(x + 2\alpha - 1)(1 - \lambda)}{2 - 2\alpha} \bar{z}^2 \right) \right. \right. \\ \left. \left. \times ((1 - \bar{z})^2)^{-1} \right] \right| \\ > |z| \left[1 - \sum_{n=2}^{\infty} \frac{\mu_n}{n} \left(\frac{n - \alpha - \lambda \alpha (n - 1)}{1 - \alpha} \right) a_n \right. \\ \left. - \sum_{n=2}^{\infty} \frac{\nu_n}{n} \left(\frac{n + \alpha - \lambda \alpha (n + 1)}{1 - \alpha} \right) b_n \right], \end{aligned} \tag{23}$$

which is nonnegative by hypothesis. \square

Theorem 6. *Let $f = h + \bar{g}$ of the form (18). Then $f \in TSP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ if and only if*

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{\mu_n}{n} \left(\frac{2n - 1 - \alpha - \lambda(1 + \alpha)(n - 1)}{1 - \alpha} \right) a_n \\ + \sum_{n=2}^{\infty} \frac{\nu_n}{n} \left(\frac{2n + 1 + \alpha - \lambda(1 + \alpha)(n + 1)}{1 - \alpha} \right) b_n \leq 1. \end{aligned} \tag{24}$$

The following theorem gives the distortion bounds for functions in $TS_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$, and $TSP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ which gives a covering result for the classes $TS_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ and $TSP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$, respectively.

Theorem 7. *Let $f \in TS_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ and $\mu_n \geq \nu_2, \nu_n \geq \nu_2, (n \geq 2)$, then for $|z| = r < 1$, we have*

$$\begin{aligned} r - \frac{2(1 - \alpha)}{(2 - \alpha - \lambda \alpha) \nu_2} r^2 \leq |f(z)| \\ \leq r + \frac{2(1 - \alpha)}{(2 - \alpha - \lambda \alpha) \nu_2} r^2. \end{aligned} \tag{25}$$

The result is sharp with equality for $f(z) = z - (2(1 - \alpha)/(2 - \alpha - \lambda \alpha) \nu_2) z^2$.

Proof. We have

$$\begin{aligned}
 & (2 - \alpha - \alpha\lambda) \frac{\nu_2}{2} \sum_{n=2}^{\infty} (a_n + b_n) \\
 & \leq \sum_{n=2}^{\infty} [n - \alpha - \lambda\alpha(n - 1)] \frac{\mu_n a_n}{n} \\
 & \quad + \sum_{n=2}^{\infty} [n + \alpha - \lambda\alpha(n + 1)] \frac{\nu_n b_n}{n} \\
 & \leq 1 - \alpha.
 \end{aligned} \tag{26}$$

Thus,

$$\begin{aligned}
 |f(z)| & = \left| z - \sum_{n=2}^{\infty} a_n z^n + \sigma \sum_{n=2}^{\infty} b_n \bar{z}^n \right| \\
 & \leq r + r^2 \sum_{n=2}^{\infty} (a_n + b_n) \\
 & \leq r + \frac{2(1 - \alpha)}{(2 - \alpha - \lambda\alpha)\nu_2} r^2.
 \end{aligned} \tag{27}$$

The proof of left-hand inequality follows in lines similar to that of right-hand side inequality. \square

Theorem 8. Let $f \in TSP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ and $\mu_n \geq \nu_2, \nu_n \geq \nu_2, (n \geq 2)$, then for $|z| = r < 1$, we have

$$\begin{aligned}
 r - \frac{2(1 - \alpha)}{(3 - \alpha - \lambda(1 + \alpha))\nu_2} r^2 & \leq |f(z)| \\
 & \leq r + \frac{2(1 - \alpha)}{(3 - \alpha - \lambda(1 + \alpha))\nu_2} r^2.
 \end{aligned} \tag{28}$$

The result is sharp with equality for $f(z) = z - (2(1 - \alpha)/(3 - \alpha - \lambda(1 + \alpha))\nu_2)z^2$.

Corollary 9. Let $f \in TSH^0(\Phi, \Psi, \sigma, \lambda, \alpha)$, then

$$\left\{ w : |w| < 1 - \frac{2(1 - \alpha)}{(2 - \alpha - \lambda\alpha)\nu_2} \right\} \subset f(U). \tag{29}$$

Corollary 10. Let $f \in TSP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$, then

$$\left\{ w : |w| < 1 - \frac{2(1 - \alpha)}{(3 - \alpha - \lambda(1 + \alpha))\nu_2} \right\} \subset f(U). \tag{30}$$

Finally, we determine the extreme points of the class $TSH^0(\Phi, \Psi, \sigma, \lambda, \alpha)$, and $TSP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$.

Theorem 11. Let

$$\begin{aligned}
 h_1(z) & = z, \quad h_n(z) = z - \frac{n(1 - \alpha)}{[n - \alpha - \lambda\alpha(n - 1)]\mu_n} z^n, \\
 g_n(z) & = z - \frac{n(1 - \alpha)}{\sigma[n + \alpha - \lambda\alpha(n + 1)]\nu_n} \bar{z}^n \quad (n = 2, 3, \dots).
 \end{aligned} \tag{31}$$

A function $f \in TSH^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ if and only if f can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} (\beta_n h_n + \gamma_n g_n), \tag{32}$$

where $\beta_n \geq 0, \gamma_n \geq 0, \beta_1 = 1 - \sum_{n=2}^{\infty} (\beta_n + \gamma_n)$ and $\gamma_1 = 0$. In particular, the extreme points of $TSH^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Let

$$\begin{aligned}
 f(z) & = \sum_{n=2}^{\infty} (\beta_n h_n + \gamma_n g_n) \\
 & = z - \sum_{n=2}^{\infty} \beta_n \frac{n(1 - \alpha)}{[n - \alpha - \lambda\alpha(n - 1)]\mu_n} z^n \\
 & \quad + \sigma \sum_{n=2}^{\infty} \gamma_n \frac{n(1 - \alpha)}{[n + \alpha - \lambda\alpha(n + 1)]\nu_n} \bar{z}^n.
 \end{aligned} \tag{33}$$

Since

$$\begin{aligned}
 & \sum_{n=2}^{\infty} \frac{\mu_n}{n} \left(\frac{n - \alpha - \lambda\alpha(n - 1)}{1 - \alpha} \right) \beta_n \frac{n(1 - \alpha)}{[n - \alpha - \lambda\alpha(n - 1)]\mu_n} \\
 & \quad + \sum_{n=1}^{\infty} \frac{\nu_n}{n} \left(\frac{n + \alpha - \lambda\alpha(n + 1)}{1 - \alpha} \right) \gamma_n \frac{n(1 - \alpha)}{[n + \alpha - \lambda\alpha(n + 1)]\nu_n} \\
 & = \sum_{n=2}^{\infty} (\beta_n + \gamma_n) = 1 - \beta_1 \leq 1,
 \end{aligned} \tag{34}$$

from Theorem 5, $f \in TSH^0(\Phi, \Psi, \sigma, \lambda, \alpha)$.

Conversely, if $f \in TSH^0(\Phi, \Psi, \sigma, \lambda, \alpha)$, then $a_n \leq n(1 - \alpha)/(n - \alpha - \lambda\alpha(n - 1))\mu_n$ and $b_n \leq n(1 - \alpha)/(n + \alpha - \lambda\alpha(n + 1))\nu_n$. Set $\beta_n = (\mu_n/n)(n - \alpha - \lambda\alpha(n - 1)/(1 - \alpha))a_n, \gamma_n = (\nu_n/n)(n + \alpha - \lambda\alpha(n + 1)/(1 - \alpha))b_n, \beta_1 = 1 - \sum_{n=2}^{\infty} (\beta_n + \gamma_n)$, and $\gamma_1 = 0$. Then,

$$\sum_{n=2}^{\infty} (\beta_n h_n + \gamma_n g_n) = z - \sum_{n=2}^{\infty} a_n z^n + \sigma \sum_{n=2}^{\infty} b_n \bar{z}^n = f(z). \tag{35}$$

\square

Theorem 12. Let

$$\begin{aligned}
 p_1(z) & = z, \\
 p_n(z) & = z - \frac{n(1 - \alpha)}{[2n - 1 - \alpha - \lambda(1 + \alpha)(n - 1)]\mu_n} z^n, \\
 q_n(z) & = z - \frac{n(1 - \alpha)}{\sigma[2n + 1 + \alpha - \lambda(1 + \alpha)(n + 1)]\nu_n} \bar{z}^n \\
 & \quad (n = 2, 3, \dots).
 \end{aligned} \tag{36}$$

A function $f \in TSP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ if and only if f can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} (\beta_n p_n + \gamma_n q_n), \quad (37)$$

where $\beta_n \geq 0$, $\gamma_n \geq 0$, $\beta_1 = 1 - \sum_{n=2}^{\infty} (\beta_n + \gamma_n)$, and $\gamma_1 = 0$. In particular, the extreme points of $TSP_H^0(\Phi, \Psi, \sigma, \lambda, \alpha)$ are $\{p_n\}$ and $\{q_n\}$.

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